Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators

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Particle-vortex duality is a powerful theoretical tool that has been used to study bosonic systems. Here, we propose an analogous duality for Dirac fermions in 2+1 dimensions. The physics of a single Dirac cone is proposed to be described by a dual theory, QED$_3$, with again a single Dirac fermion but coupled to a gauge field. This duality is established by considering two alternate descriptions of the three-dimensional topological insulator (TI) surface. The first description is the usual Dirac fermion surface state. The dual description is accessed via an electric-magnetic duality of the bulk TI coupled to a gauge field, which maps it to a gauged chiral topological insulator. This alternate description ultimately leads to a new surface theory, QED$_3$, which provides a simple description of otherwise intractable interacting electronic states. For example, an explicit derivation of the T-Pfaffian state, a proposed surface topological order of the TI, is obtained by simply pair condensing the dual fermions. The roles of time-reversal and particle-hole symmetries are exchanged by the duality, which connects some of our results to a recent conjecture by Son on particle-hole symmetric quantum Hall states.

I. INTRODUCTION

Following the prediction and classification of topological insulators (TIs) and topological superconductors (TSc) based on free-fermion models [1], the conceptual frontier has now shifted to studying strongly interacting topological phases. For low-dimensional phases, one can utilize powerful nonperturbative techniques to describe the (1+1)-dimensional [(1+1)D] edge or bulk to obtain qualitatively new physics introduced by interactions [2–8]. However, for (3+1)-dimensional [(3+1)D] systems, we have few nonperturbative tools; nevertheless, remarkable theoretical progress has been made in recent years. For example, entirely new phases that only appear in interacting systems have been predicted [9–16]. Furthermore, phases that were predicted to be distinct by the free-fermion classification, and whose edge states are stable to weak interactions, can sometimes be smoothly connected in the presence of strong interactions. A striking example of this phenomenon is provided by the 3D topological superconductors with time-reversal invariance (class DIII), whose integer free-fermion classification is broken down to Z$_{16}$ [12,17–21] by strong interactions. The discovery of strongly correlated TIs such as 3D topological Kondo insulators [22] may provide an experimental window into the effects of strong interactions.

A useful theoretical tool that was introduced to study surfaces of strongly interacting 3D topological phases is surface topological order (STO) [9,17–19,23–28]. In the early days of the field it was assumed that the surface of a topological phase, such as a topological insulator, is metallic if all symmetries are preserved. The resulting surface state of a TI, a single Dirac cone, is forbidden in a purely two-dimensional (2D) system with time-reversal invariance and charge conservation since it suffers from the parity anomaly [29,30]. If gapped, it was generally assumed that the surface must break one of the protecting symmetries such as time-reversal symmetry. However, with strong interactions, new possibilities arise. A

gapped, insulating surface state of the TI can preserve all symmetries if it is topologically ordered, i.e., if the surface supports anyonic excitations with fractional quantum numbers. This topological order must encode the parity anomaly that ensures it is a bona fide surface state of the topological insulator bulk. In this sense it encodes the same "Hilbert space," with the same anomalies as the single Dirac cone surface state. Here, we will discuss a dual surface theory that also captures the same Hilbert space, which, in contrast to the STO, is gapless in the UV and is described by QED$_3$.

More precisely, the surface Dirac theory of a TI is given by the Lagrangian

$$\mathcal{L}_e = \bar{\Psi}_e i\gamma^\mu[\partial_\mu - iA_\mu]\Psi_e, \tag{1}$$

where $\bar{\Psi}_e = \Psi_e^\dagger \gamma^0$, γ^μ are 2×2 Dirac matrices, and we have introduced an external electromagnetic potential A_μ to keep track of the conserved $U(1)$ charge, and possibly insert a chemical potential. Then, the proposed dual surface theory is

$$\mathcal{L}_{cf} = \bar{\psi}_{cf} i\gamma^\mu[\partial_\mu - ia_\mu]\psi_{cf} - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda, \tag{2}$$

where the fermions are now coupled to an emergent fluctuating gauge field a_μ, whose flux is proportional to the electron density or, more precisely, 4π flux of a corresponds to unit electron charge.

Let us note three key points. First, we can ask the following: How do we represent the electron insertion operator Ψ_e in the dual theory? We find that Ψ_e corresponds to a double monopole operator that introduces 4π flux as expected from the previous discussion. It will be shown that this operator has all the desired properties. Second, how do we interpret ψ_{cf}, the dual fermions, in terms of electrons? ψ_{cf} will be shown to be a double $(2hc/e)$ vortex in the electron fluid bound to an electron, closely analogous to the composite fermion construction [31,32],

which accounts for the subscript. Finally, we note the action of time-reversal symmetry exchanges particles and holes of ψ_{cf}, $T\,\psi_{cf} \to \psi_{cf}^\dagger$, consistent with their interpretation as vortex like degrees of freedom. A finite chemical potential on the electronic Dirac cone translates into a finite magnetic field on the composite fermions and vice versa.

Many aspects of the duality above closely resemble particle-vortex duality for bosons [33,34]. Denoting the boson by a complex scalar field Φ, we have the XY action

$$\mathcal{L}_{XY} = |(\partial_\mu - i A_\mu)\Phi|^2 - V(|\Phi|), \tag{3}$$

which is dual to the Abelian-Higgs action

$$\mathcal{L}_{AH} = |(\partial_\mu - i\alpha_\mu)\varphi|^2 - \tilde{V}(|\varphi|) - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu\alpha_\lambda. \tag{4}$$

The dual field φ is minimally coupled to a fluctuating electromagnetic field α whose flux is the boson density. The dual field φ inserts vortices into the Bose fluid and the monopole operator of the theory (4) corresponds to Φ. The duality is believed to hold at the critical point of theories (3) and (4), i.e., at the insulator-superfluid transition of bosons. An important question then arises: Is the dual surface state in Eq. (2) dynamically equivalent to the usual free-electron Dirac cone (with chemical potential at the node)? This would be the simplest form of the correspondence, but is not something that we can prove at present. This interesting question is discussed further in Sec. III C.

The existence of the dual surface theory (2) clarifies a number of earlier mysteries. Two different surface topological orders were put forward for the TI. The first, the Pfaffian-antisemion state consisting of 12 nontrivial anyons, was obtained using a vortex condensation method [26,27]. Another, simpler topological order, the T-Pfaffian, with half as many anyons was also proposed [25,28], but despite its apparent simplicity could not be "derived" in an analogous fashion, or directly connected back to the superconducting surface state of the topological insulator. We will see that T-Pfaffian is readily derived from the dual surface theory. Another observation that was previously mysterious was the close relation between *chiral* topological insulators (class AIII) with $\nu = 1$ (single surface Dirac cone) and conventional topological insulators (class AII) [35]. In both cases, there is a $U(1)$ symmetry that can be spontaneously broken at the surface; the statistics of vortices in the resulting surface superfluid can be determined. A striking observation is that vortex statistics on the conventional TI surface is closely related to the STO on the chiral TI (cTI) surface, and vice versa. For example, vortices on the chiral TI surface have the same statistics and transformation properties under time reversal as the T-Pfaffian topological order. The dual surface theory sheds light on this apparent coincidence.

We note recent works which have a significant conceptual overlap with this paper. In Ref. [36], Mross, Essin, and Alicea explicitly construct a gapless surface state for the TI called the composite Dirac liquid (CDL). Like the present dual Dirac theory, pairing the CDL leads to the T-Pfaffian state. However, in contrast to our dual theory, charge fluctuations are gapped in the CDL, and the gapless Dirac fermions have short-ranged interactions. Another insightful development is Son's proposal in Ref. [37] for a dual description of the particle-hole

symmetric half-filled Landau level (see also Ref. [38]). At first sight, this purely 2D problem seems unrelated to the anomalous surface theories we are discussing in this paper, which always occur on a higher-dimensional topological bulk. However, particle-hole symmetry of a Landau level is a nonlocal symmetry. Hence, it can evade restrictions imposed on usual symmetries, and thereby realize the equivalent of an anomalous surface theory in the same dimension. Indeed, our work can be viewed as a "derivation" of the conjecture in Ref. [37], in a setting where symmetries are conventionally implemented (such as on the surface of a topological phase where Landau levels with locally implemented particle-hole symmetry can be realized).

This paper is organized as follows. We derive the surface QED$_3$ theory in Eq. (2) in two stages. First, in Sec. II, we present an unconventional construction of a 3D topological insulator. This construction starts with a gapless $u(1)$ spin-liquid phase, with emergent fermionic quasiparticle realizing a *chiral* topological insulator band structure. The 3D TI is then obtained after a confinement transition (see Fig. 1). Next, in Sec. III A we derive the surface theory that follows from this bulk construction and show it is given by QED$_3$. In Sec. III B, we present a more heuristic derivation of QED$_3$ based solely on the surface physics, which provides a transparent physical interpretation of the dual fields. Section III C
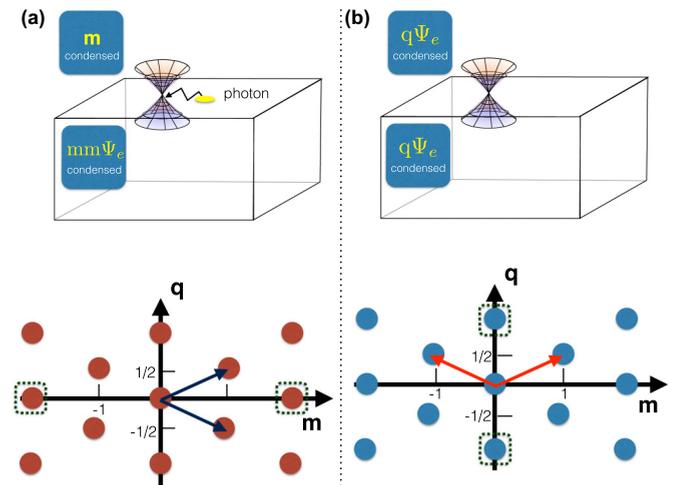

FIG. 1. (a) Dual derivation of topological insulator using fermionic partons in a *chiral* topological insulator band structure ($\nu = 1$ of class AIII), where time reversal flips the sign of the gauge *electric* charge. The bulk topological insulator phase is obtained by condensing a pair of monopoles (0,2) bound to an electron. The surface state consists of the parton Dirac cone coupled to photons that only propagate on the surface, i.e., QED$_3$. (b) The 3D TI derived more directly from the partons in the topological insulator band structure, which is Higgsed by condensing the unit electric charge (1,0) (bound to an electron). The surface is the regular single Dirac cone. The gauged versions of the chiral topological insulator (a) and ordinary topological insulator (b) are related by electric-magnetic (EM) duality, as seen from the lattice of electric and magnetic charges by identifying the basis vectors shown. The two basis vectors are exchanged by time-reversal symmetry in both (a) and (b). The EM duality relates the double monopole condensate and the Higgs condensate, consistent with obtaining a TI from both descriptions.

discusses possible scenarios for the low-energy dynamics of QED$_3$: in particular, in this section we contrast weak (Hilbert space and symmetry) and strong (dynamical) forms of duality between free Dirac theory and QED$_3$. In Sec. IV, we show how previously known surface phases of the 3D TI, including the time-reversal symmetry-broken insulator and Fu-Kane superconductor surface states, as well as the surface topological order, can be obtained in the dual description. In Secs. V and VI, we show how the particle-vortex duality of the 2D surface theory can be understood as a descendant of electric-magnetic duality of 3D $u(1)$ gauge theory.

II. A PARTON CONSTRUCTION OF A TOPOLOGICAL INSULATOR

In this section, we will use parton techniques to construct a 3D gapped state of electrons with no intrinsic topological order and $U(1) \rtimes T$ symmetry (the T symmetry acts on the electrons in a conventional Kramers manner). The 2D surface of this state is described exactly by the gauge theory in Eq. (2). We will argue that the constructed bulk state is continuously connected to a noninteracting topological insulator, therefore, the theory (2) provides a description of the TI surface.

The ingredients we will utilize are as follows:

(1) a trivial ($\theta_{EM} = 0$) band insulator of electrons.

(2) a spin-liquid state of neutral bosons SL$_\times$.

While we are ultimately interested in constructing a T-invariant state of electrons (fermions) charged under the electromagnetic $U(1)$ symmetry, as a first step we will build a T-invariant state of neutral bosons (spins). We will label this state as SL$_\times$. This state will be described by an emergent $u(1)$ gauge theory and will possess a gapless photon excitation. One can think of SL$_\times$ as a spin liquid with global time-reversal symmetry (thus the notation SL; we will explain the meaning of the subscript $\times$ shortly). We will then reintroduce the charged electrons and drive a confinement transition in the $u(1)$ gauge theory, obtaining the desired gapped electronic state that is in the same phase as the noninteracting TI.

A. The $u(1)$ spin liquid of bosons SL$_\times$

To construct the spin-liquid state SL$_\times$ we start with a Hilbert space built out of neutral bosons (spins) B. We use the standard parton approach where B is decomposed into fermionic constituents ψ as

$$B = \psi^\dagger \Gamma \psi \tag{5}$$

with Γ a matrix acting on components of ψ. (The precise component structure of ψ and the form of Γ will not be important in the discussion below.) The representation (5) is invariant under local $u(1)$ gauge rotations

$$u(1): \psi(x) \rightarrow e^{i\alpha(x)}\psi(x). \tag{6}$$

The gauge symmetry (6) will be manifested in the low-energy theory of partons and will give rise to an emergent $u(1)$ gauge field a_μ. [We use lower case letters to distinguish the emergent $u(1)$ gauge symmetry from the physical $U(1)$ charge symmetry.] We take the partons ψ to transform under time reversal as

$$T: \psi \rightarrow U_T \psi^\dagger, \quad \psi^\dagger \rightarrow U_T^* \psi \tag{7}$$

with $U_T^2 = -1$, i.e.,

$$T^2 \psi (T^\dagger)^2 = -\psi. \tag{8}$$

Note that the action of T inverts the charge of the partons under the $u(1)$ gauge symmetry. Since T is an antiunitary symmetry, T and $u(1)$ commute, so the total symmetry group of the parton theory, which includes both the gauge symmetry and the global T symmetry, is $u(1) \times T$. (Our notation SL$_\times$ emphasizes this direct product structure.) Further note that we can combine a rotation by α in the $u(1)$ group with T to get an antiunitary symmetry $T_\alpha = u_\alpha T$, which squares to $e^{2i\alpha}$ when acting on ψ. Thus, while we have nominally chosen the partons to transform as Kramers doublets under T, this has no physical consequence and is a pure convenience. The fact that the time-reversal partners ψ and $\psi^\dagger$ have different gauge charge implies that they lie in different topological sectors of the $u(1)$ gauge theory and hence cannot be assigned a physical Kramers parity.

To complete the construction of the SL$_\times$ state, we imagine that the partons ψ form a band insulator. The transformations of ψ under $u(1) \times T$ are identical to those of an electronic system in class AIII, which we will refer to as *chiral topological insulator* class. One typically thinks of class AIII as T-invariant superconductors with a $u(1)$ symmetry, corresponding to the conservation of the S^z components of spin. In our setup, the $u(1)$ symmetry is an emergent gauge symmetry and has no relation to spin conservation. Recall that noninteracting chiral topological insulators in class AIII have an integer classification $\nu \in \mathbb{Z}$ [39,40]. The 2D boundary between the phase with $\nu \neq 0$ and the vacuum ($\nu = 0$) supports $|\nu|$ Dirac cones. In the presence of interactions, the noninteracting phases are known to collapse to a $\mathbb{Z}_8$ group [17–19]. Interactions also introduce a single novel phase absent in the noninteracting classification, bringing the total classification in class AIII to $\mathbb{Z}_8 \times \mathbb{Z}_2$ [18,41].

To build our spin-liquid state, we will place the partons ψ into a noninteracting band structure with $\nu = 1$. (We give an example of such a band structure in Appendix A.) Since the partons are gapped, the resulting $u(1)$ gauge theory is in the Coulomb phase with a gapless photon a_μ. The effective action of the gauge field a_μ is given by

$$S[a_\mu] = \int d^3x\, dt \left(-\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} + \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\lambda\sigma} f_{\mu\nu} f_{\lambda\sigma} \right) \tag{9}$$

with $\theta = \pi$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. The first term in Eq. (9) is the Maxwell action with a coupling constant e. The second term is the topological term generated by integrating out the partons. Similarly to electrons in an ordinary TI, for partons in a $\nu = 1$ band structure the coefficient $\theta = \pi$.

Let us discuss the spectrum of topological excitations in the SL$_\times$ state. One type of topological excitations is given by partons ψ, i.e., electric charges of a_μ. The second type of excitations is given by magnetic monopoles of a_μ. The presence of the topological term in Eq. (9) endows a monopole with magnetic flux $2\pi m$ with an electric charge [42]

$$q = n + \frac{\theta m}{2\pi} = n + \frac{m}{2}. \tag{10}$$

Here, n is an arbitrary integer which reflects the freedom of adding n electric charges ψ to a monopole. Thus, the topologically distinct excitations form a two-dimensional lattice (q,m) labeled by electric charge q and magnetic charge m satisfying Eq. (10) [see Fig. 1(a)].

The excitations have the following statistics [10]. A single charge $\psi = (1,0)$ is a fermion. A single monopole $(q,m=1)$ of arbitrary electric charge q is a boson. A general dyon (q,m) has statistics $(-1)^{(q-m/2)(m+1)}$ with $+1$ corresponding to bosonic statistics and -1 to fermionic. In particular, a neutral double monopole $(q=0, m=2)$ is a fermion. Two dyons (q,m) and (q',m') experience a nontrivial statistical interaction. Namely, if we place (q',m') at the origin and let (q,m) move along a closed contour $\mathcal{C}$, (q,m) will accumulate a statistical phase $(qm' - q'm)\Omega/2$, where Ω is the solid angle subtended by $\mathcal{C}$. In addition to the statistical interaction, dyons also experience a $1/r$ Coulomb interaction.

As already mentioned, time-reversal symmetry (7) maps electric charge $q \rightarrow -q$. Furthermore, due to the $u(1) \times T$ group structure the magnetic flux is preserved under T: $m \rightarrow m$. Let us next discuss the physical Kramers parity T^2 of the excitations. Kramers parity can only be assigned to topological sectors which are left invariant under T. In the present case, this corresponds to $(q=0,m)$ with m even. As has been discussed in Refs. [18,19], the neutral double monopole $(q=0, m=2)$ is actually a Kramers doublet fermion. This is required by the consistency of the theory since $(0,2)$ can be obtained by fusing the time-reversal partners $(\frac{1}{2},1)$ and $(-\frac{1}{2},1)$. The presence of a nontrivial statistical interaction between $(\frac{1}{2},1)$ and $(-\frac{1}{2},1)$ forces their fusion product $(0,2)$ to be a Kramers doublet fermion [15,26].

Next, let us discuss the surface of $\text{SL}_\times$ phase. Let us imagine that space is divided into two regions by an interface at $z = 0$. We will place our partons ψ into the $\nu = 1$ band structure for $z < 0$ and into the trivial $\nu = 0$ band structure for $z > 0$. The interface then supports a single gapless Dirac cone of ψ:

$$S_{2d} = \int d^2x \, dt \, \bar{\psi}_{cf} i\gamma^\mu (\partial_\mu - ia_\mu)\psi_{cf} \qquad (11)$$

with ψ_{cf} now denoting the surface Dirac fermion. Under T, ψ_{cf} transforms as

$$T: \psi_{cf} \rightarrow \epsilon \psi_{cf}^\dagger, \quad i \rightarrow -i \qquad (12)$$

where $\epsilon = i\sigma^y$ and we are using the basis of γ matrices $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_y, -i\sigma_z, i\sigma_x)$. Again, we stress that this is different from the T transformation of the electron on the free Dirac surface of a TI (1):

$$T: \Psi_e \rightarrow \epsilon \Psi_e, \quad i \rightarrow -i. \qquad (13)$$

The $z < 0$ region realizes our $\text{SL}_\times$ phase. The $z > 0$ also realizes a spin-liquid described by a $u(1)$ gauge theory with a gapless photon. Let us briefly discuss the properties of the spin liquid in the $z > 0$ region. Since here the partons are in a trivial band structure, the topological angle $\theta = 0$ and the dyon spectrum is given by (q,m) with q integer and m integer. The (q,m) dyon has statistics $(-1)^{q(m+1)}$, in particular, all neutral monopoles are bosons. Time-reversal symmetry again acts as $T : (q,m) \rightarrow (-q,m)$, however, the single monopole $(0,1)$ is now a Kramers singlet.

So far, we have constructed an interface between two $u(1)$ spin-liquid phases: one with $\theta = 0$ and one with $\theta = \pi$. The 2D gapless Dirac fermion appearing on the interface (11) interacts with a 3D gapless photon living on both sides of the interface. In order to construct an interface between the spin liquid with $\theta = \pi$ and the vacuum, we need to drive a confinement transition in the region $z > 0$. This can be done by condensing the single neutral monopole $(0,1)$ in the region $z > 0$. Since this monopole is a boson, it can condense. Furthermore, since the monopole is a Kramers singlet the condensation process preserves the T symmetry. The only deconfined excitations are $(0,m)$, and since these are multiples of the condensed monopole $(0,1)$ the resulting phase has no topologically nontrivial excitations. Hence, after monopole condensation the $z > 0$ region realizes the trivial T-invariant vacuum phase. The Dirac cone on the interface survives the monopole condensation, however, it now interacts with a gauge field a_μ, which lives only in the $z < 0$ region.

B. Confinement to a topological insulator

We next describe how to confine the $\text{SL}_\times$ spin-liquid phase with $\theta = \pi$ described in the previous section to a T-symmetric, fully gapped insulator of electrons with no intrinsic topological order. As a first step, we will now need to work in a Hilbert space which includes the physical charged electron Ψ_e which is a Kramers doublet under time-reversal symmetry.

Let us begin by taking a noninteracting "mixture" of a trivial band insulator of electrons and the $\text{SL}_\times$ state of neutral bosons with $\theta = \pi$ constructed in the previous section. Consider a bound state D of the electron Ψ_e and the neutral double monopole of the spin liquid $(q=0, m=2)$. As we discussed above, $(q=0, m=2)$ is a Kramers doublet fermion. Therefore, D is a Kramers singlet boson, which can condense preserving T. What are the properties of the resulting phase? Recall that generally condensation of a dyon with charges (q,m) gives rise to an analog of a Meissner effect for the gauge field combination $q\vec{b} - 2\pi m\vec{e}$, with $\vec{b} = \nabla \times \vec{a}$ the magnetic field of a_μ, and $\vec{e} = \partial_t \vec{a} - \nabla a_t$ the electric field of a_μ. All excitations which are sources of this gauge field combination will be confined, i.e., a dyon (q',m') is confined if $qm' - mq' \neq 0$ (i.e., only dyons which possess trivial mutual statistics with (q,m) are deconfined) [43–45]. Now, since the electron has no charge under a_μ, our condensing dyon D still has electric gauge charge $q = 0$ and magnetic charge $m = 2$. Its condensation will gap out the photon giving rise to the "Meissner" effect $\vec{e} = 0$. Therefore, all excitation with gauge charge $q \neq 0$ will be confined. Remembering that $q = n + m/2$, only excitations with $q = 0$ and m even are deconfined. These excitations are multiples of the condensing dyon D (possibly with electrons Ψ_e added on top). Therefore, the condensed phase has no nontrivial deconfined excitations and so possesses no intrinsic topological order.

What is the fate of the physical $U(1)$ charge symmetry in the D-condensed phase? First, all excitations in the Coulomb phase can be labeled by $(q,m; Q)$, with (q,m) being the emergent $u(1)$ electric and magnetic quantum numbers coming from the $\text{SL}_\times$ sector, and Q being the physical $U(1)$ charge coming from the electron band-insulator sector. Nominally, D has quantum numbers $(q=0, m=2; Q=1)$. Therefore,

one might naively think that the D-condensed phase breaks the $U(1)$ symmetry and is a superfluid. However, this is not the case. Indeed, one cannot build any local observable (i.e., one with $q = 0$ and $m = 0$) with nonzero Q out of D. More physically, recall that the dyons D experience a long-range $1/r$ interaction. In the D-condensed phase, the dyons D form a Debye plasma with short-range correlations, so the resulting state is gapped. This fact is completely insensitive to D's carrying a global $U(1)$ charge. So, the D-condensed phase cannot possibly be a superfluid since a superfluid would necessarily possess a gapless Goldstone mode; rather, it is an insulator. Now, let us imagine inserting a gapped double monopole $(0,2)$ with $Q = 0$ into the Debye plasma of D's. This double monopole will be Debye screened by the D's; it will be surrounded by a cloud of D's and $D^\dagger$'s with a total D number equal to -1. Now, as each D carries a $U(1)$ charge $Q = 1$, the screening cloud has a total electric charge $Q = -1$. Therefore, we conclude that a deconfined double monopole sucks up a physical electric charge -1 in the D-condensed phase. More generally, the true physical $U(1)$ charge of an excitation with "nominal" quantum numbers $(q,m; Q)$ in the D-condensed phase is

$$Q_{\text{EM}} = Q - m/2. \tag{14}$$

Note that since only dyons with even m are deconfined, the electric charge Q_{EM} is always an integer.

Let us now argue that this phase has a response to the physical electromagnetic field A_μ characterized by $\theta_{\text{EM}} = \pi$. This is most conveniently done via the Witten effect, by calculating Q_{EM} of an inserted monopole of A_μ. Before the D condensation, we can label all excitations by $(q,m; Q,M)$ where M now represents the magnetic charge under A_μ. Since the response of our initial Coulomb phase to A_μ comes entirely from the trivial electron band insulator, the $U(1)$ sector is characterized by a θ angle $\theta_{\text{EM}} = 0$, so Q and M are both integers. Now, D has quantum numbers $(q = 0, m = 2; Q = 1, M = 0)$, so its condensation leads to a Meissner effect for $4\pi \vec{e} - \vec{B}$, where $\vec{B} = \nabla \times \vec{A}$ is the magnetic field strength of A_μ. Thus, deconfined excitations must have $2q = M$. In particular, an $M = 1$ external monopole of A_μ must carry a_μ electric charge $q = \frac{1}{2}$. Since $q - m/2$ is an integer, we conclude that the $M = 1$ external monopole must bind an odd number m of monopoles of a_μ. From Eq. (14), we conclude that an external $M = 1$ monopole will bind a half-odd-integer physical electric charge Q_{EM}. This implies that the D-condensed phase has a response to the external $U(1)$ gauge field with $\theta_{\text{EM}} = \pi$, the same as the EM response of a noninteracting topological insulator.

By an argument of Ref. [15], a phase of electrons with no intrinsic topological order and $\theta_{\text{EM}} = \pi$ is identical to a noninteracting topological insulator up to an SPT phase of neutral bosons with T invariance. In Sec. VI, we will give another viewpoint on why the D-condensed phase must be identical to a noninteracting TI up to a bosonic SPT phase. Moreover, by strengthening the argument in Sec. VI, we will be able to show that the D-condensed phase is continuously connected to a noninteracting TI with *no* additional bosonic SPT [41].

III. SURFACE THEORY QED$_3$

A. Derivation of surface theory from parton construction

Let us now turn to the surface of the D-condensed phase. As before, we imagine putting partons into a $\nu = 1$ band structure of class AIII in the region $z < 0$ and into the trivial $\nu = 0$ band structure in the region $z > 0$. The electrons Ψ_e are placed into a trivial band-insulator band structure everywhere in space. The interface at $z = 0$ supports a single Dirac cone of partons ψ_{cf} coupled to a bulk $u(1)$ gauge field a_μ living on both sides of the interface [see Eq. (11)]. We drive the $z > 0$ side of the interface into a trivial insulating state (vacuum) by condensing the single monopole ($q = 0$, $m = 1$; $Q = 0$). We condense the D dyon on the $z < 0$ side of the interface, driving it into the topological insulator phase. Both sides of the interface are now confined and exhibit the Meissner effect $\vec{e} = 0$. The Dirac cone of partons ψ_{cf} on the surface survives the condensation since the bulk gap to partons ψ persists during the condensation process. Due to the bulk Meissner effect, the field lines of the electric field $\vec{e}$ cannot penetrate into the bulk and can only stretch along the surface. On the other hand, a surface magnetic field b_z perpendicular to the interface is allowed; such a magnetic field gets Debye screened by the condensed monopoles/dyons on both sides of the interface. Thus, the gauge field a_μ is now confined to live on the surface becoming a $(2+1)$-dimensional $u(1)$ gauge field, so the surface theory is simply given by QED$_3$ with a single flavor of Dirac fermions.

Let us discuss the response of the surface to the $U(1)$ gauge field A_μ. Imagine that there is a magnetic field $b_z = \partial_x a_y - \partial_y a_x$ perpendicular to the surface. As already noted, this magnetic field will be Debye screened by monopoles/dyons on the two sides of the interface. On the $z > 0$ side, the condensed monopoles ($q = 0$, $m = 1$; $Q = 0$) will form a screening layer with 2D density $\rho_m = -\frac{1}{2\pi} b_z$. Since these monopoles carry no $U(1)$ charge, they do not contribute to the physical electric charge density. On the $z < 0$ side, the condensed D dyons [quantum numbers ($q = 0$, $m = 2$; $Q = 1$)] will form a screening layer with 2D density $\rho_D = \frac{1}{4\pi} b_z$. Since each D has electric charge $Q = 1$, this screening layer creates a $U(1)$ charge density $\rho_{\text{EM}} = \frac{1}{4\pi} b_z$.

Similarly, imagine that an electric field e_i ($i = x,y$) along the interface is present. This electric field must be Meissner screened by monopole currents on both sides of the interface (analogous to how a magnetic field along the surface of a superconductor is Meissner screened by electric currents). On the $z > 0$ side of the interface, this results in a monopole current $j_i^m = \frac{1}{2\pi} \epsilon_{ij} e_j$. Since these monopoles are neutral, the monopole current does not contribute to the $U(1)$ current. On the $z < 0$ side of the interface, the electric field is screened by a current of D dyons $j_i^D = -\frac{1}{4\pi} \epsilon_{ij} e_j$, which translates into a $U(1)$ electric current $j_i^{\text{EM}} = -\frac{1}{4\pi} \epsilon_{ij} e_j$. Thus, we conclude that the surface gauge field a_μ induces a $U(1)$ current $j_{\text{EM}}^\mu = \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$, and the effective action of the surface theory in the presence of an external $U(1)$ gauge field A_μ is

$$\mathcal{L}_{cf} = \bar{\psi}_{cf} i \gamma^\mu (\partial_\mu - i a_\mu) \psi_{cf} - \frac{1}{2(2\pi)} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda. \tag{15}$$

We immediately see that dynamical instantons of a_μ are prohibited in the surface theory as they do not preserve the

$U(1)$ charge. However, instantons of flux $\phi = 2\pi m$, with m even, do correspond to physical operators with electric charge $Q_{\mathrm{EM}} = m/2$ in the surface theory. In fact, a flux 4π instanton is identified with the electron insertion operator Ψ_e. To see this, imagine we create an electron Ψ_e on the surface. Our parton construction had Ψ_e gapped everywhere (including on the boundary). However, Ψ_e can decay into gapless boundary degrees of freedom as follows: it can grab a double monopole from the $z > 0$ side of the interface (where monopoles are condensed) and tunnel across the interface to the $z < 0$ region vanishing into the condensate of $D = \Psi_e \times (0,2)$. An a_μ flux of 4π and $U(1)$ charge $Q_{\mathrm{EM}} = 1$ is created on the surface in the process. Thus, an electron creation operator Ψ_e corresponds to a flux 4π instanton in the surface theory.

Note that single (flux 2π) instanton events do not correspond to physical operators in the surface theory. Indeed, a flux 2π instanton would correspond to a single monopole tunneling across the $z = 0$ interface. But, single monopoles are confined in the D-condensed region, so single instanton events are not local operators in the surface theory.

A complementary picture to the above discussion can be obtained by studying instanton events directly in surface QED$_3$ theory. Let us imagine that the TI phase obtained by D condensation occupies a solid ball of radius R and the trivial vacuum occupies the region outside this ball. The surface theory then lives on a sphere S^2. A strength m instanton event in the surface theory will create a flux $2\pi m$ on the surface. For simplicity, imagine this flux spreads uniformly across the surface. The single-particle spectrum of a Dirac fermion on S^2 in the background of a uniform flux $2\pi m$ possesses $N_0 = m$ zero modes. Recall that time-reversal symmetry inverts the $u(1)$ charge density $T\rho(x)T^\dagger = -\rho(x)$. This implies that the total $u(1)$ charge of a state with all the negative energy modes filled and all the zero and positive energy modes empty is $q = -N_0/2 = -m/2$ [46]. On a compact space such as S^2 the total $u(1)$ gauge charge q must be zero. Therefore, we must fill $m/2$ out of m zero energy modes. If $m = 1$, there is only a single zero-energy level, so we cannot half-fill it: the state with the zero mode empty has $q = -\frac{1}{2}$ and the state with the zero mode filled has $q = \frac{1}{2}$. Therefore, the flux 2π instanton is not a local operator in QED$_3$. However, if $m = 2$, we have two zero modes: we can fill either one or the other, obtaining two degenerate $q = 0$ states. In fact, these states transform in the $j = \frac{1}{2}$ representation of the rotation group on the sphere. Since the surface theory QED$_3$ is Lorentz invariant, the fact that we have changed the spin of the system by $\frac{1}{2}$ implies that we have added a fermion to the system. This is consistent with our identification of the flux 4π instanton operator with the electron Ψ_e.

It is often stated that a single Dirac fermion in $2+1$ dimensions suffers from the parity anomaly, namely, it cannot be consistently coupled to a $u(1)$ gauge field preserving the time-reversal symmetry [29]. Our surface theory (15) has a dynamical $u(1)$ gauge field a_μ, which is confined to live just on the surface (we switch off the background electromagnetic field A_μ for now). Thus, the standard argument for the evasion of the anomaly via the $(3+1)$-dimensional bulk does not directly apply in this case. Rather, the anomaly is resolved by modifying the compactification of the gauge field a_μ in the surface theory. This important point is discussed in detail in Appendix B.

TABLE I. Vortex defects on the surface of a topological insulator. The vortex statistics are described by Ising $\times U(1)_{-8}$ theory and the table lists the topological spins of vortices. The column index is the flux $k\frac{hc}{2e}$, which coincides with the $U(1)_{-8}$ charge and the row index is the Ising charge. Only ψ_0 has a well-defined Kramers parity $T^2 = -1$. The fermionic vortex ψ_4 has trivial mutual statistics with all other vortices.

$k \rightarrow$	0	1	2	3	4	5	6	7
I	1		$-i$		1		$-i$	
σ		1		-1		-1		1
ψ	-1		i		-1		i	

B. Heuristic derivation of dual surface theory

We now give a more heuristic derivation of the dual surface QED$_3$ theory, which starts directly with the Dirac cone of electrons (1) and does not rely on the bulk construction in Sec. II.

Consider the TI surface state (single Dirac cone) with $U(1)$ charge and time-reversal symmetry $U(1) \rtimes T$. For simplicity, we assume that the chemical potential is at the Dirac point. Now, consider surface superconductivity and the statistics of vortices induced in the superconductor. This was studied in Refs. [26,27] where the vortex theory in Table I was derived. The vortex statistics can be described by a topological quantum field theory (TQFT) Ising $\times U(1)_{-8}$. The anyons in this TQFT are labeled by the Ising charge $\{I, \sigma, \psi\}$ and a $U(1)_{-8}$ charge k, which will be noted as a subscript on the Ising charge. The $U(1)_{-8}$ charge coincides with the vorticity (the $hc/2e$ vortex is the unit vortex). Not all sectors of Ising $\times U(1)_{-8}$ TQFT are realized by vortices: vortices with odd vorticity always have an Ising charge σ, and vortices with even vorticity have an Ising charge I or ψ. Time-reversal symmetry reverses the vorticity.

Note that the Ising $\times U(1)_{-8}$ TQFT is the same as one describing the T-Pfaffian surface topological order of a TI (Table II), however, the action of time-reversal symmetry on the $U(1)_{-8}$ charge k is different. In a T-Pfaffian state k is preserved by T, while in the vortex theory it is reversed. A further difference is that anyons in the T-Pfaffian state also carry charge $k/4$ under the physical $U(1)_{\mathrm{EM}}$ global symmetry. Let us define the CT-Pfaffian to be a topological order of

TABLE II. T-Pfaffian$_\eta$ topological orders with $\eta = \pm 1$. The top table lists the topological spins of anyons; the column and row indices denote the $U(1)_{-8}$ charge and the Ising charge, respectively. The physical electric charge of anyons $Q_{\mathrm{EM}} = k/4$ with k the $U(1)_{-8}$ charge. Time reversal maps k to itself. The bottom row lists the T^2 assignment of anyons (where defined). The T^2 assignment is independent of the Ising charge. ψ_4 is the physical electron. The CT-Pfaffian topological order has identical anyon content and charge assignments, but T maps $k \rightarrow -k$ and ψ_0 has $T^2 = -1$.

$k \rightarrow$	0	1	2	3	4	5	6	7
I	1		$-i$		1		$-i$	
σ		1		-1		-1		1
ψ	-1		i		-1		i	
T^2	1	η		$-\eta$	-1	$-\eta$		η

electrons with $U(1) \times T$ global symmetry with the same anyon content and $U(1)$ charge assignments as the T-Pfaffian, but reversed action of time-reversal symmetry on the $U(1)_{-8}$ charge. One may suspect that the CT-Pfaffian is the surface topological order of the $\nu = 1$ class AIII chiral topological insulator. Indeed, precisely this proposal has been made in Ref. [17] (modulo a bosonic SPT phase, which we will return to). Note that the fermion ψ_4 in the T-Pfaffian and CT-Pfaffian states has trivial mutual statistics with all other anyons and is identified with the physical electron.

Let us come back to the vortices on the surface of a TI. To encode logarithmic interactions between vortices we can take them to carry gauge charge of an emergent $u(1)$ gauge field a_μ [in addition to charges in the Ising $\times U(1)_{-8}$ TQFT]. For notational convenience, let us normalize the charge of a unit vortex under a_μ to be $\frac{1}{4}$. Let us for a moment ignore the fluctuations of a_μ, and treat the vorticity as a charge under a global $u(1)$ symmetry. Then, we can regard the vortex theory as a topological order CT-Pfaffian with $u(1) \times T$ symmetry, which is the surface state of the $\nu = 1$ chiral TI in class AIII. In this identification, the quadruple vortex ψ_4 is identified with the electron of the class AIII TI.

With this connection in hand, we can discuss other possible surface states of the class AIII TI. The simplest one of course is the single Dirac cone, which will now be composed of the ψ_4 fermions. Reinstating the fact that the $u(1)$ charge is actually gauge charge implies that the fermions are coupled to a gauge field as described by the Lagrangian (2). This line of reasoning provides a physical picture of the dual fermionic field $\psi_{cf} = \psi_4$, i.e., it is the strength 4 fermionic vortex on the surface of a topological insulator. Also, if one prefers to view the vortex as a boson, then this is a bound state of a $4\frac{hc}{2e}$ vortex and a neutral Bogoliubov quasiparticle. Due to its vorticity, it is minimally coupled to a gauge field a. Since each electron appears as a flux of 4π to a strength 4 superconductor vortex ($2hc/e$ vortex), we have $(\partial_x a_y - \partial_y a_x) = 4\pi \rho_e$, the electron density. The terminology "composite fermion" to describe the fermionic vortex ψ_4 should be clear now. Recall, in the original definition of composite fermions each electron is attached to a pair of hc/e vortices, as in ψ_4. While this is usually discussed in the context of quantum Hall states [31,32], the present discussion shows that this duality is also relevant to describe the surface of a topological insulator.

One may think that the Dirac theory of composite fermions represents a different gapless surface state from the original Dirac model (1), however, the simplest conclusion that the two models are dynamically equivalent also remains an intriguing possibility as discussed below.

C. Weak versus strong form of the duality

So far, we have ignored the issue of the dynamics of the surface QED$_3$ theory. Instead, our discussion was focused on issues of symmetries and quantum numbers of operators. In principle, the surface theory (15) can be perturbed by an arbitrary local symmetry-preserving operator. This gives rise to a large landscape of possible surface phases, some of which will be discussed in Sec. IV. Since the D-condensed phase in our bulk construction is continuously connected to a noninteracting TI, one of these surface phases must be the

gapless (ungauged) free Dirac cone of electrons. Thus, the Dirac cone is dual to QED$_3$ in the sense of duality of Hilbert spaces, operators, and symmetries.

One can ask whether a stronger version of duality holds, namely, is weakly coupled QED$_3$ dual in the infra-red to the free Dirac cone. By weakly coupled QED$_3$ we mean the theory

$$\mathcal{L}_{\text{QED}_3} = \bar{\psi}_{cf} i\gamma^\mu (\partial_\mu - ia_\mu)\psi_{cf} - \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} \qquad (16)$$

with the coupling constant g^2 much smaller than the UV cutoff Λ_{UV}. The theory (16) is well defined and analytically controlled in the UV (i.e., for energy scales $g^2 \ll \omega \ll \Lambda_{\text{UV}}$) where it reduces to a Dirac fermion ψ_{cf} interacting weakly with a massless photon a_μ. The fate of the theory in the IR is not known. One can envision three different scenarios, which we list here in the order of increasing exoticity:

(i) The theory (16) spontaneously breaks time-reversal symmetry in the infrared, dynamically generating a fermion mass term $m\bar{\psi}_{cf}\psi_{cf}$. The IR theory is then a trivial gapped state with no intrinsic topological order (see Sec. IV B 2 and Appendix B). It is identical to the phase obtained from the free Dirac cone by spontaneously breaking T symmetry and generating a mass $m\bar{\Psi}_e\Psi_e$. This is the most conventional scenario. In fact, the standard (although unproven) expectation is that noncompact QED$_3$ with a small number of fermion flavors N_f and a symmetry group $SU(N_f)$ does generate a fermion mass in the IR. However, as we review in Appendix B, conventional (2+1)-dimensional QED$_3$ with the standard large gauge transformations is only consistent with T symmetry for even N_f. For odd N_f, one must add a Chern-Simons term with a half-odd-integer level k to the massless Dirac theory in order to preserve the standard large gauge transformations (see Appendix B). In this case, the T symmetry is absent so a fermion mass term is allowed by symmetry and is, in fact, generated already in perturbation theory. Now, our $N_f = 1$ surface theory has no Chern-Simons term due to the modification of allowed large gauge transformations. Therefore, it does not fit into the conventional folklore and perhaps can remain gapless in the IR. This brings us to the other two scenarios.

(ii) Weakly coupled QED$_3$ with $N_f = 1$ flows in the infrared to a free Dirac theory of ψ_e with $N_f = 1$. This would be a "strong" version of particle-vortex duality for fermions. Such a strong version of duality is believed to hold for bosons (we review the status of the bosonic duality in Appendix C).

(iii) Weakly coupled QED$_3$ with $N_f = 1$ flows in the infrared to a CFT distinct from a free Dirac theory.

At present, we can make no statement regarding which of these scenarios is realized.

IV. DUAL DESCRIPTIONS OF THE TI SURFACE PHASES

Let us describe how the different phases of the topological insulator surface are realized in the dual fermion description. Let us begin by considering symmetry-preserving surface states, and then discuss those that break symmetry. This discussion has significant overlap with a previous paper by Son [37], with some relabeling of time-reversal and particle-hole symmetries, and the physical context. Also, Ref. [37]

adopts a minimal coupling of dual fermions to gauge field with twice the charge used in this paper.

A. Surface phases preserving all symmetries

1. Fermi liquid and Halperin-Lee-Read state of dual fermions

Consider the situation when we preserve physical symmetries, charge conservation, and time reversal, with the metallic surface state of the topological insulator. Typically, the metallic surface state will be at finite filling, implying the chemical potential is away from the Dirac node. How is this Fermi liquid state described in the dual language? The finite chemical potential on the electrons implies a finite magnetic field on the composite fermions via the equation $(\partial_x a_y - \partial_y a_x) = 4\pi \rho_e$. The dual fermions are at neutrality as a consequence of physical time-reversal symmetry, so they will be at half-filling of the zeroth Landau level. Fermions in a magnetic field which fill half a Landau level can be in a variety of states. One possibility is the "composite Fermi liquid" or Halperin-Lee-Read (HLR) [31] state. Here, however, since the fermions are themselves composite fermions, performing the duality twice leads us back to the original electrons. This is nothing but the original Fermi liquid of electrons.

2. Surface topological order and superconductivity of dual fermions

Another possible way to preserve charge $U(1)$ and T on the TI surface is through surface topological order. Two different symmetric topologically ordered surface states of a TI have been identified. The first is the Pfaffian-antisemion state, with 12 particles, and the other is the T-Pfaffian state, with 6 particles (see Table II). While the Pfaffian-antisemion state can be derived by a vortex condensation argument starting from the single Dirac cone surface state of a topological insulator, the T-Pfaffian cannot be analogously derived (see, however, Ref. [36]). Rather, it is argued to be a consistent surface termination that captures all relevant anomalies of the TI surface. Here, we are able to derive this surface topological order directly from the duality (2). To do so, we simply consider the composite fermions to be in the Higgs phase, where they have paired and condensed $\langle \psi_{cf\uparrow} \psi_{cf\downarrow} \rangle \neq 0$. This gaps out the photon a_μ, and the vortices in this phase trap quantized gauge flux $\int d^2x (\partial_x a_y - \partial_y a_x) = \pi$. This simply means that the unit vortex carries electric charge $Q_{\rm EM} = \frac{1}{4}$ [since from (2) a flux 2π of a corresponds to charge $\frac{1}{2}$]. The statistics of vortices in this superfluid have previously been worked out [18,19], and they precisely correspond to the T-Pfaffian state, with the same transformation properties under time reversal as proposed in Refs. [25,28]! In fact, the T-Pfaffian appears in two varieties, T-Pfaffian$_\pm$ which differ in the transformation properties under time reversal (see Table II). While one of them corresponds to the topological insulator surface, the other differs from it by addition of the $eTmT$ SPT state of neutral bosons. The $eTmT$ phase admits a toric code surface state where both the e and m anyons are Kramers doublets. Previously, the exact correspondence was unknown. Now, pair condensation of composite fermions in the dual theory (2), in fact, gives rise to the T-Pfaffian$_+$ topological order [19]. Thus, the duality allows to resolve the

long-standing T-Pfaffian$_+$/T-Pfaffian$_-$ puzzle. We, however, remind the reader that to establish the duality (2) we had to argue that the bulk phase constructed in Sec. II is continuously connected to the noninteracting TI and does not differ from it by an $eTmT$ state. The details of this argument will be given in Sec. VI and in Ref. [41].

B. Breaking symmetries

Now, we consider surface phases that break symmetry.

1. Surface superfluid and dual surface topological order

When electrons pair and condense to form a surface superfluid, we have noted that surface vortices have the same statistics and transformation properties under T as the CT-Pfaffian topological order. As discussed in Sec. III B, this can simply be interpreted as the topological order of composite fermions ψ_{cf} of the dual surface theory. An additional feature here is the coupling to the gauge field. When all gauge charges are gapped, the photon is free to propagate, which is just the dual description of the Goldstone mode of the electronic superfluid.

A different way to motivate this connection is the following. Consider the metallic surface state of electrons at finite chemical potential. A natural instability of this Fermi liquid is the BCS instability towards pairing. In the dual description, this corresponds to a finite effective magnetic field applied to a half-filled Landau level of ψ_{cf}. Natural consequences are various non-Abelian topological orders such as Pfaffian, anti-Pfaffian, etc. These, however, break particle-hole symmetry, which is just the time-reversal symmetry of the electrons. A topological order that preserves particle-hole symmetry is the CT-Pfaffian, which corresponds to the surface superfluid of electrons.

2. Breaking time-reversal symmetry

Consider breaking time-reversal symmetry on the TI surface while maintaining the chemical potential at the electronic Dirac node. This induces a mass term $\bar{\Psi}_e \Psi_e$ leading to an insulating surface with surface Hall conductance $\sigma_{xy} = \frac{1}{2}\frac{e^2}{h}$. The effective response theory on the surface is given by

$$\mathcal{L} = -\frac{1}{2(4\pi)} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \tag{17}$$

The dual composite fermions also acquire a mass gap due to breaking of T symmetry via the mass term $\bar{\psi}_{cf} \psi_{cf}$. Integrating out the single Dirac cone of ψ_{cf} leads to the following effective action:

$$\mathcal{L} = \frac{1}{2(4\pi)} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2(2\pi)} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda. \tag{18}$$

Integrating out the dynamical gauge field a_μ then produces the same Lagrangian as Eq. (17). Other important aspects of the theory (18), such as statistics of excitations and ground-state degeneracy on a torus, are discussed in Appendix B, and are shown to precisely coincide with those in the trivial insulating T-broken surface phase.

V. PARTON THEORIES WITH INDEX $\nu \neq 1$

We have seen in Sec. II how to obtain a T-invariant electronic insulator with $\theta_{EM} = \pi$ by confining the spin-liquid phase $SL_\times$. This spin-liquid phase was obtained through a parton construction, with partons ψ placed into a noninteracting $\nu = 1$ band structure of class AIII. As we already mentioned, noninteracting electron phases in class AIII have an integer classification $\nu \in \mathbb{Z}$, which collapses to a $\mathbb{Z}_8$ group upon adding interactions. We now ask what happens if we place our partons into a noninteracting band structure with $\nu \neq 1$ and ν odd?

One can quickly see that the properties of the electronic $\nu = 1$ phase that we used in our bulk construction in Sec. II are shared by all phases with ν odd. In particular, they all have a response to a $u(1)$ gauge field characterized by $\theta = \pi$. The quantum numbers of dyons in the bosonic spin liquids $SL_\times^\nu$ based on a parton band structure with ν odd will, therefore, be identical. Consequently, the associated confined phases obtained by condensing the D dyon $\Psi_e \times (0,2)$ will all have a $\theta_{EM} = \pi$ response to the $U(1)$ gauge field A_μ. The resulting surface theories are, however, different, consisting of $N_f = |\nu|$ flavors of gapless Dirac fermions interacting with a $u(1)$ gauge field a_μ.

We now discuss whether the bulk $SL_\times^\nu$ phases with different ν and the associated confined phases are, in fact, different. First of all, due to the collapse of the noninteracting classification, the bulk phase only depends on ν mod 8. Furthermore, we claim that once the $u(1)$ symmetry of class AIII is gauged, $\nu = 1$ and $\nu = -1 \sim 7$ phases are identical. Recall that when the $u(1)$ symmetry is a global symmetry of an electronic theory, $\nu = 1$ and $\nu = -1$ phases differ in the action of the T symmetry on a single monopole ($q = \frac{1}{2}$, $m = 1$). Under T, $(\frac{1}{2}, 1) \leftrightarrow (-\frac{1}{2}, 1)$. In a theory of electrons, these two time-reversal partners differ by a local object, the electron $(1,0)$, and one can, therefore, assign a T^2 value to them. In the $\nu = 1$ state, $(\frac{1}{2}, 1)$ has $T^2 = +i$ and $(-\frac{1}{2}, 1)$ has $T^2 = -i$ (with the choice $T^2 = -1$ when acting on the electron). In the $\nu = -1$ state the T^2 assignments of these monopoles are reversed [19]. However, once we treat the $u(1)$ symmetry as a gauge symmetry, the parton $\psi = (1,0)$ is no longer a local object. Therefore, the dyons $(\frac{1}{2}, 1)$ and $(-\frac{1}{2}, 1)$ belong to different topological sectors and one cannot assign a value of T^2 to them. Similarly, $\nu = 3$ and $\nu = -3 \sim 5$ phases collapse after gauging. [For a more formal proof that phases ν and $-\nu$ coincide after gauging the $u(1)$ symmetry, see Ref. [41].]

It remains to see whether $SL_\times^{\nu=1}$ and $SL_\times^{\nu=5}$ bulk phases are distinct. In fact, they are. Recall that before the $u(1)$ symmetry is gauged, $\nu = 1$ and 5 phases differ by the $eTmT$ SPT phase of neutral bosons [18,19]. But, neutral bosons are not affected by the gauging of $u(1)$ symmetry. Therefore, $SL_\times^{\nu=1}$ and $SL_\times^{\nu=5}$ spin-liquid phases also differ by an $eTmT$ phase of neutral bosons. Likewise, the associated confined phases also only differ by an $eTmT$ phase. At the level of confined phases, this can also be seen by considering the symmetric topological surface states in the $\nu = 1$ and 5 constructions. As we discussed in Sec. IV, these states can be obtained by pair condensing the composite fermions ψ_{cf} in the surface QED$_3$ theory. In the case of both $\nu = 1$ and 5, the resulting intrinsic topological order is given by the T-Pfaffian, however,

$\nu = 1$ and 5 differ in the action of time-reversal symmetry on the anyons of T-Pfaffian. In the $\nu = 1$ case, the charge $e/4$ anyon σ_1 is a Kramers singlet and the associated state is known as T-Pfaffian$_+$. In the $\nu = 5$ case, the charge $e/4$ anyon is a Kramers doublet and the associated state is known as T-Pfaffian$_-$ (see Table II). These two T-Pfaffian states are known to differ precisely by the $eTmT$ surface topological order (i.e., T-Pfaffian$_+$ + $eTmT$ can be driven via a surface phase transition to T-Pfaffian).

As we already mentioned, since the confined phases have $\theta_{EM} = \pi$ they differ from a noninteracting TI at most by an SPT phase of neutral bosons with T invariance [15]. Such T-invariant boson SPT phases have a $\mathbb{Z}_2^2$ classification [9,14,24,47]. The two $\mathbb{Z}_2$ root phases are best understood via their symmetric topologically ordered surface states. One of the root phases admits the aforementioned $eTmT$ surface topological order. The other root phase admits a surface topological order with anyons $\{1, f_1, f_2, f_3\}$, where f_1, f_2, f_3 are fermions and the fusion rules are the same as in a toric code. This phase (and the above topological order) is abbreviated as fff. Thus, our confined phases are identical to a noninteracting electron TI up to these bosonic SPT phases. In fact, one can rule out the scenario where the confined phases differ from the ordinary TI by the fff state (or $fff + eTmT$). Indeed, if one strongly breaks the T symmetry on the surface of an fff state, one drives the surface into a topologically trivial phase with thermal Hall response $\kappa_{xy}/T = 4$ and electric Hall response $\sigma_{xy}/T = 0$. However, the trivial T-broken surface phase of an ordinary TI has $\sigma_{xy} = \kappa_{xy}/T = \frac{1}{2}$. Similarly, if we break T strongly starting from the T-Pfaffian$_\pm$ surface states of $\nu = 1$, $\nu = 5$ confined phases, we obtain $\sigma_{xy} = \kappa_{xy}/T = \frac{1}{2}$. Strictly 2D phases of fermions with no intrinsic topological order always have $\sigma_{xy} - \kappa_{xy}/T \equiv 0$ (mod 8). Therefore, our $\nu = 1$ and 5 confined phases differ from the noninteracting TI at most by the $eTmT$ phase. Since $\nu = 1$ and 5 themselves differ by the $eTmT$ phase, we conclude that one of them is continuously connected to the TI. By strengthening the arguments presented in Sec. VI, one can show that it is actually the $\nu = 1$ phase (surface topological order T-Pfaffian$_+$), which corresponds to the ordinary TI [41].

From the above discussion, we obtain a family of novel surface theories for the ordinary TI. Since all ν's of the form $\nu = 8k \pm 1$ give rise to the same bulk phase, QED$_3$ with $N_f = 8k \pm 1$ flavors provides a description of the TI surface. Let us start with the weakly coupled QED$_3$ in the UV and ask about its fate in the IR. If one considers the $SU(N_f)$ invariant situation, in the limit $N_f \to \infty$ the IR theory is a CFT. This CFT is under complete theoretical control and one can systematically compute scaling dimensions of operators in powers of $1/N_f$. For instance, the T-odd "mass" operator $\bar{\psi}_{cf}\psi_{cf}$ has scaling dimension $\Delta_{\bar{\psi}\psi} \approx 2 + \frac{128}{3\pi^2 N_f}$ [48]. The flux 4π instanton operator corresponding to the physical electron Ψ_e has scaling dimension $\Delta_{\Psi_e} \approx 0.673 N_f$ [46]. Clearly, the CFTs with large N_f are distinct from a free Dirac cone. Furthermore, these CFTs are stable to $SU(N_f)$-breaking velocity anisotropy terms [48,49] and, therefore, give rise to a family of stable novel surface phases of a TI. The strong version of particle-vortex duality discussed in Sec. III C would require that when $N_f = 1$ the IR CFT becomes a free Dirac cone.

VI. BULK DUALITY

In Sec. II, we have constructed a $(3+1)$D SPT phase of electrons with symmetry $U(1) \rtimes T$ and electromagnetic response with $\theta_{EM} = \pi$. As we already mentioned, by general arguments of Ref. [15] this phase can differ from the noninteracting TI at most by a bosonic SPT phase with T symmetry. We now give a different argument for this. In the process, we will demonstrate that the particle-vortex duality of $(2+1)$-dimensional Dirac fermions can be understood as a descendant of electromagnetic duality of the $(3+1)$-dimensional $u(1)$ gauge theory.

Our construction in Sec. II started with a T-symmetric spin-liquid phase of neutral bosons $SL_\times^{\nu=1}$. This phase was obtained by using the parton decomposition (5), assigning partons T transformations (7) resulting in an overall symmetry group $u(1) \times T$, and then placing the partons into a $\nu = 1$ band structure of class AIII. Now, consider a (seemingly) different T-symmetric spin-liquid phase of neutral bosons obtained through the decomposition

$$B = \tilde{\psi}^\dagger \tilde{\Gamma} \tilde{\psi} \tag{19}$$

with $\tilde{\psi}$ a fermionic parton. The decomposition again has a $\tilde{u}(1)$ gauge symmetry

$$\tilde{u}(1): \tilde{\psi}(x) \rightarrow e^{i\alpha(x)} \tilde{\psi}(x), \tag{20}$$

which will give rise to an emergent gauge field $\tilde{a}_\mu$. (We use the tilde superscript to distinguish the present construction from the one in Sec. II.) We assign the parton $\tilde{\psi}$ the following transformation properties under T:

$$T: \tilde{\psi} \rightarrow \tilde{U}_T \tilde{\psi} \tag{21}$$

with $\tilde{U}_T \tilde{U}_T^* = -1$, so that $T^2 \tilde{\psi}(T^\dagger)^2 = -\tilde{\psi}$. Since now T does not change the $\tilde{u}(1)$ charge of $\tilde{\psi}$, $\tilde{\psi}$ is a true Kramers doublet. The time-reversal symmetry and the gauge symmetry now do not commute: if $\tilde{u}_\alpha$ is a gauge rotation by a phase α, $T\tilde{u}_\alpha T^\dagger = \tilde{u}_{-\alpha}$, so the overall symmetry group is $\tilde{u}(1) \rtimes T$. This symmetry group is the same as for familiar topological insulators in class AII. To complete the construction of the spin-liquid phase, we place the partons $\tilde{\psi}$ into a noninteracting TI band structure. We label the resulting spin liquid $SL_\rtimes$.

We will now argue that the two states $SL_\rtimes$ and $SL_\times^{\nu=1}$, in fact, belong to the same phase. Let us first discuss the excitations of the $SL_\rtimes$ state. Integrating the partons out, we obtain an effective action for $\tilde{a}_\mu$,

$$S[\tilde{a}_\mu] = \int d^3x \, dt \left(-\frac{1}{4\tilde{e}^2} \tilde{f}_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\lambda\sigma} \tilde{f}_{\mu\nu} \tilde{f}_{\lambda\sigma} \right) \tag{22}$$

with $\theta = \pi$ and $\tilde{f}_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu$. There is again a topological term in the effective action with $\theta = \pi$. Thus, the spectrum of dyon excitations can again be labeled by electric and magnetic charges $(\tilde{q}, \tilde{m})$ with $\tilde{m}$ integers and $\tilde{q} - \tilde{m}/2$ integers. As in the $SL_\times$ phase, the self-statistics of dyons is $(-1)^{(\tilde{q} - \tilde{m}/2)(\tilde{m}+1)}$. Two dyons $(\tilde{q}, \tilde{m})$ and $(\tilde{q}', \tilde{m}')$ experience the usual statistical interaction, with a statistical phase $\exp[i(\tilde{q}\tilde{m}' - \tilde{q}'\tilde{m})\Omega/2]$, as well as a $1/r$ Coulomb interaction.

Under time reversal, $T: (\tilde{q}, \tilde{m}) \rightarrow (\tilde{q}, -\tilde{m})$. Only excitations whose topological sector is not modified by T can be assigned a Kramers parity. In the present case, these are the

pure-charge excitations $(q, 0)$. The single parton $\tilde{\psi} = (1, 0)$ is (by construction) a Kramers doublet.

In order to compare the two spin-liquid phases $SL_\times^{\nu=1}$ and $SL_\rtimes$ it is convenient to choose the following basis for the lattice of dyon excitations. Starting with the $SL_\times$ case, let us choose as a basis the two dyons: $d_+ = (\frac{1}{2}, 1)$ and $d_- = (-\frac{1}{2}, 1)$. These dyons are both bosons, and have a nontrivial mutual statistical interaction: d_+ sees d_- as a charge $(1, 0)$ would see a monopole $(0, 1)$ at $\theta = 0$. Under $T: d_+ \leftrightarrow d_-$. These time-reversal partners fuse to a double monopole $(0, 2)$, which is a Kramers doublet fermion as is required by the presence of a nontrivial statistical interaction between them. Decomposing a general dyon as $\mathcal{D} = d_+^{n_+} d_-^{n_-}$, two dyons with quantum numbers (n_+, n_-) and (n'_+, n'_-) have a static interaction

$$E = \frac{1}{4\pi r} \left(e^2 qq' + \frac{(2\pi)^2}{e^2} mm' \right)$$

$$= \frac{1}{4\pi r} \left(\frac{e^2}{4} (n_+ - n_-)(n'_+ - n'_-) \right.$$

$$\left. + \frac{(2\pi)^2}{e^2} (n_+ + n_-)(n'_+ + n'_-) \right). \tag{23}$$

In the $SL_\rtimes$ case, let us choose a different dyon basis: $\tilde{d}_+ = (\frac{1}{2}, -1)$ and $\tilde{d}_- = (\frac{1}{2}, 1)$. Again, these dyons are both bosons and have a nontrivial mutual statistical interaction: $\tilde{d}_+$ sees $\tilde{d}_-$ as a charge sees a monopole at $\theta = 0$. Furthermore, under $T: \tilde{d}_+ \leftrightarrow \tilde{d}_-$ and these two time-reversal partners fuse to a single charge $(1, 0)$, which is a Kramers doublet fermion. Decomposing a general dyon $\tilde{\mathcal{D}} = \tilde{d}_+^{\tilde{n}_+} \tilde{d}_-^{\tilde{n}_-}$, two dyons with quantum numbers $(\tilde{n}_+, \tilde{n}_-)$ and $(\tilde{n}'_+, \tilde{n}'_-)$ have a static interaction

$$E = \frac{1}{4\pi r} \left(\tilde{e}^2 \tilde{q}\tilde{q}' + \frac{(2\pi)^2}{\tilde{e}^2} \tilde{m}\tilde{m}' \right)$$

$$= \frac{1}{4\pi r} \left(\frac{\tilde{e}^2}{4} (\tilde{n}_+ + \tilde{n}_-)(\tilde{n}'_+ + \tilde{n}'_-) \right.$$

$$\left. + \frac{(2\pi)^2}{\tilde{e}^2} (\tilde{n}_+ - \tilde{n}_-)(\tilde{n}'_+ - \tilde{n}'_-) \right). \tag{24}$$

We see that the properties of excitations in the $u(1) \times T$ case and in the $\tilde{u}(1) \rtimes T$ case are the same if we identify $d_+ \sim \tilde{d}_+$, $d_- \sim \tilde{d}_-$, and $e = \frac{4\pi}{\tilde{e}}$. In fact, this duality is just an element of the general $SL(2, \mathbb{Z})$ duality of the $u(1)$ gauge theory [43,44,50,51]. The only nontrivial fact is that this element of the duality preserves the time-reversal symmetry.

Based on the above discussion, one is tempted to conclude that the two spin liquids $SL_\times^{\nu=1}$ and $SL_\rtimes$, in fact, belong to the same phase. One caveat is that, in principle, these two phases could differ by a bosonic SPT phase with T symmetry (i.e., $eTmT$ phase or fff phase). Indeed, an "addition" of such an SPT phase will not alter the properties of the excitations charged under the gauge symmetry. In fact, as discussed in Sec. V, the $SL_\times^{\nu=5}$ phase based on a $\nu = 5$ band structure of partons ψ differs from the $SL_\times^{\nu=1}$ phase precisely by an $eTmT$ phase. *A priori*, it is not clear if $SL_\rtimes$ is dual to $SL_\times^{\nu=1}$ or $SL_\times^{\nu=5}$. In Ref. [41], we argue that the duality is, in fact, between $SL_\rtimes$ and $SL_\times^{\nu=1}$. We will briefly summarize the strategy for showing this in Sec. VI A.

Having established the duality between two spin-liquid phases, we proceed to confine these phases and obtain a duality between SPT phases of electrons. To do so, imagine adding a trivial band insulator of electrons to each of the spin-liquid phases. In the $SL_\rtimes$ construction, condense the bound state of the physical electron Ψ_e and the single charge $(1,0) = \tilde{d}_+\tilde{d}_-$, i.e., the single fermionic parton $\tilde{\psi}$. This bound state is a Kramers singlet boson, so its condensation does not break the T symmetry. The effect of the condensation is to Higgs the dynamical gauge field, effectively ungauging the TI. Indeed, once $\Psi_e\tilde{\psi}$ is condensed, the parton $\tilde{\psi}$ and the electron $\Psi_e^\dagger$ become identified, so the resulting phase is continuously connected to a noninteracting TI. Now, in the dual $SL_\times^{\nu=1}$ description, the single charge $(1,0) = \tilde{\psi} = \tilde{d}_+\tilde{d}_-$ corresponds to the double monopole $(0,2) = d_+d_-$. Hence, condensing $\Psi_e\tilde{\psi}$ in the $SL_\rtimes$ construction is equivalent to condensing the dyon $D = \Psi_e \times (0,2)$ in the $SL_\times^{\nu=1}$ construction, which is precisely the confinement transition discussed in Sec. II. We, therefore, conclude that the state obtained by confining the $SL_\times^{\nu=1}$ spin liquid is continuously connected to a noninteracting TI.

A. Fixing the *eTmT* factor in the duality

We would like to show that $SL_\rtimes$ and $SL_\times^{\nu=1}$ spin liquids are identical as T-symmetric bosonic phases; in particular, that they do not differ by either the *eTmT* or the *fff* phase.

Typically, to detect an SPT phase with a unitary symmetry G using bulk probes only we must "weakly gauge" G, effectively studying the response of the bulk SPT to fluxes of G. In the case when the symmetry G is the time-reversal symmetry it has been suggested that the equivalent of "weakly gauging" the symmetry is placing the system on a nonorientable manifold [14,47,52]. For instance, the partition function of the *eTmT* phase on the nonorientable manifold RP^4 is equal to -1 [14,47]. Thus, we can detect whether two phases differ by the *eTmT* phase by comparing their partition functions on RP^4. Similarly, the partition function of the *fff* phase on an arbitrary oriented manifold is given by $(-1)^{\sigma(M)}$, where $\sigma(M)$ is the signature of the manifold M [24,47,53]. Thus, we can detect whether two phases differ by the *fff* phase by comparing their partition functions on CP^2, which has signature $\sigma(CP^2) = 1$. In Ref. [41], we show that the partition functions of $SL_\rtimes$ and $SL_\times^{\nu=1}$ spin liquids are equal on both RP^4 and CP^2, provided that the coupling constants of the two gauge theories are related by $e = \frac{4\pi}{\tilde{e}}$. This supports the proclaimed duality.

VII. DUALITY, DIRAC COMPOSITE FERMIONS, AND THE HALF-FILLED LANDAU LEVEL

Thus far we have focused on the dual description of the topological insulator surface, which is a gauged chiral topological insulator surface. Indeed, this was appropriate to study the strongly interacting surface phases of the topological insulator. However, given the bulk duality in Sec. VI, one can equally well run the connection in reverse, and establish a dual description for the chiral topological insulator surface, which is now given by a gauged TI surface. What is the physical relevance of this connection? Although chiral TIs may not be naturally realized in electronic systems, it turns out that the half-filled Landau level of a two-dimensional electron gas (2DEG) can effectively be considered equivalent to the surface of a chiral topological insulator (with $\nu = 1$). This can be readily seen by studying the particle-hole symmetry of the half-filled Landau level and relating it to the particle-hole symmetry of chiral TI surface.

Particle-hole symmetry of the half-filled Landau level. Consider a purely 2D nonrelativistic electron in a uniform magnetic field: $H_L = (\vec{p} - \vec{A})^2/2m$, where a Landau level spectrum results. Consider electrons partially filling the lowest Landau level (LLL) at a fractional filling $0 \leqslant f \leqslant 1$, and define a Hilbert space where all higher Landau levels are projected out (formally $m \to 0$). The resulting problem has a particle-hole symmetry, where one can either consider electrons or holes partially filling the Landau level. Formally, this is a combination of charge conjugation $\mathcal{C}$ and time reversal $\mathcal{T}$, where the first transformation exchanges electron creation and annihilation operators $c_r \leftrightarrow c_r^\dagger$, while the second antiunitary operator involves complex conjugation. If we choose a basis $\phi_i(r)$ for the orbitals in the LLL, then the operators that create, $c_i^\dagger$, and destroy, c_i, LLL states are exchanged:

$$c_i = \int_r \phi_i^*(r)c_r \xrightarrow{\mathcal{C}} \int_r \phi_i^*(r)c_r^\dagger \xrightarrow{\mathcal{T}} \int_r \phi_i(r)c_r^\dagger = c_i^\dagger. \quad (25)$$

Since the kinetic energy is quenched, the Hamiltonian only contains interaction terms. If we restrict to two-body interactions, after projection to the LLL the system is invariant under particle-hole symmetry. Indeed, if $\delta\rho(r)$ represents the density deviation in the LL from half-filling $[\delta\rho(r) = \rho(r) - \frac{1}{2}\frac{1}{2\pi l_B^2}]$, then under particle hole $\delta\rho(r) \to -\delta\rho(r)$. Two-body interactions $H_{\text{int}} = \int_{r_1,r_2} V(r_1,r_2)\delta\rho(r_1)\delta\rho(r_2)$ are therefore invariant. Furthermore, the filling $f \to 1 - f$, so particle-hole symmetry applies to the half-filled Landau level $f = \frac{1}{2}$. Therefore, H_{int}, when projected into the LLL at half-filling, admits a particle-hole symmetry. Furthermore, we will see that this coincides with the particle-hole symmetry of the surface state of the $\nu = 1$ chiral TI.

Connection to $\nu = 1$ chiral TI. The surface state of $\nu = 1$ cTI is usually taken to be a single Dirac cone Ψ_e [Eq. (1)] with particle-hole symmetry acting as

$$\mathcal{CT}: \Psi_e \to \epsilon\Psi_e^\dagger, \quad i \to -i. \quad (26)$$

In our previous discussion of the cTI, we referred to this antiunitary symmetry simply as time reversal, however, the term particle hole is more appropriate in the present context. We observe that an orbital magnetic field can be applied in Eq. (1), preserving all the symmetries of the cTI. This leads to Dirac Landau levels $E_n = \text{sgn}(n)\sqrt{2|Bn|}, n = 0,\pm1,\pm2,\ldots$, with the positive energy levels empty and the negative energy levels filled. Additionally, the zeroth Landau level is automatically half-filled due to particle-hole symmetry. Here, even without projection into the zeroth Landau level, particle-hole symmetry is present. Moreover, the problem is essentially identical to the nonrelativistic 2DEG at $f = \frac{1}{2}$ if in addition we impose projection into the zeroth Landau level, with the particle-hole symmetry of the Dirac fermion (26) mapping to the particle-hole symmetry of the LLL (25).

There are, however, two differences between the surface and purely 2D cases that are important to note. First, usually particle-hole symmetry ($\mathcal{CT}$) forbids a Hall conductance, and, indeed, cTI surface states with opposite values of σ_{xy} are related by this symmetry, i.e., $\sigma_{xy} \xrightarrow{\mathcal{CT}} -\sigma_{xy}$. However, in the 2D LLL context we have $\sigma_{xy} \xrightarrow{\mathcal{CT}} 1 - \sigma_{xy}$, that is, a particle-hole symmetric state has $\sigma_{xy} = \frac{1}{2}$ (in units of e^2/h). Therefore, for the 2D LLL problem, we must add to the Lagrangian that is manifestly particle-hole symmetric an additional piece that keeps track of this fact:

$$\mathcal{L}[A] = \mathcal{L}_{ph}[A] + \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda. \qquad (27)$$

One way to view this is to imagine the opposite surfaces of a slab of $\nu = 1$ cTI, the top surface provides the first term, and is manifestly particle-hole symmetric, while the bottom surface has a broken $\mathcal{CT}$ symmetry, leading to a gap for the Dirac node which yields the Hall conductance of $\frac{1}{2}$ [second term in Eq. (27)]. Importantly, $\mathcal{CT}$-symmetry breaking on the bottom surface does not "contaminate" the perfect $\mathcal{CT}$ symmetry of the top surface.

It may seem surprising that we can simulate the surface of a 3D topological phase, purely in two dimensions, while maintaining the relevant symmetries. The key observation here is that the particle-hole symmetry when realized in 2D is a nonlocal symmetry, and is not implemented as a site local symmetry as in the 3D chiral TI. Indeed, it is only present in the artificial limit of LLL projection; in any physical 2D system it is necessarily broken by inter-Landau-level mixing. The nonlocal nature of the symmetry can be divined from the following facts: particle-hole symmetry proceeds by exchanging filled and empty LLL states. However, these are necessarily not well-localized states [54], hence, the symmetry cannot be implemented in a local way. Furthermore, unlike any other local symmetry, particle hole of the LLL interchanges $\sigma_{xy} = 0$ and 1 states that differ in their topological properties. Obviously, these states have different edge degrees of freedom, but due to the nonlocal nature of LLL particle-hole symmetry it is necessarily broken in samples with boundaries. In contrast, local symmetries can be defined in samples of any geometry. Finally, we note that local symmetries allow for a symmetric "vacuum" state which is a product state; one simply makes singlets from a finite set of the existing degrees of freedom. However, there is no such symmetric vacuum for LLL particle hole since it will necessarily involve a Hall conductance $\sigma_{xy} = \frac{1}{2}$, for which no short-range entangled state exists.

Duality. Let us now discuss the consequences of the previously discussed duality for the LLL problem with particle-hole symmetry. First, we have noted the relation to the surface of the $\nu = 1$ cTI. Next, utilizing the duality in Sec. VI, which relates the gauged $\nu = 1$ cTI with the gauged ordinary TI, we can consider obtaining the ungauged version by a Higgs condensation process, entirely analogous to our duality in Sec. II. However, in this case we assume we have a trivial band of "electrons" Ψ_e that transform under $\mathcal{CT} \times U(1)$ (rather than our previous discussion where they were Kramers doublets under time reversal). Binding electrons to the electric charges of the gauged cTI and condensing removes the dynamical gauge field and yields a chiral TI. However, on the other side of

the duality we have confined the gauged topological insulator by condensing a bound state of a double monopole and electron Ψ_e. The resulting bulk phase is again the chiral TI, except it inherits a different surface state: the topological insulator surface but coupled to a (2+1)-dimensional dynamical gauge field.

Thus, we have

$$\mathcal{L}_e = \bar{\Psi}_e i\gamma^\mu[\partial_\mu - iA_\mu]\Psi_e \qquad (28)$$

is dual to

$$\mathcal{L}_{cf} = \bar{\psi}_{cf}i\gamma^\mu[\partial_\mu - ia_\mu]\psi_{cf} - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda, \qquad (29)$$

except that now Ψ_e are electrons that respect particle-hole symmetry (26), which translates into time reversal when acting on the composite fermions ψ_{cf}:

$$\mathcal{CT}: \psi_{cf} \to \epsilon\psi_{cf}, \quad i \to -i. \qquad (30)$$

As already discussed, a finite background magnetic field B in Eq. (28) preserves the $\mathcal{CT}$ symmetry and leads to a half-filled Landau level. On the dual side, this implies that the dual fermion ψ_{cf} is at a finite filling set by the relation $\psi_{cf}^\dagger\psi_{cf} = B/4\pi$. In other words, electrons in the half-filled Landau level can be mapped to a finite density of composite fermions in zero magnetic field. Indeed, this was the content of the Halperin-Lee-Read (HLR) theory of the half-filled Landau level [31]. However, the additional insight that has been gained here is the role of particle-hole symmetry, that has remained mysterious within the HLR theory. Here, we see that particle hole is simply implemented as *time-reversal* symmetry (30) on the composite fermions. Moreover, to account for the anomaly of the half-filled Landau level, the composite fermions must be Dirac like, that is, they are like the surface of a topological insulator at finite density. Their defining physical characteristic is a Berry phase of π on circulating once around the composite fermion Fermi surface. Thus, putting together all the pieces, we obtain for the effective dual theory of the half-filled Landau level in an external field B and coupled to a vector potential $A_{\text{ext}} = A_B + \delta A$, where the first term yields the background field, we have

$$\mathcal{L}_{cf} = \bar{\psi}_{cf}i\gamma^\mu[\partial_\mu - ia_\mu]\psi_{cf} - \frac{B}{4\pi}a_0$$

$$- \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\delta A_\mu\partial_\nu a_\lambda + \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda, \qquad (31)$$

where we have included the offset in the Hall conductance as described in Eq. (27).

Precisely such a Dirac composite fermion picture was conjectured by Son in a prescient recent paper [37], where he showed the theory (31) consistently accounted for several physical facts while maintaining particle-hole symmetry. Indeed, our discussion here can be viewed as providing a derivation of the same result, and is another consequence of the duality in Sec. VI.

CT-Pfaffian: A particle-hole symmetric quantized Hall state at half-filling. Duality is most useful in accessing a strongly interacting state of the original variables, which has a simple description in dual vortex fields. Indeed, just such a situation occurs here: we can describe an incompressible plateau state with $\nu = \frac{1}{2}$ by Cooper pairing of composite fermions.

In particular, since composite fermions form the surface of a topological insulator, we can consider Cooper pairing of composite fermions with opposite momentum and spin, $\langle \psi_{cf,\uparrow}(\vec{p})\psi_{cf,\downarrow}(-\vec{p})\rangle \sim \Delta$, as envisaged by Fu and Kane [55]. Since the Higgs phenomenon gaps out the gauge field, and pairing gaps out fermions, this is a gapped quantum Hall phase. Moreover, this pairing respects time-reversal symmetry (30), which in the original electron variables implies that it is particle-hole symmetric. Combining these observations, we readily see $\sigma_{xy} = \frac{1}{2}$. The excitations in this phase correspond to vortices and quasiparticles of the Fu-Kane superconductor. The unit vortex ($hc/2e$) corresponds, by the conversion between a flux and global charge, to a $e/4$ excitation. Moreover, by the Fu-Kane analysis it traps a Majorana mode. Indeed, a more detailed analysis [26,27] reveals that this is just the CT-Pfaffian state discussed previously, which has the same topological order as the T-Pfaffian in Table II, but differs in the symmetry action under $\mathcal{CT}$, which reverses the electric charge. Indeed, exactly this topological order was proposed to describe the surface of $\nu = 1$ chiral TI in Ref. [17] based on satisfying various physical requirements. However, here we have obtained a derivation of this surface topological order, which was missing until the present time. Indeed, the $\mathcal{CT}$-Pfaffian may be viewed as a candidate for an incompressible state with denominator 2 (such as $\frac{5}{2}$ state). However, numerical simulations seem to show that with Coulomb interactions in the $n = 1$ LL, the $\frac{5}{2}$ state spontaneously breaks particle-hole symmetry, leading to the Read-Moore Pfaffian state (and its degenerate particle-hole conjugate, the anti-Pfaffian) [56–60]. These correspond to pairing of composite fermions in $l = \pm 2$ angular momentum channel, $\langle \psi_{cf,\uparrow}(\vec{p})\psi_{cf,\downarrow}(-\vec{p})\rangle \sim \Delta(p_x \pm ip_y)^{|l|}$, that breaks time-reversal symmetry in the dual description [37]. Nevertheless, it is interesting whether the CT-Pfaffian may be realized in the more broad phase diagram of two-dimensional electron systems in a magnetic field.

VIII. CONCLUSIONS AND FUTURE DIRECTIONS

In summary, we have derived a description of the surface of an electronic topological insulator, given by QED$_3$, with a single gapless two-component Dirac fermion. We argued that these fermions are related to $2hc/e$ vortices of the electron fluid. QED$_3$ represents a dual description of the surface Dirac electrons at the level of Hilbert spaces, operators, and symmetries. The dual description allows us to derive well-known surface phases of the TI, and also to derive a previously proposed surface topological order, the T-Pfaffian.

An interesting question for future research is whether a "strong" version of this particle-vortex duality holds, i.e., whether a dynamical equivalence exists between $N_f = 1$ QED$_3$ and $N_f = 1$ free Dirac fermion, with N_f denoting the number of fermion flavors of the two component fermion fields. Although the conventional folklore holds that QED$_3$ is unstable at small values of N_f, we note that our $N_f = 1$ surface theory has no Chern-Simons term. Therefore, it does not fit into the conventional folklore and perhaps can remain gapless in the IR. If so, information about this conformal field theory may be available from the conformal bootstrap [61]. Much work on particle-vortex dualities (or mirror symmetry) has been on supersymmetric (SUSY) theories, which may

appear to be irrelevant to the present discussion. However, in Refs. [62,63] it was argued that the critical point between the Dirac surface state of a TI and a surface superconductor is described by a Wess-Zumino model with emergent $\mathcal{N} = 2$ SUSY. Thus far, a dual theory of this precise model has not appeared, although closely related models have successfully been dualized [64,65].

Note added. Recently, Wang and Senthil [66] pointed out the duality between $u(1)$ gauge theories discussed in Sec. VI. The same authors have also communicated to us a forthcoming preprint with independently derived results on the dual surface theory which agree with our work [67]. Another related work which appeared recently is Ref. [68]; it discusses a dual gauge theory for a half-filled Landau level with two fermions flavors. Finally, Ref. [69] presents an explicit derivation of dynamical duality between a single free Dirac cone and $N_f = 1$ QED$_3$. This derivation gives a direct map between the operators in the two theories. An important point is that the dual QED$_3$ theory of Ref. [69] is in the strong coupling limit [$g^2 \sim \Lambda_{\mathrm{UV}}$ in Eq. (16)]. Thus, Ref. [69] does not, strictly speaking, establish the strong form of the duality, as defined in Sec. III C, between the free Dirac theory and *weakly* coupled QED$_3$. Nevertheless, it makes such a strong form of duality not implausible.

APPENDIX A: BAND STRUCTURE OF CLASS AIII CHIRAL TOPOLOGICAL INSULATOR

Here, we write an explicit band structure for fermions in class AIII chiral topological insulator phase. Consider a cubic lattice model with four orbitals per site labeled by $\tau_z = \pm 1$ and $\nu_z = \pm 1$. Consider the one-particle Bloch Hamiltonian

$$H_0 = t[\sin k_x \alpha_x + \sin k_y \alpha_y + \sin k_z \alpha_z]$$
$$+ m[\lambda - (\cos k_x + \cos k_y + \cos k_z)]\beta_5.$$

We have $x_i x_j + x_j x_i = 2\delta_{ij}$ where $x_i \in (\alpha_x, \alpha_y, \alpha_z, \beta_0, \beta_5)$. An explicit representation is $(\alpha_x, \alpha_y, \alpha_z, \beta_0, \beta_5) = (\tau_x, \tau_z\nu_x, \tau_z\nu_z, \tau_y, \tau_z\nu_y)$. This Hamiltonian has a chiral symmetry $\beta_0 H_0 \beta_0 = -H_0$. Time-reversal symmetry in the second quantized representation takes the form $\psi \rightarrow \beta_0 \psi^\dagger$ (and, being an antiunitary symmetry $i \rightarrow -i$). In contrast to regular time-reversal symmetry, particles are taken to holes, so the conserved $U(1)$ is like spin rather than charge.

For $\lambda > 3$, the model is in a trivial phase. However, for $1 < \lambda < 3$ the sign of the mass term changes sign at the origin in momentum space indicating this is a $\nu = 1$ topological phase in the AIII class. In order to access SL$_\times$, the gauged chiral

topological insulator, we require the fermionic partons to take up a band structure with the same topology.

APPENDIX B: COMPACTNESS OF THE GAUGE FIELD IN THE SURFACE THEORY

It is often stated that a single Dirac fermion in $2 + 1$ dimensions suffers from the parity anomaly, namely, it cannot be consistently coupled to a $U(1)$ gauge field preserving the time-reversal symmetry [29]. When the Dirac fermion appears as the surface state of a $(3 + 1)$-dimensional insulator [either the ordinary TI in class AII with symmetry $U(1) \rtimes T$ or class AIII with symmetry $U(1) \times T$], this anomaly has a well-known resolution: when one gauges the $U(1)$ symmetry, the $U(1)$ gauge field lives in the $(3 + 1)$-dimensional bulk, and the $\theta = \pi$ bulk EM response "cancels" the anomaly of the surface. Now, our surface theory (15) has a dynamical $u(1)$ gauge field a_μ, which is confined to live just on the surface (we switch off the background electromagnetic field A_μ for now). Thus, the standard argument for the evasion of the anomaly via the $(3 + 1)$-dimensional bulk does not directly apply in this case. Rather, the anomaly is resolved by modifying the compactification of the gauge field a_μ in the surface theory. For instance, we already saw that only configurations of a_μ with magnetic flux $2\pi m$ with m even are allowed on the surface. There is a related restriction on the electric fluxes that one can place through the space-time 2-cycles of the surface.

As an example, imagine that our D-condensed phase occupies a solid torus, so that its boundary is a torus T^2, with periodic x and y directions of length L. We will choose the x cycle to wrap the hole of the solid torus, while the y cycle can be contracted within the solid torus. To simplify the discussion, let us break the T symmetry on the surface by adding a mass term $m\bar{\psi}_{cf}\psi_{cf}$ to the surface theory. The low-energy surface action then becomes

$$\mathcal{L} = \frac{ik}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \qquad (B1)$$

with $k = \frac{1}{2}\mathrm{sgn}(m)$ (for definiteness, let us choose $m > 0$ so that $k = \frac{1}{2}$). It is a standard statement that the level k of the $(2 + 1)$-dimensional Chern-Simons theory must be an integer, which seems inconsistent with our finding of $k = \frac{1}{2}$. Let us recall what this statement is based on. Let us integrate out the temporal component of the gauge field a_τ. This enforces the constraint $\partial_x a_y - \partial_y a_x = 0$. Then, $a_i(\vec{x},\tau) = \partial_i\alpha(\vec{x},\tau) + \frac{\theta_i(\tau)}{L}$, so that the only remaining physical degrees of freedom are θ_1 and θ_2 corresponding to the flux of a through the x and y 1-cycles. The effective action then takes the form

$$L = -\frac{ik}{2\pi}\theta_1\partial_\tau\theta_2. \qquad (B2)$$

Now, in a standard $(2 + 1)$-dimensional theory, large gauge transformations $a_1 \to a_1 + \frac{2\pi}{L}$, $a_2 \to a_2 + \frac{2\pi}{L}$ are allowed, corresponding to $\theta_1 \to \theta_1 + 2\pi$, $\theta_2 \to \theta_2 + 2\pi$. In the path-integral treatment, these transformations are implemented by allowing $\theta_{1,2}$ to wind by 2π around the temporal circle: $\theta_i(\beta) = \theta_i(0) + 2\pi n_i$, with n_i integers. This corresponds to placing electric fluxes $2\pi n_i$ through the space-time 2-cycles of the system. Now, imagine there is an electric flux 2π through the τ-y cycle. We see that in this case the action (B1)

changes by $S \to S - 2\pi i k$ as we shift $\theta_1 \to \theta_1 + 2\pi$. Thus, the partition function remains invariant only if k is an integer. In particular, for $k = \frac{1}{2}$ the partition function acquires a phase -1. While we have demonstrated this effect in the T-broken surface theory (B1), it is also present for the T-invariant gapless Dirac fermion [29].

One encounters the same difficulty if one attempts to quantize the theory (B2) in real time. The commutation relation between θ_1 and θ_2 reads as

$$[\theta_1,\theta_2] = -\frac{2\pi i}{k}. \qquad (B3)$$

The operators $U_{1,2}$ which implement the large gauge transformations $\theta_{1,2} \to \theta_{1,2} + 2\pi$ read as $U_1 = e^{-ik\theta_2}$, $U_2 = e^{ik\theta_1}$. Now,

$$U_1 U_2 = e^{2\pi ik} U_2 U_1. \qquad (B4)$$

Thus, the large gauge transformations along the two directions commute only if k is an integer. In our theory with $k = \frac{1}{2}$, U_1 and U_2 anticommute.

The above anomaly is resolved in our surface theory in the following way. While large gauge transformation U_1 shifting $\theta_1 \to \theta_1 + 2\pi$ is allowed, only the transformation U_2^2, shifting $\theta_2 \to \theta_2 + 4\pi$, is permitted. Thus, we are only allowed to place electric flux $2\pi n$ with n even along the y-τ 2-cycle, while a flux $2\pi n$ with arbitrary integer n can be placed along the x-τ 2-cycle. Note that from the bulk point of view, the two cycles are not equivalent: the x cycle is uncontractible in the 3D solid torus, while the y cycle is contractible. Now, U_1 and U_2^2 commute and we can compute the ground-state degeneracy. Working in the θ_1 basis and imposing $U_2^2 = 1$ we must have $\theta_1 = 2\pi l$ with l integer. Since θ_1 is identified modulo 2π, we have a unique physical ground state given by $\theta_1 = 0$. This is consistent with our expectations. Indeed, the T-broken surface state has no intrinsic topological order so it should possess no ground-state degeneracy on a torus. The only excitation is the gapped ψ_{cf}. The Chern-Simons field (B1) attaches flux 4π to ψ_{cf}, which preserves its fermionic statistics. Now, a flux 4π instanton will create ψ_{cf} with an attached flux 4π. Recalling that a flux 4π instanton corresponds to the electron creation operator, in the T-broken phase ψ_{cf} is identified with the electron.

We can directly understand the restriction on the allowed large gauge transformations of the surface theory using our bulk construction. Let us imagine a process where $\theta_2(\tau) = 2\pi\tau/\beta$, i.e., θ_2 winds by 2π along the temporal cycle. This gives rise to an electric field $\vec{e} = (0, \frac{2\pi}{\beta L})$ along the surface. As discussed in the previous section, this electric field will be Meissner screened by a current of D dyons along the surface with surface density $j_i^D = -\frac{1}{4\pi}\epsilon_{ij}e_j = (-\frac{1}{2}\frac{1}{\beta L}, 0)$. Note that the dyon current is along the uncontractible x cycle of the solid torus. The total number of D dyons that have passed through the $x = 0$ cross section of the solid torus in the time $0 < \tau < \beta$ is $N_D = -\frac{1}{2}$. Now, if the system at time $\tau = \beta$ comes back to its initial $\tau = 0$ configuration, then an integer number of D dyons must have passed through the $x = 0$ cross section. Therefore, we conclude that the system has not returned to its initial configuration at $\tau = \beta$. Therefore, $\theta_2 = 0$ and $\theta_2 = 2\pi$ are not identical, rather $\theta_2 \sim \theta_2 + 4\pi$. On the other hand, θ_1 is, indeed, periodic modulo 2π. Indeed, when an electric field

is applied along the x direction, the dyons move along the y cycle. Since this cycle is contractible, the number of dyons that pass through any cross section of the solid torus is now zero. Thus, $\theta_1 \sim \theta_1 + 2\pi$.

We conclude that the surface QED_3 theory differs from the conventional $(2 + 1)$-dimensional $u(1)$ gauge theory in the allowed large gauge transformations. Once the set of large gauge transformations is restricted, QED_3 with a single Dirac cone becomes fully consistent with T symmetry.

APPENDIX C: PARTICLE-VORTEX DUALITY OF BOSONS

In this Appendix, we review the status of particle-vortex duality of bosons in Eqs. (3) and (4). As a field theory, the vortex theory (4) is given by the Abelian-Higgs model

$$\mathcal{L} = |(\partial_\mu - i\alpha_\mu)\varphi|^2 + m^2|\varphi|^2 + u|\varphi|^4 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda)^2 \tag{C1}$$

with couplings $u, e^2 \ll \Lambda_{UV}$. The properties of this theory depend on the dimensionless parameter $\kappa^2 = u/e^2$. For κ smaller than a critical value κ_c, analytical arguments [70] and Monte Carlo (MC) simulations [71–73] indicate that the transition between the Higgs and Coulomb phases is of first order. Thus, the interesting region is $\kappa > \kappa_c$. Here, MC indicates that the transition is second order and in the inverted XY universality class [72,73]. [The term "inverted XY" stems from the dual boson Φ of Eq. (3) having $\langle\Phi\rangle \neq 0$ in the Coulomb phase, on the other hand, "condensation" of the vortex field φ occurs in the Higgs phase.] We, however, note that it is difficult to access the weakly coupled regime $u, e^2 \ll \Lambda_{UV}$ in MC simulations since to reach the true infrared physics one then needs to go to large system size $L \gg 1/u, 1/e^2 \gg \Lambda_{UV}^{-1}$. Nevertheless, an attempt to extrapolate to the $u/\Lambda_{UV}, e^2/\Lambda_{UV} \to 0$ limit was made in MC simulations of Ref. [73], which estimate $\kappa_c = (0.76 \pm 0.04)/\sqrt{2}$.

We remind the reader that an exact duality [33] exists between a certain lattice version of theory (C1), where $|\varphi|$ is frozen and only the phase of φ fluctuates, and a lattice version of the XY model. More precisely,

$$S_{AH} = \frac{1}{2g}\sum_{j\mu}(d\theta - \alpha - 2\pi n)^2_{j\mu} + \frac{1}{4e^2}\sum_{j\mu\nu}(d\alpha)^2_{j\mu\nu} \tag{C2}$$

is exactly dual to

$$S_{XY} = \frac{e^2}{8\pi^2}\sum_{\bar{j}\mu}(d\chi - 2\pi m)^2_{\bar{j}\mu} + \frac{g}{4}\sum_{\bar{j}\mu\nu}(dm)^2_{\bar{j}\mu\nu}. \tag{C3}$$

The variable j ($\bar{j}$) labels direct (dual) sites of a cubic lattice. The vortex field $\varphi \sim e^{i\theta}$, and $\Phi \sim e^{i\chi}$ is the XY field. Integers $n_{j\mu}$ and $m_{\bar{j}\mu}$ encode the periodic nature of θ and χ.
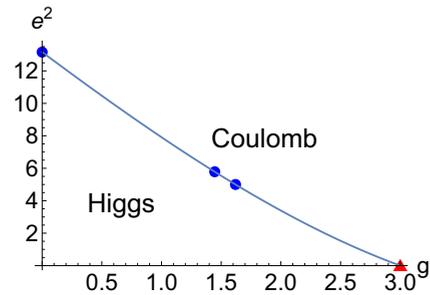

FIG. 2. Phase diagram of lattice Abelian-Higgs model (C2). Points where MC simulations have been performed are marked [33,74], solid line is a guide to the eye. An inverted XY transition has been found at points marked by blue circles, and an ordinary XY transition at the $e^2 = 0$ point marked by a red triangle. The near critical region with $e^2 \ll 1$ has the same IR fixed point as "weakly coupled" Abelian-Higgs model (C1) with $\kappa = u/e^2 \gg 1$.

Equation (C3) is the Villain version of the usual lattice XY model; based on MC, at $g = 0$ it is known to have a continuous transition in the XY universality class at $e_c^2 \approx 13$, which must survive for small g. Therefore, (C2) has an inverted XY transition in this regime of e^2, g. However, in this regime the lattice theory (C2) is far from the continuum field theory (C1). To make a connection with the continuum field theory (C1), one should instead study the regime of small e^2 in (C2). Indeed, when $e^2 = 0$, gauge fluctuations are completely frozen and (C2) has a transition in the (ordinary) XY universality class at $g_c \approx 3.00$. If we perturb this ordinary XY fixed point by a small $e^2 \ll 1$, we will recover the infrared fixed point of field theory (C1) in the regime $e^2 \ll u \ll \Lambda_{UV}$. However, again, finding out the true fate of the IR fixed point of (C2) for small e^2 with MC is difficult, as one has to go to length scales $L \gg 1/e^2$ (otherwise, one is dominated by the ordinary XY transition fixed point). Instead, MC simulations have been performed at intermediate $e^2 = 5$; the continuous inverted XY transition survives here [33] (see also a more recent, larger system size study in Ref. [74] at $e^2 \approx 5.8$.) These results are summarized in Fig. 2. A possibility generally assumed in the literature is that the inverted XY transition survives all the way to infinitesimal e^2, although again we stress that this region has not been investigated carefully. A study of a related model [75] where the Villain term $\frac{1}{2g}(d\theta - \alpha - 2\pi n)^2$ in Eq. (C2) is replaced by $-J\cos(d\theta - \alpha)$ found an inverted XY transition down to $e^2 \approx 1.7$.

We conclude by stressing that the present evidence for particle-vortex duality of bosons at the level of IR fixed point of continuum field theory relies on MC simulations (and in the authors' opinion has not been established beyond the shadow of doubt). We, thus, expect that an equivalent "strong form" of duality between weakly coupled QED_3 and a free Dirac fermion can ultimately only be settled by numerics.

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