

Finite-Frequency Fluctuation-Response Inequality

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We derive an inequality relating the finite-frequency linear response and fluctuations of an observable in a physical system in steady state. The relation holds for arbitrary observables and perturbations in general Markovian dynamics, including overdamped and underdamped Langevin systems and jump processes, both in and out of equilibrium. As a consequence, we obtain a universal upper bound on the broadband signal-to-noise ratio of noisy dynamics, which only depends on the damping constant and temperature. We further show that the inequality reduces to an equality for appropriately chosen observables or perturbations in linear systems, both overdamped and underdamped and both in and out of equilibrium.

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Fluctuations and response are equally important for measurements of small noisy systems: on the one hand, when probing a system by measuring its response to an external perturbation, the accuracy of the measurement is limited by the fluctuations of the system due to environmental noise. When the system acts as a sensor, the ratio of response and fluctuations corresponds to the signal-to-noise ratio. On the other hand, a measurement of the fluctuations themselves can also reveal information about the dynamics of the system. This connection is most prominent in equilibrium systems, where response and fluctuations are directly proportional and related by the fluctuation-dissipation theorem [1,2].

Out of equilibrium, this intimate connection between response and fluctuations is lost. A phenomenological fluctuation-dissipation theorem may be formulated by introducing an effective temperature [3–7], however, the latter depends on the type of measurement [8]. A non-equilibrium fluctuation-dissipation theorem may also be derived from the underlying dynamics [9–13], expressing the response as fluctuations of the unperturbed system. However, these fluctuations are expressed using formal conjugate observables that are not directly measurable, thus limiting practical applications.

More recent results establish relations between response and fluctuations even for out-of-equilibrium systems [14–21], whose common feature is that they are formulated in the form of inequalities rather than equalities like the fluctuation-dissipation theorem. However, from the point of view of applications, these results suffer from two drawbacks: first, most results are formulated for discrete systems [15,16,18,19,21] and not straightforward to extend to continuous systems. By contrast, experiments are frequently performed in continuous systems, be it active matter [22,23], granular materials [24], biological systems [25–27], or levitated nanoparticles [28,29].

Second, and more seriously, existing results only apply to the static response of a system, where the perturbation is constant in time. Conversely, in experiments, it is more common to work with time-periodic perturbations and to characterize response and fluctuations in frequency space [22,30–33]. This has practical reasons, as observing the static response can be challenging in small systems—a tracer particle inside a biological cell subjected to a constant force will eventually collide with the cell wall. But measuring response at different frequencies also yields additional information about the dynamics and structure of the system that is not contained in the static, zero-frequency response.

In this Letter, we derive a fluctuation-response inequality (FRI) relating the finite-frequency response and fluctuations of general observables, applicable to both discrete and continuous systems. The ratio between the magnitude of the linear response function and the power spectral density of an arbitrary observable at any frequency is upper bounded by a frequency-independent constant, depending on the temperature and friction coefficient of the environment. As a corollary, we obtain a universal upper bound on the frequency-integrated signal-to-noise ratio of noisy sensors. Moreover, we show that equality in the FRI is attained for linear dynamics, both in and out of equilibrium, which indicates that linear systems minimize the amount of fluctuations for a given magnitude of the response.

Setup—In this Letter, we focus on Langevin dynamics [34,35] describing the noisy time evolution of a set of D continuous degrees of freedom $\mathbf{x}(t) = [x_1(t), \dots, x_D(t)]$. The dynamics can be overdamped, in which case

$$\dot{\mathbf{x}}(t) = \frac{1}{\gamma}(\mathbf{f}[\mathbf{x}(t)] + \epsilon \mathbf{h}[\mathbf{x}(t), t]) + \sqrt{\frac{2T}{\gamma}}\boldsymbol{\xi}(t), \quad (1)$$

or underdamped, in which case

$$m\dot{\mathbf{v}}(t) = -\gamma\mathbf{v}(t) + \mathbf{f}[\mathbf{x}(t)] + \epsilon\mathbf{h}[\mathbf{x}(t), t] + \sqrt{2\gamma\mathbf{T}}\boldsymbol{\xi}(t). \quad (2)$$

Here, $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$ is the velocity, $\mathbf{f}(\mathbf{x})$ are coordinate-dependent forces (interactions or conservative and non-conservative external forces), and $\boldsymbol{\xi}(t)$ is a vector of uncorrelated white Gaussian noises. All degrees of freedom have the same mass m and friction coefficient γ and are in contact with equilibrium environments, whose temperatures are the entries of the diagonal matrix $\mathbf{T}$. More general situations, including nonthermal environments, velocity-dependent forces, and jump processes on a discrete state space are discussed in Supplemental Material (SM) A [36]. $\mathbf{h}(\mathbf{x}, t)$ is a perturbation force; we focus on the linear response regime $\epsilon \ll 1$. We assume that, for $\epsilon = 0$, the dynamics described by Eqs. (1) or (2) is in a steady state with probability density $p_{\text{st}}(\mathbf{x})$ or $p_{\text{st}}(\mathbf{x}, \mathbf{v})$.

We measure a set of K observables $z_k(t) = z_k(\mathbf{x}(t))$, $k = 1, \dots, K$, which are functions of the coordinates of the system. We quantify the fluctuations of the observables using the spectral-density matrix $\mathcal{S}(\omega)$, defined as the Fourier transform of the steady-state covariance [39]

$$\mathcal{S}_{kl}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \text{Cov}[z_k(t), z_l(0)]. \quad (3)$$

We further consider a set of Q time-dependent perturbations to Eq. (1) or Eq. (2),

$$\mathbf{h}(\mathbf{x}, t) = \sum_{q=1}^Q \mathbf{g}_q(\mathbf{x}) \phi_q(t). \quad (4)$$

Here, $\mathbf{g}_q(\mathbf{x})$ are coordinate-dependent forces and $\phi_q(t)$ are time-dependent functions. In the limit $\epsilon \rightarrow 0$, the response of the observables to the perturbations can to leading order be expressed using the linear-response matrix $\mathbf{R}(t)$ [10],

$$\lim_{\epsilon \rightarrow 0} \frac{\langle z_k \rangle_{t,\epsilon} - \langle z_k \rangle_{\text{st}}}{\epsilon} = \sum_{q=1}^Q \int_0^t dt' R_{kq}(t-t') \phi_q(t'), \quad (5)$$

where we assume that the system is initially in the steady state with average $\langle z_k \rangle_{\text{st}}$. $\langle z_k \rangle_{t,\epsilon}$ denotes the average value of $z_k(t)$ when applying the perturbation. $R_{kq}(t-t')$ expresses the impact of the perturbation q at time t' on the average of the observable k at the later time t . Equivalently, we have in Fourier space,

$$\mathcal{R}(\omega) = \int_0^{\infty} dt e^{i\omega t} \mathbf{R}(t), \quad (6)$$

where we used that $\mathbf{R}(t) = 0$ for $t < 0$.

Fluctuation-response inequality—The first main result is the inequality between the response- and spectral-density matrices,

$$\mathcal{R}^H(\omega) \mathcal{S}^{-1}(\omega) \mathcal{R}(\omega) \leq \mathcal{A}, \quad (7)$$

with $\mathcal{A}_{qr} = \frac{1}{2\gamma} \langle \mathbf{g}_q \cdot \mathbf{T}^{-1} \mathbf{g}_r \rangle_{\text{st}},$

where H denotes the Hermitian conjugate. The proof is provided in Appendix A for overdamped Langevin systems and in SM, Sec. A [36] for underdamped Langevin and jump dynamics. The matrix inequality Eq. (7) means that the difference of the right-hand and left-hand side is positive semidefinite. Since the right-hand side is independent of frequency and the choice of the observables, this relation yields a strong constraint between the response and fluctuations of observables in Markovian systems, irrespective of whether the system is in or out of equilibrium. This generalizes the linear fluctuation-response inequality (FRI) derived in Ref. [14] to more general observables and finite frequencies; we therefore refer to Eq. (7) as the finite-frequency FRI. We remark that for jump processes, the matrix $\mathcal{A}$ is closely related to dynamical activity (see SM, Sec. A [36] for details). For a single perturbation $\mathbf{g}(\mathbf{x})\phi(t)$, the response matrix reduces to the vector of response functions and the matrix $\mathcal{A}$ to the scalar $\mathcal{A} = \langle \mathbf{g} \cdot \mathbf{T} \mathbf{g} \rangle_{\text{st}} / (2\gamma)$. In this case, we define the response efficiency

$$\eta^R(\omega) = \frac{\mathcal{R}^*(\omega) \cdot \mathcal{S}^{-1}(\omega) \mathcal{R}(\omega)}{\mathcal{A}} \leq 1, \quad (8)$$

where $*$ denotes complex conjugation. The dimensionless response efficiency satisfies $0 \leq \eta^R(\omega) \leq 1$ and quantifies the magnitude of the observed response in relation to the fluctuations of the observables. It allows characterizing the sensitivity to perturbations in a unified manner, enabling comparisons between different systems, observables, and perturbations. For example, a weakly trapped particle is susceptible to perturbations and exhibits a large response, however, it is also susceptible to fluctuations and its measurement does necessarily not yield more information about the perturbation than a strongly trapped particle with small response and small fluctuations. For a single observable and a single perturbation ($K = Q = 1$), Eqs. (7) and (8) simplify to

$$|\mathcal{R}(\omega)|^2 \leq \mathcal{A} \mathcal{S}(\omega), \quad \eta^R(\omega) = \frac{|\mathcal{R}(\omega)|^2}{\mathcal{S}(\omega) \mathcal{A}}. \quad (9)$$

Thus, the response of any observable to a perturbation with frequency ω is bounded by the fluctuations of the observable at the same frequency times a constant. If the functional form of the perturbation is known, then the right hand side can be evaluated from the steady-state statistics of the system. Therefore, Eq. (9) allows us to infer a bound on the response from a passive measurement of the system.

Response and fluctuations in and out of equilibrium—To demonstrate this bound, we consider an underdamped

particle in a tilted periodic potential

$$m\dot{v}(t) = -\gamma v(t) - U'[x(t)] + f_0 + \epsilon g_0 \phi(t) + \sqrt{2\gamma T} \xi(t), \quad (10)$$

where $U(x) = U_0[1 - \cos(2\pi x/L)]$. The constant force f_0 tilts the potential and drives the system into a nonequilibrium steady state with a finite average velocity $\langle v \rangle_{\text{st}}$ at long times. We remark that, despite the apparent simplicity of Eq. (10), underdamped diffusion in periodic potentials is not completely understood from a theoretical point of view and possesses a rich structure of interacting dynamical effects [40–46]. We focus on the fluctuations and response of the velocity and compute the power spectrum $S_v(\omega)$ and response $\mathcal{R}_v(\omega)$ from numerical simulations (see SM, Sec. D [36] for details). The results are shown in Fig. 1. In equilibrium, the velocity power spectrum (black dashed line) shows a pronounced peak at $\omega \approx 2\pi$, corresponding to oscillations of a trapped particle. Since the particle can escape from the potential by thermal fluctuations, at low frequencies, the velocity spectrum saturates at the long-time diffusion coefficient of the particle,

$$D_x = \frac{1}{2} \lim_{\tau \rightarrow \infty} \frac{\text{Var}(x)}{\tau} = \frac{1}{2} \lim_{\omega \rightarrow 0} S_v(\omega). \quad (11)$$

At high frequencies, the velocity spectrum decays as $S_v(\omega) \simeq 2\gamma T / (m^2 \omega^2)$. The response spectrum (black solid line) likewise exhibits a peak near the oscillation frequency of the particle, whose height, in agreement with Eq. (9), is reduced compared to the one in the power spectrum. We remark that this effect stems from the nonlinearity of the trapping force—for a linear oscillator, Eq. (9) turns onto an

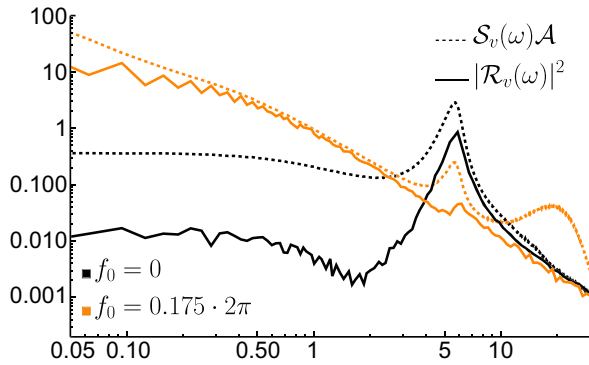


FIG. 1. Power spectral density (dashed lines) and magnitude of the response (solid lines) of the velocity of an underdamped particle in a periodic potential, Eq. (10). The black lines are for vanishing tilt $f_0 = 0$, where the system is in equilibrium. The orange lines are for a nonequilibrium steady state with $f_0 = 0.175 \cdot 2\pi$. Other parameter values are $m = 1, \gamma = 1/3, T = 1/2, U_0 = 1$, and $L = 1$. Both in and out of equilibrium, the power spectral density upper bounds the response, as predicted by Eq. (9).

equality. Equality in Eq. (9) is observed at high frequencies, where fluctuations and response are dominated by the properties of the environment. Driving the system out of equilibrium by tilting the potential decreases the effective barrier height, allowing the particle to more easily escape from a minimum of the potential. This reduces the power spectrum at the oscillation frequency, while an additional peak appears at the frequency at which the particle traverses the potential. Tilting is known to greatly increase the diffusion coefficient [43,47], leading to a significant increase in the low-frequency spectrum. Both the reduction in the power spectrum at the oscillation frequency and the increased low-frequency fluctuations are mirrored in the behavior of the response. In particular, the value of non-equilibrium power spectrum near the oscillation frequency is lower than the response in equilibrium, and thus Eq. (9) predicts a reduction of the nonequilibrium response from a passive measurement of the fluctuations.

Bound on signal-to-noise ratio—The second main result is a consequence of Eq. (9) for a single observable $z(t)$ and an equilibrium environment $\mathbf{T} = \mathbf{TI}$. In this case, the relation

$$\frac{1}{\pi} \int_0^\infty d\omega S(\omega) = \text{Var}_{\text{st}}(z) \quad (12)$$

holds, where $\text{Var}_{\text{st}}(z)$ denotes the steady-state fluctuations of $z(t)$. Integrating Eq. (9) over frequency yields

$$\frac{1}{\pi} \int_0^\infty d\omega |\mathcal{R}(\omega)|^2 \leq \frac{1}{2\gamma T} \langle \|g\|^2 \rangle_{\text{st}} \text{Var}_{\text{st}}(z). \quad (13)$$

The ratio between the magnitude of the response and the fluctuations is commonly referred to as the signal-to-noise ratio (SNR) [48],

$$\text{SNR}(\omega) = \frac{|\mathcal{R}(\omega)|}{\sqrt{\text{Var}_{\text{st}}(z)}}, \quad (14)$$

which allows rewriting the above inequality as

$$\int_0^\infty d\omega (\text{SNR}(\omega))^2 \leq \frac{\pi \langle \|g\|^2 \rangle_{\text{st}}}{2\gamma T}. \quad (15)$$

Thus, the frequency-integral of the SNR is bounded by the friction coefficient and temperature of the environment, and the overall magnitude of the perturbation, independent of the precise details of the system. The left-hand side of Eq. (15) quantifies the broadband SNR of the system over all frequencies. As we show below, equality is attained when the system consists of a single particle in a harmonic confinement. In that sense, a harmonically trapped particle in equilibrium is the best possible broadband sensor, achieving the largest possible frequency-integrated SNR. In practice, however, one is often more interested in the

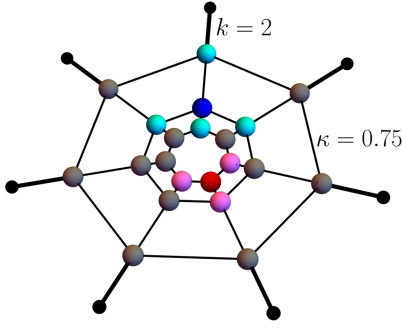


FIG. 2. A linear network consisting of 21 oscillators (spheres), coupled to each other by springs with spring constant $\kappa = 0.75$; the outermost ring is attached to the black points by springs with spring constant $k = 2$. The mass of each oscillator is $m = 1$, the damping constant is $\gamma = 0.01$, and the temperature $T = 1$. The perturbation is applied to the red oscillator and measured at the red and blue oscillators (curves of corresponding color in Fig. 3). The neighbors of these oscillators are indicated in light red and light blue, respectively.

SNR at specific frequencies, which may be substantially enhanced out of equilibrium [49]. In such a case, Eq. (15) states that enhancing the SNR beyond the single particle value in a given frequency range necessarily leads to a suppression of the SNR at other frequencies.

Conservation of response in linear networks—The third main result is that equality in Eq. (7) is attained for linear forces, $\mathbf{f}(\mathbf{x}) = -\mathbf{K}(\mathbf{x} - \mathbf{x}_{\text{eq}})$, where $\mathbf{K}$ is a matrix and $\mathbf{x}_{\text{eq}}$ is the equilibrium position. In that case, when the observables are the D positions $\mathbf{x}(t)$ of the particles and the perturbation is a spatially constant force $\mathbf{g}(\mathbf{x}) = \mathbf{g}_0$, we obtain

$$\mathcal{R}_x^*(\omega) \cdot \mathcal{S}_x^{-1}(\omega) \mathcal{R}_x(\omega) = \mathcal{A} = \frac{1}{2\gamma} \mathbf{g} \cdot \mathbf{T}^{-1} \mathbf{g}. \quad (16)$$

This relation is shown in Appendix B for overdamped systems and in SM, Sec. B [36] for underdamped systems. Importantly, this identity holds irrespective of whether the dynamics is in or out of equilibrium. For general perturbations, equality only holds when all degrees of freedom of the system are observed. Moreover, since Eq. (16) is an inequality for systems with nonlinear forces, it implies that, for a given magnitude of the response, nonlinear systems exhibit larger fluctuations. Equation (16) also applies a single degree freedom $x(t)$ in a parabolic potential $U(x) = k(x - x_{\text{eq}})^2/2$, resulting in equality in Eq. (15) after integrating over frequency.

As an example, we consider a linear network of coupled oscillators, depicted in Fig. 2. When applying a perturbation to one or several oscillators, its effect spreads across the network and we can measure the corresponding response of each oscillator. Equation (16) states that the overall response of all oscillators, weighted by their fluctuations, is a conserved quantity. In other words, measuring the response and fluctuations of all oscillators

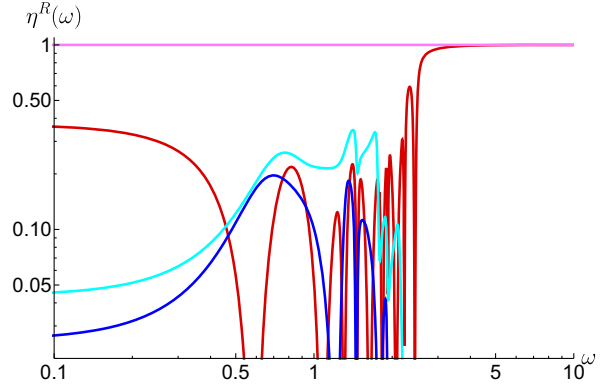


FIG. 3. Response efficiency of the network depicted in Fig. 2. The response efficiency Eq. (8) of the perturbed oscillator (red line) approaches 1 in the high-frequency limit, where the perturbation is localized. At intermediate frequencies, the response efficiency in other parts of the network (blue) can be larger. Also measuring the response of the neighbors (light blue) yields additional information about the perturbation and increases the response efficiency. In particular, measuring the perturbed oscillator and its neighbors (light red) results in a response efficiency of unity at all frequencies.

allows recovering the magnitude of an arbitrary perturbation; measuring a subset of oscillators generally only results in a lower bound. The portion of the perturbation that can be recovered from the measurement is quantified by the response efficiency Eq. (8). Figure 3 shows the response efficiency of the perturbed oscillator (red) and another oscillator in the network (blue). We observe that, in the high-frequency limit, the response efficiency of the perturbed oscillator approaches unity; at high frequencies, response and fluctuations are determined by the local properties of the environment. At intermediate frequencies, the perturbation excites collective modes in the network and causes a nonvanishing response of other oscillators. Interestingly, the response efficiency can be larger for oscillators other than the perturbed one, indicating that a nonlocal measurement can yield more information about the perturbation.

Measuring more than one oscillator simultaneously provides additional information about the perturbation. Measuring the neighboring (light blue) oscillators in addition to the blue one results in increased response efficiency (light blue line in Fig. 3). Strikingly, when measuring the perturbed (red) oscillator and its neighbors (light red) simultaneously, we find a response efficiency of unity for all frequencies (light red line in Fig. 3). As we prove in SM, Sec. C [36], this finding is generic: in linear networks, the magnitude of the perturbation can always be inferred from a measurement of the perturbed oscillators and their neighbors.

Discussion—In this Letter, we derived a universal trade-off relation between the frequency response and the frequency-resolved fluctuations of discrete and continuous

Markovian dynamics. This fluctuation-response inequality quantifies the intuition that response and fluctuations are related even out of equilibrium, since they are both properties of the same underlying physical system. We stress that Eq. (7) does not reduce to an equality in equilibrium and is thus distinct from the fluctuation-dissipation theorem. Instead, we found that equality is attained for linear dynamics, independent of whether they are in or out of equilibrium. Whether and how Eq. (7) can be related to the fluctuation-dissipation theorem and to the computation of dissipation from its violation [50,51] is an open question.

The derivation of Eq. (7) only requires the response to be represented as a correlation function of the unperturbed system. Since this relation holds for general Markovian dynamics and possibly beyond, we expect that the result can be readily generalized to other dynamics, such as open quantum systems, where the corresponding zero-frequency result was recently obtained in Ref. [21]. A more challenging generalization is to derive similar bounds on nonlinear response. Recently, a relation between static nonlinear response and first passage times was derived in Ref. [52], but whether a similar result also applies to time-periodic perturbations is unclear. Another approach is to characterize the system in the presence of a finite-strength periodic perturbation using Floquet theory and then apply the nonlinear FRI derived in Ref. [14].

The finite-frequency FRI Eq. (7) does not distinguish between systems that are in or out of equilibrium. However, the relation between response and fluctuations is clearly different out of equilibrium, where the fluctuation-dissipation theorem does not hold. Therefore, an important question is whether there are bounds on the response that explicitly take into account the nonequilibrium nature of the system. One such relation was derived for the static response of currents in Ref. [19], which is bounded by the entropy production rate. Similar relations for the finite-frequency response might be obtained starting from the variational formula for the power-spectral density derived in Refs. [53,54], which provides a formal identity between response and the power spectrum, from which bounds can be derived.

Finally, we found that the response across all degrees of freedom to a perturbation is a conserved quantity in linear systems. While we interpreted this result as being able to reconstruct the magnitude of the perturbation from the measurement, it is an open question whether it also corresponds to conservation of any physical observable. A natural system to study this question are stochastic models of circuit networks [55,56] where the perturbation corresponds to a change in the input voltage and the response is resulting change in the current through a part of the network. Within this model, it is also natural to ask whether the local conservation of response can be understood based on Kirchhoff's circuit laws. After completing

this manuscript, the author was made aware of the work by Aslyamov *et al.* [57], where the equality between the response and the power spectrum for linear systems Eq. (16) was derived independently.

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Data availability—The data that support the findings of this article are openly available [58].

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- [1] R. Kubo, The fluctuation-dissipation theorem, *Rep. Prog. Phys.* **29**, 255 (1966).
 - [2] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II: Nonequilibrium Statistical Mechanics* (Springer Science & Business Media, New York, 2012).
 - [3] L. F. Cugliandolo, J. Kurchan, and L. Peliti, Energy flow, partial equilibration, and effective temperatures in systems with slow dynamics, *Phys. Rev. E* **55**, 3898 (1997).
 - [4] D. Loi, S. Mossa, and L. F. Cugliandolo, Effective temperature of active matter, *Phys. Rev. E* **77**, 051111 (2008).
 - [5] L. F. Cugliandolo, The effective temperature, *J. Phys. A* **44**, 483001 (2011).
 - [6] G. Szamel, Self-propelled particle in an external potential: Existence of an effective temperature, *Phys. Rev. E* **90**, 012111 (2014).
 - [7] E. Lippiello, M. Baiesi, and A. Sarracino, Nonequilibrium fluctuation-dissipation theorem and heat production, *Phys. Rev. Lett.* **112**, 140602 (2014).
 - [8] S. Fielding and P. Sollich, Observable dependence of fluctuation-dissipation relations and effective temperatures, *Phys. Rev. Lett.* **88**, 050603 (2002).
 - [9] G. S. Agarwal, Fluctuation-dissipation theorems for systems in non-thermal equilibrium and applications, *Z. Phys. A* **252**, 25 (1972).
 - [10] P. Hänggi and H. Thomas, Stochastic processes: Time evolution, symmetries and linear response, *Phys. Rep.* **88**, 207 (1982).
 - [11] U. M. B. Marconi, A. Puglisi, L. Rondoni, and A. Vulpiani, Fluctuation-dissipation: Response theory in statistical physics, *Phys. Rep.* **461**, 111 (2008).
 - [12] M. Baiesi, C. Maes, and B. Wynants, Fluctuations and response of nonequilibrium states, *Phys. Rev. Lett.* **103**, 010602 (2009).
 - [13] U. Seifert and T. Speck, Fluctuation-dissipation theorem in nonequilibrium steady states, *Europhys. Lett.* **89**, 10007 (2010).
 - [14] A. Dechant and S.-i. Sasa, Fluctuation-response inequality out of equilibrium, *Proc. Natl. Acad. Sci. U.S.A.* **117**, 6430 (2020).
 - [15] J. A. Owen, T. R. Gingrich, and J. M. Horowitz, Universal thermodynamic bounds on nonequilibrium response with biochemical applications, *Phys. Rev. X* **10**, 011066 (2020).
 - [16] G. Fernandes Martins and J. M. Horowitz, Topologically constrained fluctuations and thermodynamics regulate non-equilibrium response, *Phys. Rev. E* **108**, 044113 (2023).

- [17] Q. Gao, H.-M. Chun, and J. M. Horowitz, Thermodynamic constraints on kinetic perturbations of homogeneous driven diffusions, *Europhys. Lett.* **146**, 31001 (2024).
- [18] T. Aslyamov and M. Esposito, Nonequilibrium response for Markov jump processes: Exact results and tight bounds, *Phys. Rev. Lett.* **132**, 037101 (2024).
- [19] K. Ptasiński, T. Aslyamov, and M. Esposito, Dissipation bounds precision of current response to kinetic perturbations, *Phys. Rev. Lett.* **133**, 227101 (2024).
- [20] K. Liu and J. Gu, Dynamical activity universally bounds precision of response in Markovian nonequilibrium systems, *Commun. Phys.* **8**, 62 (2025).
- [21] T. Van Vu, Fundamental bounds on precision and response for quantum trajectory observables, *PRX Quantum* **6**, 010343 (2025).
- [22] D. Loi, S. Mossa, and L. F. Cugliandolo, Effective temperature of active complex matter, *Soft Matter* **7**, 3726 (2011).
- [23] E. Fodor, C. Nardini, M. E. Cates, J. Tailleur, P. Visco, and F. van Wijland, How far from equilibrium is active matter?, *Phys. Rev. Lett.* **117**, 038103 (2016).
- [24] C. Song, P. Wang, and H. A. Makse, Experimental measurement of an effective temperature for jammed granular materials, *Proc. Natl. Acad. Sci. U.S.A.* **102**, 2299 (2005).
- [25] K. Sato, Y. Ito, T. Yomo, and K. Kaneko, On the relation between fluctuation and response in biological systems, *Proc. Natl. Acad. Sci. U.S.A.* **100**, 14086 (2003).
- [26] D. T. N. Chen, A. W. C. Lau, L. A. Hough, M. F. Islam, M. Goulian, T. C. Lubensky, and A. G. Yodh, Fluctuations and rheology in active bacterial suspensions, *Phys. Rev. Lett.* **99**, 148302 (2007).
- [27] D. Mizuno, D. A. Head, F. C. MacKintosh, and C. F. Schmidt, Active and passive microrheology in equilibrium and nonequilibrium systems, *Macromolecules* **41**, 7194 (2008).
- [28] D. Hempston, J. Vovrosh, M. Toroš, G. Winstone, M. Rashid, and H. Ulbricht, Force sensing with an optically levitated charged nanoparticle, *Appl. Phys. Lett.* **111**, 133111 (2017).
- [29] C. Gonzalez-Ballester, M. Aspelmeier, L. Novotny, R. Quidant, and O. Romero-Isart, Levitodynamics: Levitation and control of microscopic objects in vacuum, *Science* **374**, eabg3027 (2021).
- [30] A. W. C. Lau, B. D. Hoffman, A. Davies, J. C. Crocker, and T. C. Lubensky, Microrheology, Stress fluctuations, and active behavior of living cells, *Phys. Rev. Lett.* **91**, 198101 (2003).
- [31] R. R. Brau, J. M. Ferrer, H. Lee, C. E. Castro, B. K. Tam, P. B. Tarsa, P. Matsudaira, M. C. Boyce, R. D. Kamm, and M. J. Lang, Passive and active microrheology with optical tweezers, *J. Opt. A* **9**, S103 (2007).
- [32] C. Wilhelm, Out-of-equilibrium microrheology inside living cells, *Phys. Rev. Lett.* **101**, 028101 (2008).
- [33] E. Ben-Isaac, Y. K. Park, G. Popescu, Frank L. H. Brown, N. S. Gov, and Y. Shokef, Effective temperature of red-blood-cell membrane fluctuations, *Phys. Rev. Lett.* **106**, 238103 (2011).
- [34] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, Heidelberg, 1986).
- [35] W. M. Coffey and Y. P. Kalmykov, *The Langevin Equation: With Applications to Stochastic Problems in Physics, Chemistry, and Electrical Engineering* (World Scientific, Singapore, 2017).
- [36] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/3hs9-dz3d> for (A) the derivation of the finite-frequency FRI for general Langevin and jump processes, (B) the calculation of the power spectrum and response for linear systems, (C) a discussion on local perturbations and measurements, and (D) a description of the numerical simulation method, which includes Refs. [37,38].
- [37] G. Bussi and M. Parrinello, Accurate sampling using Langevin dynamics, *Phys. Rev. E* **75**, 056707 (2007).
- [38] S. W. Smith, *The Scientist and Engineer's Guide to Digital Signal Processing* (California Technical Publishing, San Diego, California, 1999).
- [39] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon & Breach Publishing Group, New York, 1963).
- [40] H. D. Vollmer and H. Risken, Eigenvalues and their connection to transition rates for the Brownian motion in an inclined cosine potential, *Z. Phys. B* **52**, 259 (1983).
- [41] K. Lindenberg, A. M. Lacasta, J. M. Sancho, and A. H. Romero, Transport and diffusion on crystalline surfaces under external forces, *New J. Phys.* **7**, 29 (2005).
- [42] I. G. Marchenko and I. I. Marchenko, Diffusion in the systems with low dissipation: Exponential growth with temperature drop, *Europhys. Lett.* **100**, 50005 (2012).
- [43] B. Lindner and I. M. Sokolov, Giant diffusion of underdamped particles in a biased periodic potential, *Phys. Rev. E* **93**, 042106 (2016).
- [44] L. P. Fischer, P. Pietzonka, and U. Seifert, Large deviation function for a driven underdamped particle in a periodic potential, *Phys. Rev. E* **97**, 022143 (2018).
- [45] L. P. Fischer, H.-M. Chun, and U. Seifert, Free diffusion bounds the precision of currents in underdamped dynamics, *Phys. Rev. E* **102**, 012120 (2020).
- [46] J. Spiechowicz, I. G. Marchenko, P. Hänggi, and J. Łuczka, Diffusion coefficient of a Brownian particle in equilibrium and nonequilibrium: Einstein model and beyond, *Entropy* **25**, 42 (2023).
- [47] P. Reimann, C. Van den Broeck, H. Linke, P. Hänggi, J. M. Rubi, and A. Pérez-Madrid, Giant acceleration of free diffusion by use of tilted periodic potentials, *Phys. Rev. Lett.* **87**, 010602 (2001).
- [48] J. Fraden, *Handbook of Modern Sensors: Physics, Designs, and Applications* (Springer Science & Business Media, New York, 2010).
- [49] A. Dechant and E. Lutz, Fundamental limits on nonequilibrium sensing, *Nat. Commun.* **16**, 10227 (2025).
- [50] T. Harada and S.-i. Sasa, Equality connecting energy dissipation with a violation of the fluctuation-response relation, *Phys. Rev. Lett.* **95**, 130602 (2005).
- [51] T. Harada and S.-i. Sasa, Energy dissipation and violation of the fluctuation-response relation in nonequilibrium Langevin systems, *Phys. Rev. E* **73**, 026131 (2006).
- [52] R. Bao and S. Liang, Nonlinear response identities and bounds for nonequilibrium steady states, [arXiv:2412.19602](https://arxiv.org/abs/2412.19602).
- [53] A. Dechant, Thermodynamic constraints on the power spectral density in and out of equilibrium, [arXiv:2306.00417](https://arxiv.org/abs/2306.00417).

- [54] V. T. Vo, A. Dechant, and K. Saito, Inverse thermodynamic uncertainty relation and entropy production, *Phys. Rev. Lett.* **135**, 237104 (2025).
- [55] N. Freitas, J.-C. Delvenne, and M. Esposito, Stochastic and quantum thermodynamics of driven RLC networks, *Phys. Rev. X* **10**, 031005 (2020).
- [56] N. Freitas, J.-C. Delvenne, and M. Esposito, Stochastic thermodynamics of nonlinear electronic circuits: A realistic framework for computing around kT , *Phys. Rev. X* **11**, 031064 (2021).
- [57] T. Aslyamov, K. Ptaszyński, and M. Esposito, companion Letter, Macroscopic fluctuation-response theory and its use for gene regulatory networks, *Phys. Rev. Lett.* **136**, 067102 (2026).
- [58] A. Dechant *et al.* (2025), GitHub, <https://github.com/AndreasDechant/FiniteFrequencyFRI>.

End Matter

Appendix A: Proof of the FRI for overdamped dynamics—In the interest of conciseness, we focus on the overdamped case Eq. (1) here; the derivation for more general overdamped dynamics, as well as underdamped Langevin and Markov jump dynamics is provided in SM, Sec. A [36]. We consider a set of K observables of the form

$$r_k(t) = z_k[\mathbf{x}(t)] + \mathbf{w}_k[\mathbf{x}(t)] \circ \dot{\mathbf{x}}(t). \quad (\text{A1})$$

This is slightly more general than the form discussed above, since it also contains a currentlike contribution, where $\circ$ denotes the Stratonovich product. We denote the steady-state average of $r_k(t)$ by $\langle r_k \rangle_{\text{st}}$. Next, the response of the observables is calculated. A central result of nonequilibrium linear response theory [9–13] is that the response function can be formally expressed as a steady-state correlation function

$$R_{kq}(t) = \langle r_k(t) \psi_q(0) \rangle. \quad (\text{A2})$$

The first crucial insight is that the quantity $\psi_q(t)$ has the same form as the observables Eq. (A1); specifically,

$$\begin{aligned} \psi_q(t) &= \tilde{z}_q[\mathbf{x}(t)] + \tilde{\mathbf{w}}_q[\mathbf{x}(t)] \circ \dot{\mathbf{x}}(t), \\ \tilde{z}_q(\mathbf{x}) &= -\frac{1}{2\gamma^2} \mathbf{g}_q(\mathbf{x}) \cdot \mathbf{T}^{-1} \mathbf{f}(\mathbf{x}) - \frac{1}{2\gamma} \nabla_{\mathbf{x}} \cdot \mathbf{g}_q(\mathbf{x}), \\ \tilde{\mathbf{w}}_q(\mathbf{x}) &= -\frac{1}{2\gamma^2} \mathbf{T}^{-1} \mathbf{g}_q(\mathbf{x}). \end{aligned} \quad (\text{A3})$$

The second crucial insight is that the correlation function of the “response observables” $\psi_q(t)$ is given by

$$\langle \psi_q(t) \psi_r(t') \rangle = \frac{1}{2\gamma} \langle \mathbf{g}_q \cdot \mathbf{T}^{-1} \mathbf{g}_r \rangle_{\text{st}} \delta(t - t'), \quad (\text{A4})$$

which can be obtained by direct calculation, see SM, Sec. A [36]. This result allows writing the joint spectral density matrix $\mathbb{S}(\omega)$ of the $K + Q$ observables $[r_1(t), \dots, r_k(t), \psi_1(t), \dots, \psi_Q(t)]$ in blockwise form

$$\mathbb{S}(\omega) = \begin{pmatrix} \mathcal{S}(\omega) & \mathcal{R}(\omega) \\ \mathcal{R}^H(\omega) & \mathcal{A} \end{pmatrix}, \quad (\text{A5})$$

where the symmetric, positive definite matrix $\mathcal{A}$ is defined as

$$\mathcal{A}_{qr} = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \psi_q(t) \psi_r(0) \rangle = \frac{1}{2\gamma} \langle \mathbf{g}_q \cdot \mathbf{T}^{-1} \mathbf{g}_r \rangle_{\text{st}}. \quad (\text{A6})$$

Since $\mathbb{S}(\omega)$ is a positive definite matrix, its blocks have to satisfy the conditions

$$\mathcal{R}^H(\omega) \mathcal{S}^{-1}(\omega) \mathcal{R}(\omega) \leq \mathcal{A}, \quad (\text{A7a})$$

$$\mathcal{R}(\omega) \mathcal{A}^{-1} \mathcal{R}^H(\omega) \leq \mathcal{S}(\omega). \quad (\text{A7b})$$

The first condition is precisely the first main result, Eq. (7). The second condition provides an alternative trade-off relation between the fluctuations and the response. Integrating the second condition with respect to frequency yields, for configuration-dependent observables with $\mathbf{w}_q = 0$,

$$\frac{1}{\pi} \int_0^{\infty} d\omega \mathcal{R}(\omega) \mathcal{A}^{-1} \mathcal{R}^H(\omega) \leq \Xi_{\text{st}}. \quad (\text{A8})$$

That is, the frequency integral of the squared response matrix is bounded by the steady-state covariance matrix of the observables. This generalizes Eq. (15) to the case of several observables and perturbations.

Appendix B: Linear dynamics—Next, the above results are applied to a system of particles with linear interactions and perturbations that are constant in space. In this case, Eq. (4) is written as

$$\dot{\mathbf{x}}(t) = \frac{1}{\gamma} (-\mathbf{K}(\mathbf{x}(t) - \mathbf{x}_{\text{eq}}) + \epsilon \mathcal{G} \boldsymbol{\phi}(t)) + \sqrt{\frac{2T}{\gamma}} \boldsymbol{\xi}(t). \quad (\text{B1})$$

The matrix $\mathcal{G} \in \mathbb{R}^{D \times Q}$ has the constant force $\mathbf{g}_q$ as its q th column and $\boldsymbol{\phi}(t) = [\phi_1(t), \dots, \phi_Q(t)]$. $\mathbf{K}$ is a matrix of spring constants, whose eigenvalues have positive real parts, so that the system has a stable steady state. If $\mathbf{K}$ is

symmetric, the resulting steady state is in equilibrium; if $\mathbf{K}$ is not symmetric, then nonreciprocal interactions drive the system into a nonequilibrium steady state. Since both the dynamics and the perturbation in Eq. (B1) are linear, the linear response matrix Eq. (5) fully describes the response of the system for arbitrary magnitude ϵ of the perturbation. The observables are likewise linear combinations of the coordinates of the particles, i.e., $\mathbf{z}(t) = \mathbf{Z}\mathbf{x}(t)$ with $\mathbf{Z} \in \mathbb{R}^{K \times D}$. Since the system is linear, explicit expressions for the spectral density and response matrices are obtained (see the SM [36] for the calculation),

$$\begin{aligned}\mathcal{S}(\omega) &= \gamma \mathbf{Z}((\mathbf{K} - i\omega\gamma\mathbf{I})^{-1}\mathbf{\Xi}_{\text{st}} + [\text{H.c.}])\mathbf{Z}^T, \\ \mathcal{R}(\omega) &= \mathbf{Z}(\mathbf{K} - i\omega\gamma\mathbf{I})^{-1}\mathcal{G}, \\ \mathcal{A} &= \frac{1}{2\gamma}\mathcal{G}^T\mathbf{T}^{-1}\mathcal{G}.\end{aligned}\quad (\text{B2})$$

Here, [H.c.] represents the Hermitian conjugate of the first matrix and $\mathbf{\Xi}_{\text{st}}$ denotes the steady-state covariance matrix of the coordinates. With these expressions, it is a matter of straightforward linear algebra (see SM, Sec. B [36]) to verify the relations

$$\mathcal{R}^H(\omega)\mathcal{S}^{-1}(\omega)\mathcal{R}(\omega) = \mathcal{A} \quad \text{if rank}(\mathbf{Z}) = D, \quad (\text{B3a})$$

$$\mathcal{R}(\omega)\mathcal{A}^{-1}\mathcal{R}^H(\omega) = \mathcal{S}(\omega) \quad \text{if rank}(\mathcal{G}) = D. \quad (\text{B3b})$$

If the observables provide the complete information about the coordinates of the particles ($\mathbf{Z}$ is an invertible matrix), then Eq. (A7a) is an equality. Conversely, if the perturbations can reproduce any constant-in-space force ($\mathcal{G}$ is an invertible matrix), then Eq. (A7b) is an equality. Equality holds for both equilibrium and out-of-equilibrium situations, as long as the dynamics is linear.

Appendix C: Computing $\mathcal{A}$ from measurements—The FFFRI Eq. (7) relates the frequency-dependent response and fluctuations of observables to the frequency-independent matrix $\mathcal{A}$. If the properties of the environment (friction γ and temperature $\mathbf{T}$), as well as the functional form of the perturbations $\mathbf{g}_q(\mathbf{x})$ are known, then $\mathcal{A}$ can be estimated from trajectory data by taking steady-state averages. We remark that the temperature and friction coefficients can be obtained from high-frequency measurements of the position $\mathbf{x}$ and its response to constant perturbations in the respective direction. More precisely, for $r(t) = x_k(t)$ and $\mathbf{g} = g_0\hat{\mathbf{e}}_k$, we have

$$\lim_{\omega \rightarrow \infty} \omega^2 \mathcal{S}_{kk}(\omega) = \frac{T_k}{\gamma}, \quad \lim_{\omega \rightarrow \infty} (-i\omega)\mathcal{R}_{kk}(\omega) = \frac{g_0}{\gamma}, \quad (\text{C1})$$

or, equivalently,

$$\frac{1}{\gamma T_k} = \frac{1}{g_0^2} \lim_{\omega \rightarrow \infty} \frac{|\mathcal{R}_{kk}(\omega)|^2}{\mathcal{S}_{kk}(\omega)}. \quad (\text{C2})$$

Comparing this to Eq. (9), we therefore see that equality is attained in the high-frequency limit for specific choices of observables and perturbations. Thus, the “bare” diffusion matrix $\gamma\mathbf{T}$ can be obtained from high-frequency measurements of the coordinates, which can then in turn be used to derive a bound on the fluctuations of more general observables and their response to more general perturbations. The above can be derived by taking the high-frequency limit of the expressions in Appendix B and noting that, in the high-frequency limit, both response and fluctuations are governed by the properties of the viscous environment and therefore become independent of the systematic forces acting in the system.