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Gapless layered three-dimensional fractional quantum Hall states

Michael Levin[1] and Matthew P. A. Fisher[2]

[1]*Department of Physics, University of California, Santa Barbara, California 93106, USA*

[2]*Microsoft Research, Station Q, University of California, Santa Barbara, California 93106, USA*

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Using the parton construction, we build a three-dimensional (3D) multilayer fractional quantum Hall state with average filling $\nu=1/3$ per layer that is qualitatively distinct from a stacking of weakly coupled Laughlin states. The state supports gapped charge $e/3$ fermionic quasiparticles that can propagate both within and between the layers, in contrast to the quasiparticles in a multilayer Laughlin state which are confined within each layer. Moreover, the state has gapless neutral collective modes, a manifestation of an emergent "photon," which is minimally coupled to the fermionic quasiparticles. The surface sheath of the multilayer state resembles a chiral analog of the Halperin-Lee-Read state, which is protected against gap-forming instabilities by the topological character of the bulk 3D phase. We propose that this state might be present in multilayer systems in the "intermediate tunneling regime," where the interlayer tunneling strength is on the same order as the Coulomb energy scale. We also find that the parton construction leads to a candidate state for a bilayer $\nu=1/3$ system in the intermediate tunneling regime. The candidate state is distinct from both a bilayer of $\nu=1/3$ Laughlin states and the single layer $\nu=2/3$ state but is, nonetheless, a fully gapped fractional quantum Hall state with charge $e/3$ anyonic quasiparticles.

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I. INTRODUCTION

While the fractional quantum Hall effect is firmly rooted in two dimensions (2D), *anisotropic* three-dimensional (3D) electron systems—such as multilayer systems in a perpendicular magnetic field—can exhibit fractional quantum Hall (FQH) states, at least in principle. The simplest example of such a 3D multilayer state is a stacking of N-decoupled Laughlin states.[1,2] At the next level of complexity, one can construct states with interlayer correlations such as the (3,3,1) bilayer state.[3] More generally, multicomponent Chern-Simons theory allows one to construct a myriad of N-layer analogs of the (3,3,1) state.[4]

These states are quite general but they suffer from a limitation: they all have a fixed number of electrons in each layer. This restriction could be problematic for describing certain multilayer systems, especially those with appreciable interlayer tunneling. Therefore, alternative constructions of 3D multilayer FQH states are desirable theoretically.

On the experimental side, a number of experiments on 3D semiconductor multilayers have explored the behavior of stacked integer quantum Hall states,[5–7] including the novel vertical transport due to the conducting surface sheath.[1,8–14] More recently, experiments on bismuth crystals in high magnetic fields have revealed intriguing anomalies in the ultraquantum limit—the limit where the magnetic field is sufficiently large that only the lowest Landau level is (partially) occupied.[15] It has been suggested that a novel 3D fractional quantum Hall type state might be present. While these are not layered materials, strong electron correlations could drive a transition wherein the electrons spontaneously form a weak-layered structure, as suggested in recent work.[16] Bulk graphite also reveals transport anomalies in the ultraquantum limit,[17–19] which have been attributed to a charge-density wave (CDW) transition in this layered material. The observed quantum Hall effect in graphene,[20–23] and future prospects for graphene multilayers, provides further impetus to explore 3D layered quantum Hall phenomena.

Motivated by these experiments and the lack of previous theoretical exploration, we revisit the behavior of multilayer systems in the fractional quantum Hall regime. Generally, we are interested in addressing the following class of questions: what is the fate of a weakly coupled stacking of 2D fractional quantum Hall states when the interlayer electron tunneling becomes strong enough to close the quantum Hall gap? In particular, are new fractional quantum Hall type states possible when the Coulomb interaction is comparable to the interlayer tunneling strength? We believe that this intermediate tunneling strength regime is both experimentally accessible and theoretically novel.

Answering these questions definitively for a specific microscopic model is quite challenging and likely requires extensive numerical calculation. Here, we are less ambitious. Our goal is simply to find *candidate* ground states for the intermediate tunneling regime. This is already a nontrivial problem since, as we mentioned earlier, most multilayer states—such as those obtained from Chern-Simons mean-field theory—have a fixed number of electrons in each layer and hence are unnatural unless the interlayer tunneling is weak.

In this paper, we construct a candidate ground state for the simplest possible multilayer system: spinless (or spin polarized) electrons with an average filling of $\nu=1/3$ per layer. We speculate that the candidate state may be realized at intermediate tunneling strength. However, our arguments for the candidate state are indirect, as we do not make any detailed analysis of energetics.

We construct our candidate state using a slave-particle gauge theory approach known as the "parton construction."[24,25] The basic idea of the parton construction is to write the electron creation operator as a product of several (in our case, 3) fermionic parton creation operators.

By choosing different mean-field parton states, and including (gauge) fluctuations, one can construct different FQH states. The advantage of the parton construction is that it naturally leads to states with electron number fluctuations in each layer. Thus, parton FQH states may be particularly natural in the intermediate tunneling regime. In addition, the particular state we construct has the interesting property that one can tune from it to a decoupled multilayer Laughlin state by changing a single coupling constant in the parton gauge theory. Given that the multilayer Laughlin state is likely realized at weak interlayer tunneling, this is an additional reason we consider our state to be a candidate ground state for intermediate interlayer tunneling.

We analyze the candidate state for both a finite number of layers N and for the 3D limit $N\rightarrow\infty$. For a finite number of layers N, we find that the state is a fully gapped FQH state (when $N=1$, it is simply the Laughlin state). The quasiparticle excitations are anyonic and carry charge $e/3$. We find that the quasiparticle excitations are described by a K matrix[26] of dimension $(3N-2)\times(3N-2)$, along with a charge vector of length $(3N-2)$.

In the 3D limit, the candidate state exhibits more unusual physics. It supports two types of excitations: gapped charge $e/3$ fermionic quasiparticle excitations and gapless electrically neutral collective modes. The $e/3$ fermionic quasiparticle excitations (which are essentially the "deconfined" partons) are truly 3D quasiparticles and can move freely between layers. This should be contrasted with the $e/3$ excitations in the multilayer Laughlin state which are confined to individual layers. (In this sense, our candidate state is "more 3D" then the multilayer Laughlin state). As for the gapless electrically neutral collective modes, these are the emergent U(1) gauge bosons or "photons" which originate from fluctuations about the mean-field parton state. Unlike Maxwell photons, these excitations have only one polarization state and have an anisotropic dispersion of the form $\omega_{\mathbf{k}}^2\sim\mathbf{k}_\perp^2+k_z^4$ (for layers oriented in the xy plane). The $e/3$ fermionic quasiparticles are minimally coupled to these photon modes and thus have long-range interactions.

The edge physics of the N-layer and 3D systems is also interesting. In the case of a finite number of layers N, the edge theory is a chiral boson conformal field theory with $3N-2$ chiral modes. The edge Lagrangian can be read off from the bulk K matrix using the standard formalism.[26] The edge (or surface) physics in the 3D limit is more complex. For a 3D system with layers in the xy plane and a boundary in the xz plane, the three flavors of dispersing edge modes form a "sheath" of chiral fermions. At the mean-field level these fermions are noninteracting, but with fluctuations included are minimally coupled to the gapless bulk photons. The surface sheath, then, resembles a chiral analog of the Halperin-Lee-Read[27] state, which is protected against gap-forming instabilities by the topological character of the bulk phase.

This paper is organized as follows. In Sec. II we speculate about phase diagrams of multilayer $\nu=1/3$ systems and we describe the intermediate tunneling regime in more detail. In Sec. III we construct our candidate state. In Secs. IV–VII, we analyze the bulk physics of the candidate state for single layer, bilayer, N-layer, and 3D systems. In Sec. VIII we investigate the crossover between 2D and 3D physics in systems with a large but finite number of layers. In Sec. IX we describe the relationship between the candidate state and the multilayer Laughlin state. Finally, in Sec. X we analyze the edge physics of the candidate state for N-layer and 3D systems.

II. MODEL AND POSSIBLE PHASE DIAGRAMS

In this section, we discuss the physics of N-layer $\nu=1/3$ FQH systems in more detail. We speculate about possible phase diagrams and we explain what the intermediate tunneling regime is and why it is interesting.

Consider a geometry where the layers are oriented in the xy plane and neighboring layers are spaced a distance a in the z direction. There are four energy scales in the problem: two intralayer and two interlayer scales. The intralayer energy scales are the cyclotron energy $\hbar\omega_c=\hbar eB/m$ and the characteristic intralayer Coulomb energy scale $E_C=e^2/l_B$. The interlayer scales are the interlayer Coulomb energy e^2/a and the interlayer tunneling strength t_z. In the following discussion, we will focus on the regime where (1) $\hbar\omega_c$ is much larger than any of the other energy scales and (2) a is comparable to, but larger than l_B so that the interlayer Coulomb energy scale is smaller than but on the same order as the intralayer energy E_C. In this regime, there is only one dimensionless parameter in the problem: the ratio $g\equiv t_z/E_C$.

Let us think about the phase diagram as we vary the dimensionless ratio $g=t_z/E_C$. To begin, suppose $N=2$. When $g=0$, there is no interlayer tunneling and we expect that the ground state is given by two decoupled Laughlin $\nu=1/3$ states, with perhaps small quantitative modifications due to the interlayer Coulomb interaction. Since the Laughlin state is gapped, it will be stable to small interlayer hopping, i.e., $g\ll1$. In the opposite limit, with very large interlayer tunneling $g\gg1$, all of the electrons will be in the "bonding band:" the band consisting of symmetric combinations of Landau orbitals in the two layers. The system is thus an effective single layer system at filling $\nu=2/3$. The weak Coulomb interactions will presumably lead to an Abelian $\nu=2/3$ state with gap of order E_C.

Now, consider the regime $g\sim1$. Starting from the decoupled limit, when g is increased one expects that the quasiparticle gap in each layer will diminish and presumably be driven to zero at some critical value g_1. On the other hand, when g is brought down from large values, the gap of the "single layer" $\nu=2/3$ state will diminish (due to Landau-level mixing into the "antibonding band") and be driven to zero at some value (g_2). Generally, there is no reason to expect that $g_1=g_2$, although it is possible that there is a direct first-order transition between the $1/3+1/3$ decoupled phase and the 2/3 single layer state. If $g_2>g_1$, there will be a third FQH phase (or phases) for $g_1<g<g_2$. This potential phase is the "intermediate coupling" phase we consider in this paper.

One can imagine a similar scenario for the 3D limit $N\rightarrow\infty$. Again, at weak interlayer tunneling, $g\ll1$, the stack of Laughlin states is stable and can be readily analyzed.[1] On the other hand, the nature of the strong tunneling phase with

$g \gg 1$ is nontrivial. In the noninteracting limit, the lowest Landau level in each layer will form a band that disperses in the z direction, with energy $-t_z \cos(k_z a)$. The noninteracting ground state consists of completely filling all of the lowest Landau-level band states with $|k_z| < k_F \equiv \pi/3a$. This describes a gapless state. In a Wannier basis of orthonormalized lowest Landau-level wave functions with guiding centers sitting on the sites of a regular 2D lattice (say triangular), one can view the system as an array of one-dimensional (1D) noninteracting electron systems. Gapless particle-hole excitations exist across the two Fermi points in each of the 1D "wires." This noninteracting state is likely to be unstable in the presence of arbitrarily weak Coulomb interactions $E_C \neq 0$ due to the nested $2k_F$ backscattering interactions between nearby wires. The simplest scenario would be the development of a fully gapped $Q=2k_F=2\pi/(3a)$ CDW state, which corresponds to a tripling of the unit cell along the z axis. Naively one would expect the CDW gap to scale as $\Delta_{\text{CDW}} \sim t_z \exp(-\text{const} \cdot g)$.[28,29] This CDW state can be loosely thought of as an effective system with $N/3$ layers, each one at filling $\nu_{\text{eff}}=1$.

As in the bilayer case, increasing g from small values will presumably close the Laughlin quasiparticle gap in each of the layers, destroying the decoupled phase at some g_1. Similarly, when g is decreased from very large values down to order 1, the CDW state will become disfavored due to the increasing intralayer Coulomb repulsion. One expects the CDW state to be destroyed for some $g < g_2$. As for $N=2$, it is possible that there is a third phase for $g_1 < g < g_2$: an "intermediate tunneling phase."

The possibility of such an intermediate tunneling phase for either the finite N case or the 3D limit $N \to \infty$ is the starting point for this paper. One reason we feel it is a particularly interesting possibility is that the interlayer tunneling and the Coulomb interaction are both of paramount importance in such a putative phase. This poses a theoretical challenge since the obvious FQH states—such as those constructed from Chern-Simons mean-field theory—have a fixed number of electrons in each layer and are therefore unnatural except for very weak interlayer tunneling. On the other hand, if one tries to construct a state by treating the N-layer system as an effective single layer system at filling $N/3$, the result is unnatural except for very strong interlayer tunneling. In this paper, we use a different approach—a slave-particle construction with fermionic "partons"—to build a candidate state that overcomes these difficulties.

III. PARTON CONSTRUCTION

Our candidate state can be described most naturally using the parton construction.[24,25] Let us first describe the construction in the case of the single layer system; we will then generalize to multiple layers.

Our starting point is the single layer electron Hamiltonian. As it will be convenient in what follows, we regularize this Hamiltonian, replacing the 2D continuum by a square lattice. We take the flux through each plaquette in the lattice to be $2\pi/M$, the electron density to be $1/3M$, and we consider the limit $M \to \infty$. In this limit, the lattice model behaves like a 2D continuum with electrons at filling fraction $\nu=1/3$.

The regularized electron Hamiltonian can be written as

$$H = -\sum_{xi} (t c_x^\dagger e^{i\tilde{A}_{x,i}} c_{x+\hat{x}_i} + \text{H.c.}) + \text{interactions}, \quad (1)$$

where $\tilde{A}_{x,i}$ is a lattice gauge field with $\Delta_1 \tilde{A}_{x,2} - \Delta_2 \tilde{A}_{x,1} = 2\pi/M$, and $\tilde{A}$ is periodic with unit cell of size M. Here, Δ_i denotes a lattice derivative in the $\hat{x}_i$ direction: $\Delta_i f_x \equiv f_{x+\hat{x}_i} - f_x$.

In the parton construction, we think of the electron as a composite of three fermionic partons d^p, $p=1,2,3$,

$$c = d^1 d^2 d^3. \quad (2)$$

We then substitute the expression for c into this Hamiltonian and expand around a saddle point. The result is a noninteracting mean-field Hamiltonian for the partons. Many different saddle points can be stabilized depending on the details of the interactions. Different saddle points correspond to different FQH states.

Here, we consider a particular saddle point. The saddle point we are interested in is associated with the mean-field Hamiltonian

$$H_{\text{mf}} = -\sum_{xip} [t_p (d_x^p)^\dagger e^{iA_{x,i}} d_{x+\hat{x}_i}^p + \text{H.c.}], \quad (3)$$

where $A_{x,i}$ is a lattice gauge field with $\Delta_1 A_{x,2} - \Delta_2 A_{x,1} = 2\pi/3M$, and $A_{x,i}$ is periodic with unit cell of size $3M$. Notice that the flux $2\pi/3M$ is exactly the right size so that the partons are at filling $\nu=1$ (the partons, like the electrons, are at density $1/3M$). We have also assumed that the hopping amplitudes t_p are different for the three species of partons.

What is the physics of this parton state? At the mean-field level, the parton state is a gapped state with fermionic excitations. However, this mean-field result is not quite correct since we have not taken into account the effect of fluctuations about the saddle point. These fluctuations are described by fluctuations in the hopping amplitudes t_p of the form $t_p \to t_p e^{i\theta_p}$ with $\theta_1 + \theta_2 + \theta_3 = 0$. We can parametrize them in terms of two U(1) gauge fields A^q, $q=1,2$ by setting $\theta_p = Q_{pq} A^q$ where $q=1,2$, and $Q_{p1}=(1,0,-1)$, $Q_{p2}=(0,1,-1)$. The effect of fluctuations is thus to couple the partons to two U(1) gauge fields A^q. [Note that the structure of the gauge fluctuations is closely tied to the symmetries of the saddle point. For example, at the symmetric saddle point $t_1=t_2=t_3$ the fluctuations are described by SU(3) gauge fluctuations $t_p \to t_{p'} U_p^{p'}$ rather than the $\text{U}(1) \times \text{U}(1)$ fluctuations present here[30]]. Including these fluctuations, our Hamiltonian is given by

$$H = H_t + H_A, \quad (4)$$

where H_t describes the parton hopping and H_A describes the gauge-field dynamics,

$$H_t = -\sum_{xip} [t_p (d_x^p)^\dagger e^{iQ_{pq} A_{x,i}^q + iA_{x,i}} d_{x+\hat{x}_i}^p + \text{H.c.}], \quad (5)$$

$$H_A=\sum_{xiq}\frac{g}{2}(E^q_{x,i})^2-\sum_{xq}J\cos(\Delta_1 A^q_{x,2}-\Delta_2 A^q_{x,1}). \tag{6}$$

The parton state we are interested in is described by the above Hamiltonian in the weak gauge fluctuation regime, e.g., $g \ll J, t_p$.

Generalizing this construction to the N layer case is straightforward. In this case, the mean-field parton Hamiltonian is given by

$$H_{\mathrm{mf}}=-\sum_{xzip}[t_{p\perp}(d^p_{xz})^\dagger e^{iA_{x,i}}d^p_{(x+\hat{x}_i)z}+\mathrm{H.c.}] -\sum_{xzp}[t_{p3}(d^p_{xz})^\dagger d^p_{x(z+1)}+\mathrm{H.c.}], \tag{7}$$

where z is the layer index and $t_{p\perp}, t_{p3}$ are the intralayer and interlayer hopping amplitudes.

Including the $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge fluctuations, we arrive at

$$H=\sum_{z=1}^{N}(H_{zt}+H_{zA})+\sum_{z=1}^{N-1}(H_{z(z+1)t}+H_{z(z+1)A}), \tag{8}$$

where H_{zt}, H_{zA} describe the intralayer hopping and gauge-field terms,

$$H_{zt}=-\sum_{xip}[t_{p\perp}(d^p_{xz})^\dagger e^{iQ_{pq}A^q_{xz,i}+iA_{x,i}}d^p_{(x+\hat{x}_i)z}+\mathrm{H.c.}], \tag{9}$$

$$H_{zA}=\sum_{xiq}\frac{g_\perp}{2}(E^q_{xz,i})^2-\sum_{xq}J_\perp\cos(\Delta_1 A^q_{xz,2}-\Delta_2 A^q_{xz,1}), \tag{10}$$

and $H_{z(z+1)t}, H_{z(z+1)A}$ describe the interlayer hopping and gauge-field terms,

$$H_{z(z+1)t}=-\sum_{xp}[t_{p3}(d^p_{xz})^\dagger e^{iQ_{pq}A^q_{xz,3}}d^p_{x(z+1)}+\mathrm{H.c.}],$$

$$H_{z(z+1)A}=\sum_{xq}\frac{g_3}{2}(E^q_{xz,3})^2-\sum_{xiq}J_3\cos(\Delta_i A^q_{xz,3}-A^q_{xz,i}+A^q_{x(z+1),i}). \tag{11}$$

Again, we assume that the gauge fluctuations are weak: $g_3 \ll J_3, t_{p3}, t_{p\perp}$ and $g_\perp \ll J_\perp, t_{p3}, t_{p\perp}$. In the following sections, we analyze the physics of this state.

IV. SINGLE LAYER

We begin with the simplest case: the single layer parton state. We rederive the well-known result that the single layer parton state is precisely the Laughlin state.[24,25]

To understand the properties of the single layer parton state, we need to analyze the low energy physics of the Hamiltonian (4). One way to do this is to introduce U(1) gauge fields a^p_μ to describe the parton number currents,

$$j^{p\lambda}=\frac{1}{2\pi}\epsilon^{\lambda\mu\nu}\partial_\mu a^p_\nu. \tag{12}$$

The low-energy effective theory for the parton hopping terms H_t can then be written as

$$L=\frac{1}{4\pi}\sum_p \epsilon^{\lambda\mu\nu}a^p_\lambda\partial_\mu a^p_\nu+\text{minimal coupling to } A^q. \tag{13}$$

Including the minimal coupling to A^q gives

$$L=\frac{1}{4\pi}\sum_p \epsilon^{\lambda\mu\nu}a^p_\lambda\partial_\mu a^p_\nu+\frac{1}{2\pi}\epsilon^{\lambda\mu\nu}Q_{pq}A^q_\lambda\partial_\mu a^p_\nu. \tag{14}$$

Adding the gauge-field terms H_A, expanding the cosines to quadratic order, and taking the continuum limit, we arrive at the low-energy effective theory

$$L=\frac{1}{4\pi}\sum_p \epsilon^{\lambda\mu\nu}a^p_\lambda\partial_\mu a^p_\nu+\frac{1}{2\pi}\epsilon^{\lambda\mu\nu}Q_{pq}A^q_\lambda\partial_\mu a^p_\nu +\sum_{iq}\frac{1}{2g}(\partial_0 A^q_i-\partial_i A^q_0)^2-\sum_q\frac{Jl^2}{2}(\partial_1 A^q_2-\partial_2 A^q_1)^2, \tag{15}$$

where l is the lattice spacing. The last two terms are irrelevant to the low-energy physics since integrating out the a^p_μ field produces a Chern-Simons term for A^q (which has one less derivative then the above Maxwell terms). Dropping these terms and integrating out A leaves us with

$$L=\frac{1}{4\pi}\sum_p \epsilon^{\lambda\mu\nu}a^p_\lambda\partial_\mu a^p_\nu \tag{16}$$

together with the constraints $\partial_\mu a^1_\nu=\partial_\mu a^2_\nu=\partial_\mu a^3_\nu$. Letting $a_\nu=a^1_\nu=a^2_\nu=a^3_\nu$, we get

$$L=\frac{1}{4\pi}\epsilon^{\lambda\mu\nu}3a_\lambda\partial_\mu a_\nu. \tag{17}$$

If we include the coupling to the physical electromagnetic gauge field A_{EM}, assigning electric charges e_1, e_2, e_3 to the partons with $e_1+e_2+e_3=e$, we find

$$L=\frac{1}{4\pi}\epsilon^{\lambda\mu\nu}3a_\lambda\partial_\mu a_\nu+\frac{e}{2\pi}\epsilon^{\lambda\mu\nu}A_{\mathrm{EM},\lambda}\partial_\mu a_\nu \tag{18}$$

(irrespective of the values of e_1, e_2, e_3). This is the low-energy effective theory for the Laughlin state. We conclude that the single layer parton state is in the same universality class (e.g., quantum phase) as the Laughlin state.

V. BILAYER

In this section, we analyze the parton state in the next simplest case: a bilayer. In this case, the parton Hamiltonian (8) reduces to

$$H=\sum_{z=1}^{2}(H_{zt}+H_{zA})+(H_{12t}+H_{12A}), \tag{19}$$

where H_{zt}, H_{zA} describe the intralayer hopping and gauge-field terms,

$$H_{zt}=-\sum_{xip}[t_{p\perp}(d^p_{xz})^\dagger e^{iQ_{pq}A^q_{xz,i}+iA_{x,i}}d^p_{(x+\hat{x}_i)z}+\mathrm{H.c.}], \tag{20}$$

$$H_{zA} = \sum_{xiq} \frac{g_\perp}{2}(E^q_{xz,i})^2 - \sum_{xq} J_\perp \cos(\Delta_1 A^q_{xz,2} - \Delta_2 A^q_{xz,1}), \tag{21}$$

and H_{12t}, H_{12A} describe the interlayer hopping and gauge-field terms,

$$H_{12t} = -\sum_{xp} [t_{p3}(d^p_{x1})^\dagger e^{iQ_{pq}A^q_{x1,3}} d^p_{x2} + \text{H.c.}], \tag{22}$$

$$H_{12A} = \sum_{xq} \frac{g_3}{2}(E^q_{x1,3})^2 - \sum_{xiq} J_3 \cos(\Delta_i A^q_{x1,3} - A^q_{x1,i} + A^q_{x2,i}). \tag{23}$$

As before, we can derive the low-energy physics of this Hamiltonian by introducing U(1) gauge fields $a^p_{z\mu}$ to describe the parton number currents in each layer,

$$j^{p\lambda}_z = \frac{1}{2\pi}\epsilon^{\lambda\mu\nu}\partial_\mu a^p_{z\nu}. \tag{24}$$

Expanding the cosines in the gauge-field terms to quadratic order, and putting everything together, we arrive at the effective theory

$$\begin{aligned} L = {} & \frac{1}{4\pi}\sum_{zp} \epsilon^{\lambda\mu\nu} a^p_{z,\lambda}\partial_\mu a^p_{z,\nu} + \frac{1}{2\pi}\sum_{z} \epsilon^{\lambda\mu\nu} Q_{pq} A^q_{z,\lambda}\partial_\mu a^p_{z,\nu} \\ & + \frac{1}{2g_3 a^2}(\partial_0 A^q_{1,3} - A^q_{1,0} + A^q_{2,0})^2 \\ & - \sum_{iq} \frac{J_3}{2}(\partial_i A^q_{1,3} - A^q_{1,i} + A^q_{2,i})^2, \end{aligned} \tag{25}$$

where a is the layer spacing. (As in the single layer case, we have dropped the intralayer Maxwell terms as they are irrelevant to the low-energy physics). To proceed further, we choose the gauge $A^q_{1,3}=0$ and define new fields $A^{q\pm} = A^q_1 \pm A^q_2$. Expressing the Lagrangian in terms of these fields gives

$$\begin{aligned} L = {} & \frac{1}{4\pi}\sum_{zp} \epsilon^{\lambda\mu\nu} a^p_{z,\lambda}\partial_\mu a^p_{z,\nu} \\ & + \frac{1}{4\pi}\sum_{\pm} \epsilon^{\lambda\mu\nu} Q_{pq}[A^{q\pm}_\lambda \partial_\mu (a^p_{1,\nu} \pm a^p_{2,\nu})] + \frac{1}{2g_3 a^2}(A^{q-}_0)^2 \\ & - \sum_{iq} \frac{J_3}{2}(A^{q-}_i)^2. \end{aligned} \tag{26}$$

As in the single layer case, the final step is to integrate out the gauge fields $A^{q\pm}$. Integrating out A^{q+} generates the constraints $\Sigma_z \partial_\mu a^1_{z,\nu} = \Sigma_z \partial_\mu a^2_{z,\nu} = \Sigma_z \partial_\mu a^3_{z,\nu}$; integrating out A^{q-} generates a Maxwell term for a^p which is irrelevant to the low-energy physics due to the presence of the Chern-Simons term.

We thus arrive at the Lagrangian

$$L = \frac{1}{4\pi}\sum_{zp} \epsilon^{\lambda\mu\nu} a^p_{z,\lambda}\partial_\mu a^p_{z,\nu} \tag{27}$$

together with the constraints $\Sigma_z \partial_\mu a^1_{z,\nu} = \Sigma_z \partial_\mu a^2_{z,\nu} = \Sigma_z \partial_\mu a^3_{z,\nu}$. There are four independent gauge fields left which we can parametrize by $a^1 = a^1_1, a^2 = a^2_1, a^3 = a^3_1, a^4 = \Sigma_z a^1_z = \Sigma_z a^2_z = \Sigma_z a^3_z$. In terms of these variables, we have

$$L = \frac{1}{4\pi}\epsilon^{\lambda\mu\nu} K_{IJ} a^I_\lambda \partial_\mu a^J_\nu, \tag{28}$$

where

$$K = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}. \tag{29}$$

Including the coupling to the physical electromagnetic gauge field A_{EM}, assigning charges e_1, e_2, e_3 to the partons with $e_1 + e_2 + e_3 = e$, we find

$$L = \frac{1}{4\pi}\epsilon^{\lambda\mu\nu} K_{IJ} a^I_\lambda \partial_\mu a^J_\nu + \frac{e}{2\pi}\epsilon^{\lambda\mu\nu} t_I A_{\text{EM},\lambda}\partial_\mu a^I_\nu, \tag{30}$$

where $t^T = (0,0,0,1)$ (irrespective of the values of e_1, e_2, e_3).

The parton state is completely specified by the above K matrix and charge vector t (or more accurately, the *universal* properties of this quantum state are completely specified). We now analyze the basic properties of this state.

We begin with the quasiparticle statistics. According to the K-matrix formalism, the quasiparticle excitations can be labeled by integer vectors l. The exchange statistics of a quasiparticle l is given by $\theta_{\text{ex}} = \pi(l^T K^{-1} l)$. The mutual statistics of two quasiparticles l and l', e.g., the phase associated with braiding one particle around another, is given by $\theta_{\text{mut}} = 2\pi(l^T K^{-1} l')$.

In principle, these formulas completely specify the quasiparticle statistics of the parton state. However, it is convenient to describe the statistics of the parton state in a more concise way. A general way to do this is to find a subset of quasiparticles with the property that one can generate all topologically distinct quasiparticles by taking composites of these basic quasiparticles. One can then describe the complete quasiparticle statistics by specifying the statistics of this generating subset of quasiparticles. For the above state, the three parton excitations corresponding to $l_1 = (1,0,0,0), l_2 = (0,1,0,0), l_3 = (0,0,1,0)$ generate all the others. [One way to see this is to note that the excitation (0,0,0,1) is topologically identical to (2,0,0,0)]. Simple algebra shows that the three parton excitations have exchange statistics

$$\theta_p = \pi(l_p^T K^{-1} l_p) = \frac{2\pi}{3} \tag{31}$$

and mutual statistics

$$\theta_{pp'} = 2\pi(l_p^T K^{-1} l_{p'}) = \frac{(1+3\delta_{pp'})\pi}{3}. \tag{32}$$

This gives a complete description of the quasiparticle statistics of the bilayer parton state. The electric charges of the quasiparticles are also easy to obtain. Again, it suffices to specify the parton charges, which are given by

$$q_p = e(t^T K^{-1} l_p) = \frac{e}{3}. \tag{33}$$

Now that we have computed these properties, we can see that the bilayer parton state is distinct from a bilayer of decoupled $\nu=1/3$ Laughlin states as well as the conventional $\nu=2/3$ state. Indeed, one can distinguish the states by noting that the $e/3$ excitation in the parton state has a statistical angle $2\pi/3$, while the $e/3$ excitation in the other two states has an angle $\pi/3$.

One can also distinguish the states by their ground-state degeneracy on a torus. This quantity is particularly easy to measure in numerical calculations. The ground-state degeneracy for the bilayer parton state is just the determinant of K which is 12. On the other hand, the degeneracy of a bilayer of decoupled $\nu=1/3$ Laughlin states is 9, and the degeneracy of the $\nu=2/3$ state is 3.

A final way to distinguish the states is via their thermal Hall conductances. Recall that each chiral boson edge mode gives a contribution of $\pm\frac{\pi^2 k_B^2}{3h}T$ to the thermal Hall conductance, with the sign determined by the chirality of the mode. Thus, the thermal Hall conductance can be computed by counting the number of positive and negative eigenvalues of K. In the case of the bilayer parton state, there are four positive eigenvalues so the thermal Hall conductance is 4 (in units of $\frac{\pi^2 k_B^2}{3h}T$). On the other hand, the thermal Hall conductance for the bilayer of Laughlin states is 2 and the thermal Hall conductance for the $\nu=2/3$ state is 0.

VI. N-LAYER SYSTEM

The bilayer results can be easily generalized to the N-layer case. For general N, one finds a K matrix of dimension $(3N-2)\times(3N-2)$. The result is shown below for the case $N=3$,

$$K = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 3 \end{pmatrix}. \tag{34}$$

The corresponding charge vector is $t^T=(0,0,0,0,0,0,1)$. The generalization to arbitrary N is clear: along the diagonals there are three $(N-1)\times(N-1)$ blocks of the form $1+\delta_{ij}$, while the last row and column is made up of -1's with a 3 in the bottom right-hand corner.

As before, the K matrix and charge vector determine all the universal properties of the FQH state such as the quasiparticle statistics and charges. Also, just as before, one can summarize the quasiparticle charges and statistics more concisely by specifying the statistics and charges of the three parton species [which correspond to $l_1=(1,0,0,0,0,0,0)$, $l_2=(0,0,1,0,0,0,0)$, and $l_3=(0,0,0,0,1,0,0)$ in the $N=3$ case].

One finds that the parton excitations have exchange statistics

$$\theta_p = \frac{(3N-2)\pi}{3N}, \tag{35}$$

mutual statistics

$$\theta_{pp'} = \frac{[2+(6N-6)\delta_{pp'}]\pi}{3N}, \tag{36}$$

and charge

$$q_p = \frac{e}{3}. \tag{37}$$

VII. 3D LIMIT

In this section, we analyze the parton construction in the 3D limit $N\to\infty$. Recall that the parton Hamiltonian is given by

$$H = \sum_z (H_{zt} + H_{zA} + H_{z(z+1)t} + H_{z(z+1)A}), \tag{38}$$

where

$$H_{zt} = -\sum_{xip} [t_{p\perp}(d_{xz}^p)^\dagger e^{iQ_{pq}A_{xz,i}^q + iA_{x,i}} d_{(x+\hat{x}_i)z}^p + \text{H.c.}], \tag{39}$$

and

$$H_{zA} = \sum_{xiq} \frac{g_\perp}{2}(E_{xz,i}^q)^2 - \sum_{xq} J_\perp \cos(\Delta_1 A_{xz,2}^q - \Delta_2 A_{xz,1}^q), \tag{40}$$

and

$$H_{z(z+1)t} = -\sum_{xp} [t_{p3}(d_{xz}^p)^\dagger e^{iQ_{pq}A_{xz,3}^q} d_{x(z+1)}^p + \text{H.c.}],$$

$$H_{z(z+1)A} = \sum_{xq} \frac{g_3}{2}(E_{xz,3}^q)^2 - \sum_{xiq} J_3 \cos(\Delta_i A_{xz,3}^q - A_{xz,i}^q + A_{x(z+1),i}^q). \tag{41}$$

As usual, we derive a low-energy effective theory by introducing U(1) gauge fields $a_{z\mu}^p$ to describe the parton number currents in each layer,

$$j_z^{p\lambda} = \frac{1}{2\pi}\epsilon^{\lambda\mu\nu}\partial_\mu a_{z\nu}^p. \tag{42}$$

Expanding the cosines in the gauge-field terms to quadratic order, and putting everything together, we arrive at the effective Lagrangian

$$L = L_a + L_A + L_{aA}, \tag{43}$$

where

$$L_a = \frac{1}{4\pi}\sum_{zp} \epsilon^{\lambda\mu\nu} a_{z,\lambda}^p \partial_\mu a_{z,\nu}^p, \tag{44}$$

and

$$L_A = \sum_{ziq} \frac{1}{2g_\perp}(\partial_0 A_{z,i}^q - \partial_i A_{z,0}^q)^2 - \sum_{zq} \frac{J_\perp l^2}{2}(\partial_1 A_{z,2}^q - \partial_2 A_{z,1}^q)^2 + \sum_{zq} \frac{1}{2g_3 a^2}(\partial_0 A_{z,3}^q - A_{z,0}^q + A_{z+1,0}^q)^2 - \sum_{ziq} \frac{J_3}{2}(\partial_i A_{z,3}^q - A_{z,i}^q + A_{z+1,i}^q)^2, \tag{45}$$

and

$$L_{aA} = \frac{1}{2\pi}\sum_{z} \epsilon^{\lambda\mu\nu} Q_{pq} A_{z,\lambda}^q \partial_\mu a_{z,\nu}^p. \tag{46}$$

To proceed further, we integrate out the gauge fields a_z^p corresponding to the parton excitations. Note that this is different from the approach we took in the single layer and bilayer cases where we integrated out the gauge fields A^q instead. We could have used this approach in those cases as well. The advantage of this approach is that it leads to a simpler description of the bulk low-energy physics: the low-energy effective theory for the N-layer case is simply a 2×2 Chern-Simons theory coupled to fermionic partons [instead of a $(3N-2)\times(3N-2)$ Chern-Simons theory coupled to bosons]. The disadvantage is that the edge physics cannot be easily read off from the bulk effective theory. Here our primary interest is in the bulk physics; thus we choose to integrate out the fields a_z^p.

Integrating out the a field (e.g., the partons) produces a Chern-Simons term for the A field. We can then drop the $J_\perp$ and $g_\perp$ Maxwell terms, as they are irrelevant at long distances. The resulting Lagrangian is given by

$$L_{A,\text{eff}} = \frac{1}{4\pi}\sum_{z} \epsilon^{\lambda\mu\nu} K_{qq'} A_{z,\lambda}^q \partial_\mu A_{z,\nu}^{q'} + \sum_{zq} \frac{1}{2g_3 a^2}(\partial_0 A_{z,3}^q - A_{z,0}^q + A_{z+1,0}^q)^2 - \sum_{ziq} \frac{J_3}{2}(\partial_i A_{z,3}^q - A_{z,i}^q + A_{z+1,i}^q)^2, \tag{47}$$

where

$$K_{qq'} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}. \tag{48}$$

The full low-energy effective theory is described by fermionic partons minimally coupled to this gauge theory with gauge charges Q_{pq},

$$L = L_{\text{part}} + L_{A,\text{eff}}, \tag{49}$$

where

$$L_{\text{part}} = \sum_{pz} (d_z^p)^\dagger (i\partial_0 + Q_{pq}A_{z,0}^q) d_z^p - \sum_{ipz} \frac{1}{2m_{p\perp}} (d_z^p)^\dagger (\partial_i - iQ_{pq}A_{z,i}^q)^2 d_z^p - \sum_{pz} t_{p3} (d_z^p)^\dagger e^{iQ_{pq}A_{z,3}^q} d_{z+1}^p + \text{H.c.} \tag{50}$$

[Here, $m_{p\perp} = 1/(2t_{p\perp}l^2)$]. We now analyze the physics of this low-energy effective theory. We begin with the excitations in the bulk. There are two types of excitations: gapped parton excitations described by d^p and gapless gauge boson excitations described by the above $U(1)\times U(1)$ gauge theory.

Let us try to understand the gapless gauge boson excitations in greater detail. We can derive the dispersion relation for these gapless modes by going to Fourier space. Going to Fourier space and taking k small, we have

$$L = \frac{1}{4\pi a}\epsilon^{\lambda\mu\nu} K_{qq'} A_\lambda^q i k_\mu A_\nu^{q'} + \sum_q \frac{1}{2g_3 a}(k_0 A_3^q - k_3 A_0^q)^2 - \sum_{iq} \frac{J_3 a}{2}(k_i A_3^q - k_3 A_i^q)^2. \tag{51}$$

Defining $A^\pm = \frac{1}{\sqrt{2}}(A^1 \pm A^2)$, K is diagonalized and our Lagrangian becomes

$$L = \frac{1}{4\pi a}\sum_q \epsilon^{\lambda\mu\nu} m_q A_\lambda^q i k_\mu A_\nu^q + \sum_q \frac{1}{2g_3 a}(k_0 A_3^q - k_3 A_0^q)^2 - \sum_{iq} \frac{J_3 a}{2}(k_i A_3^q - k_3 A_i^q)^2, \tag{52}$$

where $m_\pm = -3, -1$ are the two eigenvalues of K.

We can write this as

$$L = \sum_q (A^q)^\dagger M_q (A^q), \tag{53}$$

where

$$M_q = \begin{pmatrix} \frac{k_3^2}{2g_3 a} & \frac{im_q k_2}{4\pi a} & -\frac{im_q k_1}{4\pi a} & -\frac{k_0 k_3}{2g_3 a} \\ -\frac{im_q k_2}{4\pi a} & -\frac{J_3 a k_3^2}{2} & \frac{im_q k_0}{4\pi a} & \frac{J_3 a k_1 k_3}{2} \\ \frac{im_q k_1}{4\pi a} & -\frac{im_q k_0}{4\pi a} & -\frac{J_3 a k_3^2}{2} & \frac{J_3 a k_2 k_3}{2} \\ -\frac{k_0 k_3}{2g_3 a} & \frac{J_3 a k_1 k_3}{2} & \frac{J_3 a k_2 k_3}{2} & \frac{k_0^2}{2g_3 a} - \frac{J_3 a (k_1^2 + k_2^2)}{2} \end{pmatrix}. \tag{54}$$

Choosing the temporal gauge $A_0 = 0$, we can reduce M_q to the 3×3 submatrix

$$M_q = \begin{pmatrix} -\frac{J_3 a k_3^2}{2} & \frac{im_q k_0}{4\pi a} & \frac{J_3 a k_1 k_3}{2} \\ -\frac{im_q k_0}{4\pi a} & -\frac{J_3 a k_3^2}{2} & \frac{J_3 a k_2 k_3}{2} \\ \frac{J_3 a k_1 k_3}{2} & \frac{J_3 a k_2 k_3}{2} & \frac{k_0^2}{2g_3 a} - \frac{J_3 a (k_1^2 + k_2^2)}{2} \end{pmatrix}. \tag{55}$$

Setting the determinant to 0, we find one gapless mode for each $q = \pm$ with dispersion

$$\omega^2 = J_3 g_3 a^2 (k_1^2 + k_2^2) + \frac{4\pi^2 J_3^2 a^4}{m_q^2} k_3^4. \tag{56}$$

In principle, these gapless modes should be visible in inelastic light-scattering measurements. Thus, such measurements could be used to distinguish the 3D parton state from other candidate states such as the decoupled Laughlin state. In Sec. X we describe another experimental signature of the 3D parton state, involving surface excitations.

In addition to the gapless gauge excitations, the 3D state also contains gapped parton excitations. These excitations are fermions and carry electric charge $q = e/3$ as in the N-layer case (37). They are minimally coupled to the gapless gauge bosons and therefore have long-range interactions. Note that these excitations are truly 3D quasiparticles; they can propagate freely both within layers and between layers. This should be contrasted with the charge $e/3$ particles in the decoupled Laughlin state, which are confined to individual layers. In this sense, the parton state is more 3D than the decoupled Laughlin state.

VIII. CROSSOVER FROM 2D TO 3D

In the previous sections, we referred to the 3D limit as the limit $N \to \infty$. However, the 3D limit can also be accessed when there are a finite number of layers provided that we probe the system at appropriate length and energy scales. In this section, we discuss this crossover from 2D to 3D physics.

Consider an N-layer system with $N \gg 1$. This system is described by three species of partons minimally coupled to the the gauge theory (47). According to the analysis following Eq. (47), the low-energy modes of this gauge theory satisfy the dispersion relation (56). Since N is finite, k_3 is quantized in multiples of π/Na. For each value of k_3, there is a corresponding 2D mode.

The mode with the smallest gap corresponds to $k_3 = \pi/Na$; the dispersion relation for this mode is

$$\omega^2 = J_3 g_3 a^2 (k_1^2 + k_2^2) + \frac{4 J_3^2 \pi^6}{m_q^2 N^4}. \tag{57}$$

We see that this mode has a gap $\Delta \sim J_3/N^2$ and a correlation length $\xi \sim N^2 a \sqrt{g_3/J_3}$.

The gap Δ and correlation length ξ are the important energy and length scales in the 2D/3D crossover. If one probes the system at energies less than Δ or lengths larger then ξ (parallel to the layers), all the modes with $k_3 \neq 0$ will freeze out and the system will behave like a gapped 2D system. The physics is then described by the gapped FQH state in Sec. VI. On the other hand, if one probes the system at energies greater than Δ or lengths smaller then ξ, the system will behave like a 3D system. In this case, the physics is described by partons coupled to the gapless gauge theory (47).

One example of this crossover is the following thought experiment. Imagine adiabatically braiding one charge $e/3$ parton excitation p around another $e/3$ parton excitation p'—say of a different species—using a braiding path parallel to the layers (see Fig. 1). First, consider the case where the separation r between the partons is kept larger than ξ. In this case, the mode (56) will be effectively frozen at this distance. While the presence of parton p' will change the gauge flux seen by parton p, the gauge flux will be localized to within a distance ξ of p' and will be exponentially suppressed near the braiding path. Thus, the only interaction between the two partons will be a statistical interaction: the presence of parton p' will change the total gauge flux enclosed by the braiding

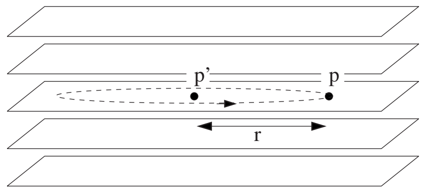

FIG. 1. If $r \gg \xi$, the Berry phase associated with braiding a parton p around another parton p' is given by the 2D formula $\theta_{pp'} = 2\pi/3N$. If $r \ll \xi$, it depends on the details of the path and can be calculated from the 3D gapless gauge theory (47).

path of parton p. The Berry phase associated with braiding one parton around the other is then given by the 2D mutual statistics formula $\theta_{pp'} = 2\pi/3N$.

Now, consider the case where the separation r between the partons is kept much smaller than ξ. In this case, the mode (56) will not be frozen out and the partons will experience long-range interactions. The phase associated with braiding one parton around the other will depend on the details of the path and can be calculated using the 3D gapless gauge theory (47).

As for the crossover between the two regimes, we expect that the Berry phase in the 3D regime scales with the separation between the partons according to some power law. When the separation is on the order ξ, we expect that the phase is on the order $1/N$ so that it agrees with the phase in the 2D regime.

IX. RELATIONSHIP TO DECOUPLED $\nu = 1/3$ LAYERS

An interesting feature of the parton construction is that it can describe the decoupled $\nu = 1/3$ layered state within the same framework as the 3D parton state. One can tune from one state to the other by changing a single coupling constant in the parton gauge theory.

To see this, let us go back to the original parton Hamiltonian (8). So far we have analyzed the physics of this Hamiltonian in the limit of weak gauge fluctuations: $g_3 \ll J_3, t_{p3}, t_{p\perp}$ and $g_\perp \ll J_\perp, t_{p3}, t_{p\perp}$. We found that in this regime, the low-energy physics was described by the 3D parton state.

However, by increasing g_3 one can also access a regime where (interlayer) gauge fluctuations are strong. More specifically, suppose that $g_3 \gg J_3, t_{p3}, t_{p\perp}$. In this case, the compactness of the U(1) gauge field becomes important. Since the lattice electric field is integer valued and g_3 is large, the interlayer field E_3^q is essentially fixed at $E_3^q = 0$. Nonzero values of E_3^q cost energy on the order g_3. As a result, the interlayer tunneling terms $H_{tz(z+1)}$ and interlayer flux terms $\cos(\Delta_i A_{xz,3}^q - A_{xz,i}^q + A_{x(z+1),i}^q)$ in Eq. (8) are suppressed and can be dropped from the Hamiltonian. At low energies, the physics is then described by $H = \Sigma_z (H_{zA} + H_{zt})$; the effective theory for *decoupled* $\nu = 1/3$ states.

On an intuitive level, the basic physics is that large interlayer gauge fluctuations prohibit partons from tunneling between the layers, leading to a decoupled layer state. One can also think about the transition between the 3D state and the decoupled layer state in terms of Higgs condensation. Consider, for example, the bilayer case. Recall that the interlayer gauge-field term (23) in the bilayer parton Hamiltonian is given by

$$H_{12A} = \sum_{xq} \frac{g_3}{2} (E_{x1,3}^q)^2 - \sum_{xiq} J_3 \cos(\Delta_i A_{x1,3}^q - A_{x1,i}^q + A_{x2,i}^q). \quad (58)$$

Let us view the operator $e^{iA_{x1,3}^q}$ as the creation operator of a boson at site x, while $E_{x1,3}^q$ is the number operator which measures the number of bosons at site x. The first term is then a potential-energy term which describes the energy associated with having a certain number of bosons on a given site, while the second term is a kinetic-energy term which describes a boson hopping from one site to a neighboring site. The presence of the combination $A_{1,i}^q - A_{2,i}^q$ in the argument of the cosine tells us that the boson is minimally coupled to the gauge field $A_{1,i}^q - A_{2,i}^q$.

It is illuminating to think about the strong and weak gauge fluctuation regimes in this language. When $g_3 \gg J_3$, the boson is massive (e.g., in a Mott insulating phase) and is therefore irrelevant at low energies. The interlayer gauge terms can then be dropped and the low-energy effective Hamiltonian (19) consists of two decoupled layers $H = \Sigma_{z=1}^2 (H_{zt} + H_{zA})$. The ground state is thus two decoupled $\nu = 1/3$ states.

On the other hand, when $g_3 \ll J_3$, the boson condenses. Since the boson is minimally coupled to the gauge field $A_{1,i}^q - A_{2,i}^q$, this boson condensation is a kind of Higgs condensation where the Higgs boson is coupled to a Chern-Simons gauge field.[31] When such a Higgs boson condenses, the result is another gapped FQH state; in this case, the bilayer parton state.

Because one can tune from the decoupled state to the parton state by changing a single coupling constant g_3, one can speculate that these states are in some sense neighboring or proximate phases. This is one of the reasons that we propose the parton state as a candidate for an intermediate tunneling phase.

X. EDGE (AND SURFACE) STATES

In this section, we discuss the edge states for the N-layer system: both for finite N and in the 3D limit $N \to \infty$. First, consider the case of finite N. In this case, the edge theory can be read off from the K matrix and charge vector t described in Sec. VI, using the standard formalism.[26] The result is a chiral boson theory with $3N-2$ modes, $\phi^I, I = 1, \ldots, 3N-2$. The Lagrangian is of the form

$$L = \frac{1}{4\pi} (K_{IJ} \partial_t \phi^I \partial_x \phi^J - V_{IJ} \partial_x \phi^I \partial_x \phi^J), \quad (59)$$

where V_{IJ} is a positive-definite velocity matrix which describes the velocities of each of the modes and the density-density interactions between different modes. Quasiparticle excitations are parametrized by integer vectors l and are created by operators of the form $\exp(i l_I \phi^I)$. The electric charge corresponding to a quasiparticle l is given by $q = e(t^T K^{-1} l)$.

As many aspects of the edge theory depend on microscopic details of the edge and can be affected by edge reconstruction, let us discuss two simple quantities which are universal. The first quantity—the electric Hall conductance—is given by

$$\sigma_{xy} = (t^T K^{-1} t)\frac{e^2}{h} = \frac{Ne^2}{3h}. \quad (60)$$

Of course, this is exactly what we expect since the parton state has $\nu=1/3$ per layer.

A more interesting quantity is the thermal Hall conductance. Recall that each chiral boson edge mode gives a contribution of $\pm\frac{\pi^2 k_B^2}{3h}T$ to the thermal Hall conductance, with the sign determined by the chirality of the mode. Thus, the thermal Hall conductance can be computed by counting the number of edge modes. Since the K matrix has $3N-2$ positive eigenvalues and no negative eigenvalues, there are $3N-2$ modes propagating in one direction and no modes propagating in the opposite direction. We conclude that the thermal Hall conductance is $3N-2$ (in units of $\frac{\pi^2 k_B^2}{3h}T$).

Before concluding this section, let us briefly discuss the 3D limit $N\to\infty$. In this case, the boundary is two dimensional and the edge states are actually *surface* states. Let us focus on the most interesting kind of boundary: a boundary in the xz plane (with layers oriented in the xy plane).

The analysis of the surface states is complicated by the fact that the bulk has gapless modes. Because of this, we will not analyze the surface in detail but rather sketch the basic qualitative picture which is evident in mean-field theory. In mean-field theory, the surface modes are given by three species of noninteracting 2D fermions, which are chiral in the x direction but nonchiral in the z direction. The modes form a sheath of chiral fermions with Fermi surface $k_x \sim t_z \cos(k_z a)$.[1] When one goes beyond mean-field theory and includes gauge fluctuations, these fermions will become minimally coupled to the bulk photon mode. The gauge fluctuations will certainly affect the surface theory; however, we know that they cannot gap out the surface modes entirely. Indeed, the gaplessness of the edge modes is protected by the nonzero electric Hall conductivity in the bulk ($e^2/3h$ per layer). The surface sheath therefore resembles a chiral analog of the Halperin-Lee-Read[27] state which is protected against gap-forming instabilities by the topological character of the bulk phase.

These surface states may provide the simplest experimental signature of the 3D parton states. In particular, consider the z-axis surface longitudinal conductance σ_{zz}. In mean-field theory, σ_{zz} behaves just like the conductance of the layered integer quantum Hall system studied in Ref. 1. Thus, $\sigma_{zz}\sim$ const as $T\to 0$. Including gauge fluctuations, we expect that this constant will be renormalized, but σ_{zz} will remain finite at zero temperature. This should be contrasted with the behavior of σ_{zz} in the decoupled $\nu=1/3$ Laughlin state. In that case, $\sigma_{zz}\sim T^3$ as $T\to 0$.[1] Thus, a measurement of σ_{zz} could, in principle, distinguish the 3D parton state from the decoupled Laughlin state.

XI. CONCLUSION

In this paper, we have constructed a candidate state for a multilayer FQH system with average filling $\nu=1/3$ per layer. We have proposed that the state may be realized in the intermediate tunneling regime where the interlayer tunneling strength is on the same order as the Coulomb energy e^2/l_B. Our construction is based on a slave-particle approach known as the "parton construction."

We have analyzed the state for both a finite number of layers N and in the 3D limit $N\to\infty$. In the case of a finite number of layers N, the state is a gapped FQH state and is described by a $(3N-2)\times(3N-2)$ K matrix. Its quasiparticle excitations are anyonic and carry charge $e/3$.

In the 3D limit $N\to\infty$, the state is more unusual. It supports two types of excitations: gapped $e/3$ fermionic quasiparticle excitations and gapless neutral collective modes. The quasiparticle excitations are truly 3D quasiparticles and can propagate freely both within and between layers. (This is in contrast to the charge $e/3$ excitations in the multilayer Laughlin state, which are confined to individual layers). The gapless neutral collective modes are emergent photon modes originating from the slave-particle gauge theory. Unlike Maxwell photons, they come in only one polarization and have an anisotropic dispersion $\omega_{\mathbf{k}}^2 \sim \mathbf{k}_\perp^2 + k_z^4$. The $e/3$ fermionic quasiparticles are minimally coupled to these photon modes so that they have long-range interactions.

The edge physics of the finite layer and 3D systems is also interesting. When N is finite, the edge theory is described by a conformal field theory with $3N-2$ chiral boson modes. In the 3D limit, the edge modes are more complex. In mean-field theory, the edge (or more accurately, surface) modes are described by three different species of noninteracting 2D fermions which propagate chirally in the x direction (e.g., the direction parallel to the layers) and nonchirally in the z direction (e.g., the direction perpendicular to the layers). Going beyond the mean-field theory, we expect that these fermions are minimally coupled to the gapless bulk photon modes. However, we have not analyzed the surface physics in detail. This is an interesting direction for future research.

Another direction for future research would be to construct other types of layered FQH states. For example, it would be interesting to build multilayer states with average filling $\nu=1/2$ per layer, in particular, multilayer states which are related to the Moore-Read[32] $\nu=1/2$ state or the composite Fermi liquid $\nu=1/2$ state[27] (instead of the Laughlin $\nu=1/3$ state which we have investigated here). One possible approach for this problem would be to employ a parton construction where one writes an electron as $c=d_1 d_2 f$, where d_1, d_2 are fermionic partons carrying charge $e/2$ and f is a neutral fermionic parton. One could then consider mean-field states where, in each layer, d_1, d_2 are in integer quantum Hall states and f is in a $p+ip$ superconducting state or a Fermi-liquid state. In this way, one may be able to construct multilayer and 3D relatives of the Moore-Read or composite Fermi-liquid states.

In general, there are clearly many possibilities for 3D multilayer FQH states, most of which have not been explored. We hope that the parton construction provides a useful tool for constructing and analyzing these states of matter.

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