

Gravity Duals for Nonrelativistic Conformal Field Theories

Koushik Balasubramanian and John McGreevy

CTP, MIT, Cambridge, Massachusetts 02139, USA

(Received 6 May 2008; published 4 August 2008)

We attempt to generalize the anti-de Sitter/conformal field theory correspondence to nonrelativistic conformal field theories which are invariant under Galilean transformations. Such systems govern ultracold atoms at unitarity, nucleon scattering in some channels, and, more generally, a family of universality classes of quantum critical behavior. We construct a family of metrics which realize these symmetries as isometries. They are solutions of gravity with a negative cosmological constant coupled to pressureless dust. We discuss realizations of the dust, which include a bulk superconductor. We develop the holographic dictionary and find two-point correlators of the correct form. A strange aspect of the correspondence is that the bulk geometry has two extra noncompact dimensions.

DOI: [10.1103/PhysRevLett.101.061601](https://doi.org/10.1103/PhysRevLett.101.061601)

PACS numbers: 11.25.Tq, 03.75.Ss

Introduction.—Many attempts have been made to use the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1] to study systems realizable in a laboratory. One does not yet have a holographic dual matching the precise microscopic details of any such system and is therefore led to try to match the universality class of the system. In general, physics in the far infrared is described by a (sometimes trivial) fixed point of the renormalization group. It has been argued that the associated zero-temperature CFT controls a swath of the finite-temperature phase diagram, namely, the region in which the temperature is the only important scale, and no dimensionful couplings have turned on (see, e.g., [2]). In trying to use gauge-string duality to study such questions, it seems urgent, then, to match the symmetries to those of the system of interest.

The AdS/CFT correspondence so far gives an effective description of relativistic conformal field theories at strong coupling. Not many of these are accessible experimentally.

However, there are many *nonrelativistic* conformal field theories which govern physical systems. Such examples arise in condensed matter physics [2], atomic physics [3], and nuclear physics [4]. In the first situation, these are called “quantum critical points”; this term refers to a second-order phase transition that is reached not by varying the temperature but by varying some coupling constants at zero temperature.

A particularly interesting example where exquisite experimental control is possible is the case of cold fermionic atoms at unitarity (e.g., [3] and references therein and thereto). These are scale-invariant in the following sense. The interactions are tuned (by manipulating the energy of a two-body bound state to threshold) to make the scattering length infinite (hence the system is strongly coupled). The material is dilute enough that the effective range of the potential can be treated as zero. While the mass of the atoms is a dimensionful quantity, the dependence of the energy and other physical quantities on the mass is determined by the symmetry algebra. A recent paper studying

the constraints from Galilean conformal symmetry on such systems is Ref. [5].

In this Letter, we set out to find a bulk dual of nonrelativistic CFTs, analogous to the AdS gravity description of relativistic CFTs, at strong coupling. We approach this question by considering the algebra of generators of the nonrelativistic conformal group, which appears in Ref. [5] (related work includes Ref. [6]). We seek a solution of string theory (in its low-energy gravity approximation) whose asymptotic boundary symmetry group is not the Poincaré symmetry group (i.e., Lorentz and translations) but rather Galilean invariance and translations. We also demand a nonrelativistic version of scale invariance and, when possible, special conformal transformations.

Clearly, we will have to introduce some background energy density to find such a solution (i.e., it will not be just a solution of gravity with a cosmological constant). Holographically, this is because the theories that we are trying to describe arise in general (perhaps not always) from relativistic microscopic theories (not necessarily conformal) upon adding some background of stuff (which fixes a preferred reference frame) and then taking some limit focusing on excitations with small velocities in the frame of the stuff.

An analogy which may be useful is the following. The plane-wave symmetry group is a contraction of the AdS isometry group; the plane-wave solution arises from AdS in considering a limit focusing on the worldline of a fast-moving particle. So it might be possible to take some limit of AdS analogous to the plane-wave limit to find our solution. The limit involved would be like the Penrose limit but would replace the single particle worldline with a uniform background number density. We leave such a derivation from a more microscopic system for the future and proceed to guess the end result.

Scale invariance can be realized in nonrelativistic theories in a number of ways. One freedom is the relative scale dimension of time and space, called the “dynamical exponent” z (see, e.g., [7]). The familiar Schrödinger case

has $z = 2$, and the conformal algebra for this case is described in Ref. [5]. Fixed points with $z \neq 2$ can be Galilean-invariant, too. We will find a metric with this symmetry group as its isometry group for each value of z [8]. To be specific, the relevant algebra is generated by angular momenta M_{ij} , momentum P_i , Galilean boosts K_i , dilatations D , and rest mass N . Here $i, j = 1, \dots, d$ label the spatial dimensions. The commutators which are neither zero nor determined by spin are

$$[K_i, P_j] = i\delta_{ij}N, \quad [D, P_i] = iP_i, \quad [D, K_i] = (1-z)iK_i, \\ [H, K_i] = -iP_i, \quad [D, H] = z i H. \quad (1)$$

For the special case $z = 2$, there is an additional special conformal generator C which enjoys the algebra:

$$[M_{ij}, C] = 0, \quad [K_i, C] = 0, \\ [D, C] = -2iC, \quad [H, C] = -iD.$$

The metrics we find are also invariant under the combined operations of time reversal and charge conjugation. That the particle number can be conserved is a consequence of the possibility of the absence of antiparticles in a non-relativistic theory [10].

In the next section, we write down a metric with this isometry group [11]. It is sourced by a stress tensor describing a negative cosmological constant plus dust. After demonstrating the action of the isometry group, we provide answers to the important question: Why does the dust not collapse due to the gravitational attraction? In the third section, we generalize the AdS/CFT dictionary to extract information about the field theory dual, including anomalous dimensions and Green's functions. To conclude, we highlight some of the many open questions raised by this construction.

Geometry.—The metric that we will study is

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2 + 2d\xi dt}{r^2} + \frac{dr^2}{r^2} \right). \quad (2)$$

Here $\vec{x}$ is a d vector, and z is the advertised dynamical exponent. L is a length scale analogous to the AdS radius. The coordinate ξ is something new, whose interpretation we will develop below [13]. The metric (2) is nonsingular. As is manifest in the chosen coordinates, it is conformal to a pp -wave spacetime.

We shall show that the isometries of the above metric comprise the $d + 1$ -dimensional nonrelativistic dilatation group with algebra (1). It is clear that the above metric is invariant under translations in $\vec{x}$ and t and under rotations of $\vec{x}$. It is also invariant under the scale transformation

$$x' = \lambda x, \quad t' = \lambda^z t, \quad r' = \lambda r, \quad \xi' = \lambda^{2-z} \xi.$$

It is invariant under the Galilean boost $\vec{x}' = \vec{x} - \vec{v}t$ if we assign the following transformation to the coordinate ξ : $\xi' = \xi + \frac{1}{2}(2\vec{v} \cdot \vec{x} - v^2 t)$. In the special case $z = 2$, special conformal transformations act as

$$x' = \frac{x}{1+ct}, \quad t' = \frac{t}{1+ct}, \quad r' = \frac{r}{1+ct}, \\ \xi' = \xi + \frac{c}{2} \frac{\vec{x} \cdot \vec{x} + r^2}{1+ct}.$$

One can check that the vector fields generating these isometries have Lie brackets satisfying the algebra (1) [16]. The only nonobvious identification is that the rest mass is identified with $N = \partial_\xi$. Note that, although the metric is not invariant under simple time reversal $t \rightarrow -t$, it is invariant under the combined operation $t \rightarrow -t$, $\xi \rightarrow -\xi$, which, given the interpretation of the ξ -momentum as rest mass, can be interpreted as the composition of charge conjugation and time reversal.

The stress tensor sourcing the metric in Eq. (2) consists of a negative cosmological constant [17] $\Lambda = -\frac{(d+1)(d+2)}{L^2}$ plus a pressureless “dust,” with constant energy density $\mathcal{E}$:

$$T_{ab} = -\Lambda g_{ab} - \mathcal{E} \delta_a^0 \delta_b^0 g_{00}.$$

In the case $d = 3$, we find $\mathcal{E} = (2z^2 + z - 3)L^{-2}$. Our next goal is to identify a physical realization of this dust.

Motivated by the plane-wave analogy described in the introduction, we include a gauge field in the bulk, with a nonzero background charge density. We observe that, because of the nonstationary form of the metric, the stress tensor for an electric field in the radial direction F_{rt} has only a 00 component. Maxwell's equation in the bulk coupled to a background current j^b is $(1/\sqrt{g})\partial_a(\sqrt{g}F^{ab}) = j^b$. To produce an electric field in the r direction, which gives the correct behavior of T_{00} , we consider a nonzero j^ξ of the form $j^\xi = \rho_0 r^{-z+2}$, which by Gauss' law generates an electrostatic potential $A_0 = \frac{\rho_0}{z(z+d)} r^{-z}$. The current and gauge field are therefore related by the London equation $j_a = m_A^2 A_a$, with $m_A^2 = \frac{z(z+d)}{L^2}$ [18].

Having gained this insight, we observe that our metric is sourced by the ground state of an Abelian Higgs model in its broken phase. The model

$$S = \int d^{d+3}x \sqrt{g} \left(-\frac{1}{4} F^2 + \frac{1}{2} |D\Phi|^2 - V(|\Phi|^2) \right),$$

with $D_a \Phi \equiv (\partial_a + ieA_a)\Phi$, with a Mexican-hat potential

$$V(|\Phi|^2) = g(|\Phi|^2 - v^2)^2 + \Lambda,$$

produces the correct dust stress tensor for arbitrary g , as long as $e^2 v^2 = m_A^2 = \frac{z(z+d)}{L^2}$ as above [21]. The parameter ρ_0 in the solution for the gauge field is determined by the Einstein equation to be $\rho_0^2 = \mathcal{E} \frac{z(z+d)^2}{2z+d} L^4$. It is tempting to relate the phase of Φ to that of the boundary wave function. We hope in the future to use this theory with finite g to describe the vortex lattice formed by rotating the cold-atom superfluids.

The fact that the metric (2) is sourced by a reasonable class of physical systems with stable ground states suggests that it need have no intrinsic instabilities.

Correspondence.—Consider a scalar field in the background (2), with action $S = -\frac{1}{2} \int d^{d+3}x \sqrt{g} (\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2)$. As in AdS, the transverse fluctuations of the metric $h_y^x(k_z)$ satisfy the same equation of motion as a massless scalar, and so we can take the $m = 0$ case as a proxy for this component of the metric fluctuations.

Translation invariance allows us to Fourier decompose in directions other than r : $\phi(r) = e^{i\omega t + i\vec{k} \cdot x + i l \xi} f_{\omega, \vec{k}, l}(r)$. Then the wave equation in the background (2) takes the form

$$\left[-r^{d+3} \partial_r \left(\frac{1}{r^{d+1}} \partial_r \right) + r^2 (2l\omega + \vec{k}^2) + r^{4-2z} l^2 + m^2 \right] f_{\omega, \vec{k}, l}(r) = 0. \quad (3)$$

To find the generalization of the gauge/gravity dictionary to this case, we first study asymptotic solutions of this equation near the boundary. Writing $f \propto r^\Delta$, we find for $z \leq 2$:

$$\Delta_\pm = 1 + \frac{d}{2} \pm \sqrt{\left(1 + \frac{d}{2}\right)^2 + m^2 + \delta_{z,2} l^2}. \quad (4)$$

As usual, these exponents $\Delta_\pm$ can be interpreted as scaling dimensions of spin-zero operators in the boundary theory. With the action S , the normalizability condition is $\Delta > 1 + \frac{d}{2}$; by modifying the action by boundary terms as in Ref. [22], this can be relaxed to $\Delta > d/2$. It would be nice to match this to a unitarity bound on operator dimensions as in the relativistic case. We look forward to extending this analysis to the fluctuations of fields of other spin.

In the case $z > 2$, there is no scaling solution near the boundary; the ξ momentum dominates the boundary behavior, and the solutions vary with r faster than any power of r . It would be interesting to understand a possible meaning of the asymptotic solutions which do arise and to interpret the critical nature of $z = 2$ in terms of the boundary theory.

For definiteness, we focus on the familiar, critical $z = 2$ case with $d = 3$ spatial dimensions. In that case, the momentum in the ξ direction simply adds to the mass of the bulk scalar; i.e., they appear in the combination $l^2 + m^2$. In this case the wave equation is solved by

$$f_{\omega, \vec{k}, l}(r) = A r^{5/2} K_\nu(\kappa r),$$

where K is a modified Bessel function, A is a normalization constant, and [23]

$$\nu = \sqrt{\left(\frac{5}{2}\right)^2 + l^2 + m^2}, \quad \kappa^2 = 2l\omega + \vec{k}^2.$$

As usual in AdS/CFT, we choose K over I because it is well-behaved near the horizon, $K(x) \sim e^{-x}$ at large x .

To compute correlators, we introduce a cutoff near the boundary at $r = \epsilon$ and normalize $f_\kappa(\epsilon) = 1$ [so $A = \epsilon^{-5/2} K_\nu(\kappa\epsilon)^{-1}$]. The on-shell bulk action is

$$S[\phi_0] = \frac{1}{2} \left[\int d^{d+2}X \sqrt{g} g^{rr} \phi(X) \partial_r \phi(X) \right]_{r=\epsilon},$$

where $X \in \{\vec{x}, t, \xi\}$ are the coordinates on the boundary. This evaluates to

$$S[\phi_0] = \frac{1}{2} \int dp \phi_0(-p) \mathcal{F}(\kappa, \epsilon) \phi_0(p),$$

with the “flux factor”

$$\begin{aligned} \mathcal{F}(\kappa, \epsilon) &= \lim_{r \rightarrow \epsilon} \sqrt{g} g^{rr} f_\kappa(r) \partial_r f_\kappa(r) \\ &= \sqrt{g} g^{rr} \partial_r [r^{1+(d/2)} \ln K_\nu(\kappa r)]|_{r=\epsilon}. \end{aligned}$$

Expanding near the boundary and keeping the leading singular term, this gives

$$\begin{aligned} \langle \mathcal{O}_1(\omega, \vec{k}) \mathcal{O}_2(\omega', \vec{k}') \rangle &\propto \delta(k + k') \frac{1}{2\epsilon^5} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \\ &\times \left(\frac{(2l\omega + \vec{k}^2)\epsilon^2}{4} \right)^\nu \end{aligned}$$

(in the case $\phi = h_{xy}$, $\mathcal{O} = T_{xy}$). Fourier transforming to position space, this is

$$\langle \mathcal{O}_1(x, t) \mathcal{O}_2(0, 0) \rangle \propto \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \delta_{\Delta_1, \Delta_2} \theta(t) \frac{1}{|\epsilon^2 t|^\Delta} e^{-ilx^2/2|t|},$$

where we used translation invariance to put the second operator at $(\vec{x}, t) = (\vec{0}, 0)$ and $\Delta_{i,j}$ are the scaling dimensions of the operators that we found in (4). This is the form one expects for the correlators in a Galilean-invariant conformal field theory in d spatial dimensions with dynamical exponent $z = 2$ [5].

Discussion.—In discussions of AdS/CFT, one often hears that the CFT “lives at the boundary” of the bulk spacetime. The spacetime (2) is conformal to a pp -wave spacetime and hence has a boundary which is one-dimensional—for $z > 1$, g_{tt} grows faster than the other components at small r . While one might be tempted to speculate that this means that the dual field theory is one-dimensional, there is no clear evidence for this statement (in the usual case, only the UV of the field theory can really be said to live at the boundary of the AdS space), and indeed we have nevertheless used it to compute correlators which look like those of a d -dimensional nonrelativistic field theory.

Perhaps the most mysterious aspect of this new correspondence is not of too few dimensions but of too many. The ξ direction is an additional noncompact dimension of the bulk geometry besides the usual radial direction, which seems to be associated to the conserved rest mass. We have computed correlators at fixed l ; this is reasonable since the theory is nonrelativistic, and so a sector with definite l can be closed (unlike in a relativistic theory, where these sectors unavoidably mix by production of back-to-back particles and antiparticles). The relation to the plane wave appears here, too: The plane wave describes a relativistic sys-

tem in the infinite momentum frame, where there is also no particle production [24].

One interesting observable, which has been *measured* as well as computed numerically for cold-fermion systems, is the analog of the famous $3/4$ in the $\mathcal{N} = 4$ super Yang-Mills theory, namely, the ratio of the energy density at unitarity to the energy density of the corresponding free theory (ξ in, e.g., Ref. [25], proportional to β in Ref. [26]) [27]. We will need to learn to compute the analog of the Arnowitt-Deser-Misner mass for spacetimes with the asymptotium of (2).

Once the finite-temperature solution with these asymptotics is found, there will be many more interesting things to compute. Of course, it will be nice to check the universality of $\frac{7}{5}$ [28], obtainable from the correlator that we have studied, at nonzero temperature. As a possible but quite ambitious example of a qualitative consequence of the nonrelativistic microscopic constituents in these systems, we offer the following. It has been noted (e.g., [29]) that the plasma made by putting any relativistic CFT at finite temperature is quite compressible. The compressibility is known to have an important effect on the turbulence cascade [30]; it can even change its direction. It is therefore of interest to have gravity descriptions of less compressible fluids, which these would presumably be.

We thank Allan Adams, Shamit Kachru, Pavel Kovtun, Hong Liu, Mukund Rangamani, Krishna Rajagopal, Brian Swingle, and Alessandro Tomasiello for discussions and encouragement. This work was supported in part by funds provided by the U.S. Department of Energy (DOE) under cooperative research agreement No. DE-FG0205ER41360.

Note added.—While this work was being completed, we learned that Son has independently found very closely related results [31].

-
- [1] J. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998); *Int. J. Theor. Phys.* **38**, 1113 (1999); E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Phys. Lett. B* **428**, 105 (1998).
 - [2] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).
 - [3] K. M. O'Hara *et al.*, *Science* **298**, 2179 (2002); C. A. Regal, M. Greiner, and D. S. Jin, *Phys. Rev. Lett.* **92**, 040403 (2004); M. Bartenstein *et al.*, *Phys. Rev. Lett.* **92**, 120401 (2004); M. Zwierlein *et al.*, *Phys. Rev. Lett.* **92**, 120403 (2004); J. Kinast *et al.*, *Phys. Rev. Lett.* **92**, 150402 (2004); T. Bourdel *et al.*, *Phys. Rev. Lett.* **93**, 050401 (2004).
 - [4] T. Mehen, I. W. Stewart, and M. B. Wise, *Phys. Lett. B* **474**, 145 (2000).
 - [5] Y. Nishida and D. T. Son, *Phys. Rev. D* **76**, 086004 (2007).
 - [6] U. Niederer, *Helv. Phys. Acta* **45**, 802 (1972); C. R. Hagen, *Phys. Rev. D* **5**, 377 (1972); R. Jackiw and S. Pi, arXiv:hep-th/9206092.

- [7] P. C. Hohenberg and B. I. Halperin, *Rev. Mod. Phys.* **49**, 435 (1977).
- [8] Gravity duals of related but distinct Lifshitz points (with time-reversal invariance) are being studied by Kachru, Liu, and Mulligan [9]. We thank them for private correspondence.
- [9] S. Kachru, X. Liu, and M. Mulligan (to be published).
- [10] That is to say, the dual field theories that we are describing must not have particle production. It is, of course, possible for nonrelativistic theories to nevertheless contain both particles and antiparticles, as indeed do the systems under study in Ref. [9], which, unlike ours, are invariant under $t \rightarrow -t$.
- [11] Earlier work on geometric realizations of the Schrödinger group include Ref. [12] and references therein, wherein the metrics that we study are called Bargmann-Einstein structures.
- [12] C. Duval, G. W. Gibbons, and P. Horvathy, *Phys. Rev. D* **43**, 3907 (1991).
- [13] In the case $d = 2$, $z = 3$, this metric has appeared previously as the near-horizon limit of D3-branes in the presence of lightlike tensor fields [14]; its causal structure was studied in Ref. [15]. The sources present in those solutions do not preserve the rotations M_{ij} .
- [14] M. Alishahiha, Y. Oz, and J. G. Russo, *J. High Energy Phys.* 09 (2000) 002.
- [15] V. E. Hubeny, M. Rangamani, and S. F. Ross, *J. High Energy Phys.* 07 (2005) 037.
- [16] We believe that no $(d + 2)$ -dimensional metric can realize this isometry group.
- [17] We have set $8\pi G_N = 1$. The indices a and b run over all $d + 3$ dimensions of the bulk.
- [18] Recent holographic studies of spontaneously broken global symmetries of the boundary theory include Refs. [19] and (at finite temperature) [20].
- [19] J. Erlich *et al.*, *Phys. Rev. Lett.* **95**, 261602 (2005).
- [20] S. S. Gubser, arXiv:0801.2977; S. S. Gubser, arXiv:0803.3483; S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, arXiv:0803.3295 [*Phys. Rev. Lett.* (to be published)].
- [21] Our metric is therefore a solution of the same system studied in the first and last references in [20], with very different asymptotics.
- [22] I. R. Klebanov and E. Witten, *Nucl. Phys.* **B556**, 89 (1999).
- [23] For general d , with $z = 2$, the solution is $f_{\omega, \vec{k}, l}(r) = Ar^{1+d/2} K_\nu(\kappa r)$, with $\nu = \sqrt{[1 + (d/2)]^2 + M^2}$, $\kappa^2 = 2l\omega + \vec{k}^2$, and $M^2 \equiv m^2 + l^2$.
- [24] We thank Allan Adams for this observation.
- [25] T. Schafer, *Prog. Theor. Phys. Suppl.* **168**, 303 (2007).
- [26] J. T. Stewart *et al.*, *Phys. Rev. Lett.* **97**, 220406 (2006).
- [27] We thank Krishna Rajagopal for teaching us about this.
- [28] A. Buchel and J. T. Liu, *Phys. Rev. Lett.* **93**, 090602 (2004); P. Kovtun, D. T. Son, and A. O. Starinets, *Phys. Rev. Lett.* **94**, 111601 (2005).
- [29] M. Van Raamsdonk, *J. High Energy Phys.* 05 (2008) 106.
- [30] M. Chertkov, I. Kolokolov, and M. Vergassola, *Phys. Rev. Lett.* **80**, 512 (1998).
- [31] D. Son, arXiv:0804.3972 [*Phys. Rev. D* (to be published)].