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Stabilizer entropies are monotones for magic-state resource theory

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Magic-state resource theory is a powerful tool with applications in quantum error correction, many-body physics, and classical simulation of quantum dynamics. Despite its broad scope, finding tractable resource monotones has been challenging. Stabilizer entropies have recently emerged as promising candidates (being easily computable and experimentally measurable detectors of nonstabilizerness) though their status as true resource monotones has been an open question ever since. In this Letter, we establish the monotonicity of stabilizer entropies for $\alpha \geqslant 2$ within the context of magic-state resource theory restricted to pure states. Additionally, we show that linear stabilizer entropies serve as strong monotones. Furthermore, we extend stabilizer entropies to mixed states as monotones via convex roof constructions whose computational evaluation significantly outperforms optimization over stabilizer decompositions for low-rank density matrices. As a direct corollary, we provide improved conversion bounds between resource states, revealing a preferred direction of conversion between magic states. These results conclusively validate the use of stabilizer entropies within magic-state resource theory and establish them as the only known family of monotones that are experimentally measurable and computationally tractable.

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Introduction. The inception of magic-state resource theory can be traced back to the seminal work of Bravyi and Kitaev [1]. They showed the feasibility of applying non-Clifford gates to a target state through *stabilizer operations* (i.e., Clifford operations and measurements) leveraging a single-qubit auxiliary state prepared in a nonstabilizer (or "magic") state. Building upon this fundamental result, magic-state resource theory was then formalized [2]. The core principle hinges on the following dichotomy: while stabilizer operations are easily implemented in a fault-tolerant fashion (i.e., transversally) in many schemes [3], achieving fault-tolerance for nonstabilizer operations proves to be challenging [4]. As a result, stabilizer operations are regarded as the *free operations* within the resource theory framework, with stabilizer states designated as the corresponding *free states*, while non-Clifford operations and nonstabilizer states are considered *resourceful*. The task at hand involves utilizing nonstabilizer resource states and exclusively employing stabilizer operations to implement non-Clifford unitary gates fault-tolerantly, a process commonly referred to as *magic-state distillation*.

Stabilizer operations are not only nonuniversal, but have also been shown to be efficiently classically simulatable [5]. Nonstabilizer states become hard to simulate classically [6]. Hence, beyond the scope of magic-state distillation, magic-state resource theory is also employed to quantify the hardness of classically simulating quantum states using stabilizer formalism. From this perspective, nonstabilizer states are regarded as resourceful since they enable quantum computations that surpass the capabilities of classical computers.

From the technical point of view, the construction of a quantum resource theory is straightforward; given the set of free states $\mathcal{T}$ uniquely defining the resource theory, free operations are quantum channels (completely positive, trace-preserving, and linear operations on a quantum system) leaving $\mathcal{T}$ invariant. The challenge lies in selecting a *resource monotone* (or measure), akin to a "thermometer," to accurately quantify the resource amount in a given state $\psi \notin \mathcal{T}$, represented by a positive scalar function. While universal resource monotones (independent from the specific resource theory) do exist [7], they are impractical to compute and experimentally measure in real scenarios. For a resource theory to be applicable in practice, such as in quantum many-body physics and quantum computation, finding a resource-specific measure is an extremely important and nontrivial task.

Stabilizer entropies (SEs) indexed by a α-Rényi index were recently proposed to probe nonstabilizerness in multiqubit pure quantum states [8]. They have garnered particular attention due to their analytical [9–12] and numerical [13,14] computability as well as experimental measurability [15,16]. Thanks to their computability advantages, stabilizer entropies have significantly advanced the study of magic-state resource theory within the context of many-body physics, allowing numerical simulations up to $\sim$100 qubits [13,14], which was previously limited to only a few bodies [17]. In addition, through stabilizer entropies, magic-state resource theory found connection to a number of topics in quantum physics, including quantum chaos and scrambling [8], as well as in quantum information with applications in cryptography [18], fidelity estimation [19], Pauli sampling [20,21], and classical shadows [22,23].

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TABLE I. Summary of the results. M_α and M_α^{lin} represent the stabilizer entropy and the linear stabilizer entropy respectively (see Definition 4). Both are extended to mixed states.

	Stabilizer monotone	Strong stabilizer monotone
$M_{\alpha<2}$	×	×
$M_{\alpha\geqslant 2}$	✓	×
$M_{\alpha<2}^{\text{lin}}$	×	×
$M_{\alpha\geqslant 2}^{\text{lin}}$	✓	✓

However, a counterexample to the monotonicity of stabilizer entropies with Rényi index $\alpha < 2$ under stabilizer protocols was provided [24]. Since then, the question of whether stabilizer entropies with $\alpha \geqslant 2$, when restricted to pure states, serve as a monotone for magic-state resource theory has remained unanswered. Given the significance of stabilizer entropies across various contexts, the question "are stabilizer entropies good monotones for magic-state resource theory?" has garnered increasing attention.

In this work, we demonstrate that stabilizer entropies with integer Rényi index $\alpha \geqslant 2$ are monotones for magic-state resource theory when restricted to pure states. We also show that their linearized versions (linear stabilizer entropies) serve as *strong* monotones. Moreover, we extend stabilizer entropies to mixed states through convex-roof construction, ensuring monotonicity under stabilizer protocols and offering substantial computational advantages over known monotones for low-rank density matrices. See Table I for a summary. As an application of our results, we improve over state-of-the-art resource conversion bounds. These results conclusively validate the use of stabilizer entropies to quantify nonstabilizerness in multiqubit quantum systems.

Setup. Throughout the Letter we consider the Hilbert space $\mathcal{H}_n$ of n qubits and denote as $d_n = 2^n$ its dimension. A natural operator basis is given by Pauli operators $P \in \mathbb{P}_n$, i.e., n-fold tensor products of ordinary Pauli matrices I, X, Y, Z. The subgroup of unitary matrices that maps Pauli operators to Pauli operators is known as the Clifford group. Stabilizer states, denoted as $|\sigma\rangle$ in this work, are pure states obtained from $|0\rangle^{\otimes n}$ with the action of unitary Clifford operators. Throughout the work, we will use ψ, ϕ to denote pure states, while ρ to denote (possibly mixed) general states.

Magic-state resource theory. A resource theory is uniquely characterized by its set of free states. The set of free states for magic-state resource theory is given by the convex hull of pure stabilizer states, hereby denoted as $\text{STAB} := \{\sum_i p_i |\sigma_i\rangle\langle\sigma_i| : p_i \geqslant 0, \sum_i p_i = 1\}$.

Free operations are the *stabilizer protocols* that are completely positive and trace-preserving maps that leave invariant the set STAB. Denoting as $\mathcal{S}$ the set of stabilizer protocols, every operation $\mathcal{E} \in \mathcal{S}$ can be built out of the following elementary operations: (i) Clifford unitaries; (ii) partial trace; (iii) measurements in the computational basis; (iv) composition with auxiliary qubits in $|0\rangle$; the above operations conditioned on (v) measurement outcomes and (vi) classical randomness. Since these operations (and combination thereof) return different states based on conditioned operations and nondeterministic strategies, the most general form of a stabilizer protocol applied on a state ρ is $\mathcal{E}(\rho) = \{(p_i, \rho_i)\}$, i.e., a collection of n_i qubit states ρ_i with probabilities p_i. A stabilizer protocol $\mathcal{E}$ is *deterministic* if $\mathcal{E}(\rho) = \tilde{\rho}$, i.e., a unique quantum state is reached with unit probability.

Given stabilizer protocols, one can define monotones for magic-state resource theory.

Definition 1 (Stabilizer monotone). A stabilizer monotone $\mathcal{M}$ is a real-valued function for all $n \in \mathbb{N}$ qubit systems (or collection thereof) such that (i) $\mathcal{M}(\rho) = 0$ if and only $\rho \in$ STAB; (ii) $\mathcal{M}$ is nonincreasing under stabilizer protocols $\mathcal{S}$, i.e.,

$$\mathcal{M}(\mathcal{E}(\rho)) \leqslant \mathcal{M}(\rho), \quad \forall \mathcal{E} \in \mathcal{S}. \tag{1}$$

As is customary in resource theories [7], there are functions that always serve as monotones regardless of the specific resource theory, which includes (but is not limited to) robustness measures (robustness of magic [25]), resource-rank measures (stabilizer rank [26]), or distance-based measures (relative entropy of magic [27]). While all of them require an impractical optimization over the set of free states (stabilizer decomposition), the challenge has always been to find a tractable resource monotone that meaningfully apply to the specific resource theory at hand, e.g., von Neumann entanglement entropy for entanglement. In magic-state resource theory, such a role is played by the stabilizer Rényi entropy [8]. As we shall see, the stabilizer entropy is naturally defined for pure states. We thus define a monotone for magic-state resource theory restricted to pure states.

Definition 2 (Pure-state stabilizer monotone). A pure-state stabilizer monotone is a real-valued function for all $n \in \mathbb{N}$ qubit systems such that (i) $\mathcal{M}(\psi) = 0$ if and only if $|\psi\rangle$ is a pure stabilizer state; (ii) for every pair $(|\psi\rangle, \mathcal{E})$, where $\mathcal{E} \in \mathcal{S}$ a stabilizer protocol, obeying $\mathcal{E}(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi|$, it holds that

$$\mathcal{M}(\phi) \leqslant \mathcal{M}(\psi). \tag{2}$$

In other words, a pure-state stabilizer monotone is a monotone when restricted to *deterministic* pure-state stabilizer protocols, i.e., those mapping pure states to pure states. Note that, as stabilizer protocols consist of multiple elementary operations, the aforementioned definition does not preclude intermediate states from becoming mixed, provided the final state remains pure. Clearly, being a pure-state monotone is a weaker condition than being a monotone and, as a matter of fact, Definition 1 implies Definition 2. However, deterministic pure-state stabilizer protocols have their own relevance: they are mainly employed for the conversion between resource states, which is crucial in the field of quantum error correction for implementing nonstabilizer operations fault-tolerantly [28]. A famous example is provided by the *magic-state injection*, i.e., $\mathcal{E}(|T\rangle \otimes |\psi\rangle) = |0\rangle \otimes T|\psi\rangle$ where $|T\rangle = T|+\rangle$ is the T state and $T := \text{diag}(1, e^{i\pi/4})$ being the T gate [29].

As a last notion, let us introduce the concept of strong monotonicity for pure states. An analogous definition for mixed states follows.

Definition 3 (Strong pure-state stabilizer monotone). Let $\mathcal{M}$ a pure-state stabilizer monotone. Let $|\psi\rangle$ be a pure state and consider the collection of pure states $\mathcal{E}(\psi) = \{(p_i, |\phi_i\rangle)\}$, obtained after applying a stabilizer protocol $\mathcal{E} \in \mathcal{S}$. $\mathcal{M}$ is a

strong pure-state stabilizer monotone if

$$\mathcal{M}(\psi) \geqslant \sum_i p_i \mathcal{M}(\phi_i). \tag{3}$$

Informally, the notion of strong monotonicity says that, on average, the nonstabilizer resource cannot increase after nondeterministic stabilizer protocols. Operationally, strong stabilizer monotones bound the optimal probability of conversion between resource states via stabilizer operations [7], as explained in greater detail below. Now that we have all the necessary definitions and tools, the next section reveals the main findings of our work.

Monotonicity of stabilizer entropies. In this section, we establish the monotonicity of stabilizer entropies under stabilizer protocols in various flavors. Let us first define SEs. Given a pure state $|\psi\rangle$, the quantity $d_n^{-1}|\langle\psi|P|\psi\rangle|^2$ forms a probability distribution on the Pauli group $\mathbb{P}_n$, known also as *characteristic distribution* [30]. SEs are defined, up to an offset, as α-Rényi entropies of this probability distribution.

Definition 4 (α-stabilizer entropy). The α-Rényi stabilizer entropy of a pure quantum state $|\psi\rangle$ on n qubits reads

$$M_\alpha(\psi) := \frac{1}{1-\alpha}\log_2 P_\alpha(\psi),\ P_\alpha(\psi) := \frac{1}{d_n}\sum_{P\in\mathbb{P}_n}|\langle\psi|P|\psi\rangle|^{2\alpha},$$

where $P_\alpha(\psi)$ is referred to as stabilizer purity. Linear stabilizer entropies are defined as $M_\alpha^{\text{lin}}(\psi) := 1 - P_\alpha(\psi)$.

Stabilizer entropies exhibit two key properties required for magic-state resource theory: (i) they are minimized, i.e., are zero, only for stabilizer states and (ii) are invariant under the action of Clifford unitary operators. Additionally, SEs possess several useful properties, such as additivity under tensor products, which will be crucial for bounding resource conversion rates, as explained later. Moreover, they serve as lower bounds for key monotones within magic-state resource theory, including the robustness of magic [25], min-relative entropy of magic [27], stabilizer nullity [28], and stabilizer extent [31]. For a comprehensive list, please refer to Ref. [32].

However, mere invariance under Clifford operators does not suffice to establish stabilizer entropies as effective monotones for magic-state resource theory, given the significantly broader scope of stabilizer protocols. In fact, in a recent study [24], a specific counterexample to deterministic pure-state stabilizer protocols involving Clifford operations conditioned on measurement outcomes has been presented for $\alpha < 2$. Whether stabilizer entropies serve as stabilizer monotones for larger values of α has remained an open question, and given the extensive list of useful features, it is indeed significant. We demonstrate below that stabilizer entropies function as monotones for deterministic stabilizer protocols for any $\alpha \geqslant 2$, thereby qualifying as pure-state stabilizer monotones according to Definition 2.

Theorem 1. Stabilizer entropies M_α are pure-state stabilizer monotones for every integer $\alpha \geqslant 2$.

Proof sketch. First of all, it is crucial to note that a deterministic pure-state stabilizer protocol only necessitates the initial and final states to be pure. We consider the following decomposition of a pure state $|\psi\rangle = \sqrt{p}|0\rangle\otimes|\phi_1\rangle + \sqrt{1-p}|1\rangle\otimes|\phi_2\rangle$. The key step of the proof is showing that $M_\alpha(\psi) \geqslant \min\{M_\alpha(\phi_1), M_\alpha(\phi_2)\}$. From this simple observation, for a general stabilizer protocol $\mathcal{E} \in \mathcal{S}$, it holds that

$$M_\alpha(\psi) \geqslant \inf_{(q_{ij},\phi_{ij})}\left\{\min_{ij} M_\alpha(\phi_{ij}) \ :\ p_i\rho_i = \sum_j q_{ij}\phi_{ij}\right\},$$

where the infimum is over the possible pure state convex decompositions of $p_i\rho_i$ where $\mathcal{E}(\psi) = \{(p_i, \rho_i)\}$. By requiring $\mathcal{E}$ to be a deterministic pure-state stabilizer protocol, the result just follows. The complete proof can be found in [32]. ■

In fact, we can present an even stronger result. We show that linear stabilizer entropies are strong monotones according to Definition 3.

Theorem 2. Linear stabilizer entropies M_α^{lin} are strong pure-state stabilizer monotones for every integer $\alpha \geqslant 2$.

Proof sketch. To show the strong pure-state monotonicity, we prove the more general result from which Theorem 3 follows. Starting from a pure state ψ, for all stabilizer operations $\mathcal{E} \in \mathcal{S}$ such that $\mathcal{E}(\psi) = \{(p_i, \rho_i)\}$, it holds that

$$M_\alpha^{\text{lin}}(\psi) \geqslant \inf_{(q_{ij},\phi_{ij})}\left\{\sum_{ij} q_{ij}M_\alpha^{\text{lin}}(\phi_{ij}) : p_i\rho_i = \sum_j q_{ij}\phi_{ij}\right\}.$$

Similarly to Theorem 1, the crucial step is to consider an arbitrary pure state $|\psi\rangle = \sqrt{p}|0\rangle\otimes|\phi_1\rangle + \sqrt{1-p}|1\rangle\otimes|\phi_2\rangle$ and show that $M_\alpha^{\text{lin}}(\psi) \geqslant pM_\alpha^{\text{lin}}(\phi_1) + (1-p)M_\alpha^{\text{lin}}(\phi_2)$. With the restriction imposed on the collection $\mathcal{E}(\psi)$ to consist solely of pure states, the result follows [32]. ■

According to Theorem 2, linear stabilizer entropies are powerful stabilizer monotones defined through entropies of the characteristic distribution. In Ref. [24], it was shown that stabilizer entropies M_α violate the strong monotonicity condition. However, since the two quantities are related to each other through a concave function, as $M_\alpha = \frac{1}{1-\alpha}\log_2(1 - M_\alpha^{\text{lin}})$, ultimately this discrepancy is due to the Jensen's inequality which is equivalent to the difference between arithmetic mean and geometric mean. Let us present a counterexample for M_α for $\alpha \geqslant 2$. We consider the state $|\psi\rangle \propto |0\rangle^{\otimes n} + |1\rangle\otimes|\phi\rangle$ with $|\phi\rangle$ a Haar random state on $n-1$ qubits. We have $M_\alpha(|\psi\rangle) = O(1)$, as $|\psi\rangle$ has a nonvanishing overlap to the stabilizer state $|0\rangle^{\otimes n}$ (see [32]). While $M_\alpha(|\phi\rangle) = \Omega(n)$ (with overwhelming probability [18]). As such, there exist n_0 such that $\forall n \geqslant n_0$ one has $M_\alpha(|\psi\rangle) \leqslant \frac{1}{2}M_\alpha(|\phi\rangle)$, thus violating strong monotonicity of M_α for some sufficiently large n.

However, as a corollary of Theorem 1, one can see that stabilizer entropies obey a weaker condition compared to strong pure-state monotonicity. Given a nondeterministic stabilizer protocol that, starting from a pure state $|\psi\rangle$, yields a collection of pure states $\mathcal{E}(\psi) = \{(p_i, |\phi_i\rangle)\}$, it follows that $M_\alpha(\psi) \geqslant \min_i M_\alpha(\phi_i)$ for every integer $\alpha \geqslant 2$.

That being said, while strong monotonicity is indeed a desirable trait with clear operational significance, it is worth noting that it is not an essential requirement for effective stabilizer monotones. Min-relative entropy of magic [27], which is a natural monotone being the (negative log) value of the overlap with the closest stabilizer state, does not obey strong monotonicity either (the counterexample shown above holds true). Consequently, the adherence of linear stabilizer entropies to strong monotonicity can be regarded as a distinctive and advantageous feature, which we employ in the subsequent section to provide improved conversion bounds.

Improved resource conversion bounds. Having established the monotonicity of stabilizer entropy, we now exploit it to improve on resource conversion bounds previously studied in the literature. Specifically, we build on the literature aiming to answer the following question: What is the rate of conversion between resource states $|R_1\rangle$ and $|R_2\rangle$, that is, the maximum number of copies of a given resource state $|R_2\rangle$ produced per copy of another resource state $|R_1\rangle$ by means of stabilizer operations?

Given a pure-state monotone $\mathcal{M}$, satisfying the additivity property, i.e., $\mathcal{M}(|\phi_1\rangle \otimes |\phi_2\rangle) = \mathcal{M}(|\phi_1\rangle) + \mathcal{M}(|\phi_2\rangle)$, the rate of conversion $r_{\to} := r[|R_1\rangle \mapsto |R_2\rangle]$ can be simply upper bounded by [28]

$$r_{\to} \leqslant \frac{\mathcal{M}(|R_1\rangle)}{\mathcal{M}(|R_2\rangle)}. \tag{4}$$

While one may expect that the rate $r_{\to}$ is simply the inverse of the rate $r_{\leftarrow} := r[|R_2\rangle \mapsto |R_1\rangle]$, it has been shown that, in magic-state resource theory, there is a clear gap between $r_{\to}$ and $r_{\leftarrow}$ [28]. In what follows, we build on this picture by analyzing conversion rates between the m-qubit resource states $|C^{m-1}Z\rangle \propto \sum_b (-1)^{b_1 \cdots b_m} |b\rangle$ for $b \in \{0,1\}^m$. In particular, we analyze the gap between the forward conversion rate $r^{(m)}_{\to} := r[|C^{m-1}Z\rangle \mapsto |C^2 Z\rangle]$ and the backward conversion rate $r^{(m)}{\leftarrow} := r[|C^2 Z\rangle \mapsto |C^{m-1}Z\rangle]$ for $m \geqslant 3$. In Ref. [28], via stabilizer nullity arguments, it has been shown that, to deterministically distill $|C^{m-1}Z\rangle$, it is not possible to consume fewer than $m/3$ $|C^2 Z\rangle$, i.e., $r^{(m)}_{\leftarrow} \geqslant m/3$. We now investigate the forward conversion rate $r^{(m)}_{\to}$. The best-known bound is $r^{(4)}_{\to} \leqslant 4/3$. While already showing a clear gap, i.e., $\Omega(m)$ versus $O(1)$, between forward and backward rates, this bound leaves open the possibility of a forward conversion rate greater than one. The following corollary not only bounds $r^{(4)}_{\to}$ away from one, but also drastically amplifies the gap between forward and backward conversion rates.

Corollary 1. The conversion rate $r^{(m)}_{\to} := r[|C^{m-1}Z\rangle \mapsto |C^2 Z\rangle] = O(2^{-m})$. In particular, $r^{(m)}_{\to} < r^{(4)}_{\to} \leqslant 0.9$ for every $m \in \mathbb{N}$.

Proof. It is sufficient to compute the α-stabilizer entropy of $|C^{m-1}Z\rangle$ for $m \geqslant 3$, which can be simply done by a direct calculation. See Ref. [32] for the complete expression. Thanks to the additivity property of M_α, we bound $r^{(m)}_{\to} \leqslant M_\alpha(|C^{m-1}Z\rangle)/M_\alpha(|C^2 Z\rangle)$, $\forall \alpha \in \mathbb{N}$ [28]. The tightest bound is obtained for $\alpha = 2$. ■

While the above bounds show an exponentially large gap, i.e., $\Omega(m)$ versus $O(2^{-m})$, between the forward and backward conversion rates, they focus on the *deterministic* conversion between resource states. One may consider probabilistic protocols between resource states and ask whether one can achieve better conversion rates. A figure of merit for probabilistic conversion is given by the optimal success probability $\pi^{\max}_{|R_1\rangle}(|R_2\rangle)$ of converting the resource state $|R_1\rangle$ into $|R_2\rangle$. The latter can be nicely upper bounded using any strong stabilizer monotone $\mathcal{M}$ (see Definition 3) as [7]

$$\pi^{\max}_{|R_1\rangle}(|R_2\rangle) \leqslant \frac{\mathcal{M}(|R_1\rangle)}{\mathcal{M}(|R_2\rangle)}. \tag{5}$$

Hence, as a direct corollary of Theorem 2, we answer the above question in the negative by showing an exponentially suppressed success probability.

Corollary 2. The maximal probability for the probabilistic resource conversion $|C^{m-1}Z\rangle \mapsto |C^2 Z\rangle$ obeys $\pi^{\max}_{|C^{m-1}Z\rangle}(|C^2 Z\rangle) = O(2^{-m})$.

Proof. Computing the stabilizer purity of $|C^{m-1}Z\rangle$ for $m \geqslant 3$ [32], it is sufficient to exploit the strong monotonicity of linear stabilizer entropies and use Eq. (5). ■

These results, among others presented in Ref. [32], strengthen the main findings of Ref. [28]: we can conclude that, even for probabilistic protocols, there is a clear preferred direction of conversion in magic-state resource theory.

Extension to mixed states. While stabilizer entropies find their natural definition within the realm of pure states, when extending magic-state resource theory to mixed states, a diverse landscape emerges. This becomes evident when considering that magic-state auxiliary qubits are often assumed to be noisy, prompting interest in extracting a single clean non-Clifford unitary from many copies of these noisy auxiliary systems [1]. To conclude, we leverage our findings to extend stabilizer entropies to mixed states, via convex-roof extension [7], ensuring adherence to the monotonicity criterion outlined in Definition 1.

Definition 5 (Convex-roof extension of SEs). Let $\mathcal{C} := \{(p_i, \rho_i)\}$ be a collection of n_i qubit quantum states. Extended stabilizer entropies are defined as

$$\widehat{M}_\alpha(\mathcal{C}) := \frac{1}{1-\alpha} \log_2 \widehat{P}_\alpha(\mathcal{C}), \quad \widehat{M}^{\mathrm{lin}}_\alpha(\mathcal{C}) := 1 - \widehat{P}_\alpha(\mathcal{C})$$
$$\widehat{P}_\alpha(\mathcal{C}) := \sup_{(q_{ij},\phi_{ij})} \left\{ \sum_{ij} q_{ij} P_\alpha(\phi_{ij}) \right\}. \tag{6}$$

The sup is over the possible convex pure-state decompositions $p_i \rho_i = \sum_j q_{ij} \phi_{ij}$ for $\{(p_i, \rho_i)\} \in \mathcal{C}$.

It is worth noting that both the *extended* stabilizer entropy and linear stabilizer entropy reduce to the entropies defined on pure states in Definition 4. The following theorem establishes the monotonicity of the above convex roof constructions.

Theorem 3. Extended stabilizer entropies $\widehat{M}_\alpha$ and linear stabilizer entropies $\widehat{M}^{\mathrm{lin}}_\alpha$ are stabilizer monotones for every integer $\alpha \geqslant 2$. Moreover, $\widehat{M}^{\mathrm{lin}}_\alpha$ is a strong stabilizer monotone.

Proof sketch. Both results for extended stabilizer entropy and linear stabilizer entropy follows by showing that, given a collection $\mathcal{C} = \{(p_i, \rho_i)\}$ of n_i qubit states and an elementary stabilizer protocol $\mathcal{E} \in \mathcal{S}$, then

$$\widehat{P}_\alpha[\mathcal{E}(\mathcal{C})] \geqslant \widehat{P}_\alpha(\mathcal{C}). \tag{7}$$

The steps follow from the proof of Theorem 2, see Ref. [32]. ■

The above result extends stabilizer entropies, along with some of their amenable properties, to the realm of mixed states, thus unlocking the exploration of the magic-state resource theory in many-body physics in more intricate situations. While finding the optimal decomposition can be challenging in the worst case, there are scenarios where computing the extended stabilizer entropy is significantly less expensive than optimizing over stabilizer states, as other known measures require [27]. Let us estimate the cost of computing $\widehat{M}_\alpha$ for a d^n-dimensional density matrix of rank r. Using a brute-force search algorithm over convex decompositions, the computational effort involves $r 4^n 2^{O(nr^2)}$ time steps. This cost includes the 4^n operations needed to compute the stabilizer entropy M_α for r pure states and the $2^{O(nr^2)}$

possible convex decompositions of ρ, resolving the stabilizer purity with $O(2^{-n})$ accuracy. Hence, for low-rank density matrices $r = o(\sqrt{n})$, this approach attains a superpolynomial advantage over the $2^{\Theta(n^2)}$ required to optimize over stabilizer decompositions.

Discussion. In this work, we prove that stabilizer entropies serve as monotones for magic-state resource theory. Furthermore, their linearized counterparts exhibit strong monotonicity, meaning they do not, on average, increase following a nondeterministic protocol. As a direct corollary of these findings, we provide enhanced conversion bounds between resource states. Aligning with the recent findings of Ref. [28], our results indicate a preferred direction of conversion between magic states.

Additionally, as a result of our investigation, we provide a convex-roof extension of stabilizer entropies to mixed states, ensuring monotonicity under stabilizer protocols and achieving computational advantages over stabilizer decomposition for low-rank density matrices. This paves the way for further exploration of magic-state resource theory beyond pure states.

While our results solidify the utility of stabilizer entropies in quantifying "magic" within multiqubit systems, there remain intriguing open questions regarding their properties, which remain mostly unexplored. For instance, despite robust metrics such as robustness of magic, stabilizer extent, and stabilizer nullity directly quantifying the cost of classical simulations via stabilizer formalism, there lacks a simulation algorithm directly assessed by stabilizer entropy (with $\alpha \geqslant 2$) costs. In this context, it is worth noting that the special case of $\alpha = 1/2$ stabilizer entropy, also known as GKP magic [33] or stabilizer norm [25], quantifies the simulation cost of classical simulation [34], thus proving to be a powerful tool. However, it does not serve a monotone for magic-state resource theory [35].

The maximal resourceful n-qubit state according to SEs remains unknown. While it is evident that the upper bound $M_2 < \log_2 \frac{d_n+1}{2}$ cannot be achieved by any characteristic distribution (for $n \neq 1, 3$), a tighter bound and the state saturating it are yet to be determined.

In conclusion, this Letter provides a definitive answer to the open question of whether stabilizer entropies serve as monotones for magic-state resource theory or merely act as weak detectors of nonstabilizerness. With this clarification, there arises a compelling need for further exploration of stabilizer entropies across diverse domains, ranging from many-body physics to classical simulation via stabilizer formalism, and within the realms of theoretical resource theory.

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