

Classical and quantum scattering in post-Minkowskian gravity

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New structural properties of post-Minkowskian (PM) gravity are derived, notably within its effective one-body (EOB) formulation. Our results concern both the mass dependence, and the high-energy behavior, of the classical scattering angle. We generalize our previous work by deriving, up to the fourth post-Minkowskian (4PM) level included, the explicit links between the scattering angle and the two types of potentials entering the Hamiltonian description of PM dynamics within EOB theory. We compute the scattering amplitude derived from quantizing the third post-Minkowskian (3PM) EOB radial potential (including the contributions coming from the Born iterations), and point out various subtleties in the relation between perturbative amplitudes and classical dynamics. We highlight an apparent tension between the classical 3PM dynamics derived by Bern *et al.* [*Phys. Rev. Lett.* **122**, 201603 (2019)], and previous high-energy self-force results [*Phys. Rev. D* **86**, 104041 (2012)], and propose several possible resolutions of this tension. We point out that linear-in-mass-ratio self-force computations can give access to the exact 3PM and 4PM dynamics.

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I. INTRODUCTION

The recent, dramatically successful, beginning of gravitational-wave astronomy [1–4], and the expected future improvements in the sensitivity of gravitational-wave detectors, give a renewed motivation for improving our theoretical knowledge of the gravitational dynamics of two-body systems in general relativity. Our current knowledge of the dynamics and gravitational-wave emission of binary systems has been acquired by combining several types of (interrelated) analytical approximations schemes, and furthermore, by completing analytical results with the results of a certain number of numerical simulations of coalescing binary black holes. The main types of analytical schemes that have been used are: post-Minkowskian (PM), post-Newtonian (PN), multipolar-post-Minkowskian, effective-one-body (EOB), black-hole-perturbation, gravitational self-force (SF), and effective-field-theory (EFT).

Recently, a new avenue for improving our theoretical knowledge of gravitational dynamics¹ has been actively pursued. It consists of translating the (classical or quantum) scattering observables of gravitationally interacting two-body systems into some Hamiltonian counterpart. The idea of mapping quantum gravitational scattering amplitudes onto some type of gravitational potential had been first explored long ago [6–11]. The idea of these works was to construct a two-body Hamiltonian of the type

$$H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) = c^2 \sqrt{m_1^2 + \frac{\mathbf{p}_1^2}{c^2}} + c^2 \sqrt{m_2^2 + \frac{\mathbf{p}_2^2}{c^2}} + V(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2), \quad (1.1)$$

such that the scattering amplitude in the momentum-dependent potential $V(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$ (given by a usual Born-type expansion) is equal to the scattering amplitude computed by means of the Feynman-diagrams defined by a (perturbative) quantum field theory comprising two scalar fields ϕ_1, ϕ_2 (of masses m_1 and m_2) interacting via perturbatively quantized Einstein gravity. This was done within the framework of the PN approximation scheme, i.e., using a small-velocity expansion, and working actually with the PN-expanded form of the Hamiltonian, up to some finite (and rather low) accuracy:

$$H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) = (m_1 + m_2)c^2 + \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{\mathbf{p}_1^4}{8m_1^3c^2} - \frac{\mathbf{p}_2^4}{8m_2^3c^2} + \dots + V^{\text{PN}}(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2), \quad (1.2)$$

with

$$V^{\text{PN}}(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) = -\frac{Gm_1m_2}{|\mathbf{x}_1 - \mathbf{x}_2|} + \text{PN corrections}. \quad (1.3)$$

This did not yield at the time results that could not be (often more efficiently) obtained by conventional PN classical

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¹We shall not discuss here the related issue of improving our knowledge of gravitational-wave emission by amplitude methods; see Ref. [5] and references therein.

computations.² A similar approach was also used in quantum electrodynamics to derive the (v^2/c^2) -accurate (first post-Coulombian) Breit Hamiltonian. See, notably, the fourth volume of the Landau-Lifshitz treatise of theoretical physics [12] which derives the Breit Hamiltonian by starting from the scattering amplitude \mathcal{A} of two massive, charged particles.

The idea of extracting classical gravitational dynamics from the scattering amplitude \mathcal{M} of two gravitationally interacting massive particles has been further explored and extended in more recent papers [13–18]. However, these works limited their ambition to extracting leading terms in the PN expansion of the dynamics.

It is only recently that the issue of linking the gravitational scattering amplitude \mathcal{M} to PM gravity, i.e., without using a small-velocity expansion, has been explored. This was done at the second post-Minkowskian (2PM) level (i.e., $O(G^2)$ or one-loop) in Refs. [19–23], and at the third post-Minkowskian (3PM) level (i.e., $O(G^3)$ or two-loop) in the breakthrough work of Bern *et al.* [24,25]. Before the latter work, the only extant two-loop result was the trans-Planckian, eikonal-approximation two-loop result of Amati, Ciafaloni and Veneziano (ACV) [26] (which was recently generalized [27,28], and confirmed [28]). [Reference [20] has extracted both 3PM and 4PM classical information from the result of ACV.] Let us also mention some further (partly conjectural) work concerning the link between the gravitational scattering amplitude of spinning particles and the classical gravitational interaction of Kerr black holes [29–33], as well as work on the computation of classically measurable quantities from on-shell amplitudes [34,35].

Those recent works dealing with PM gravity in connection with the quantum amplitude \mathcal{M} have been preceded by older investigations, using purely classical methods, of the PM expansion of the gravitational dynamics of two-body systems. The first post-Minkowskian (1PM; $O(G^1)$) dynamics was studied in Refs. [36–38], while the second post-Minkowskian (2PM; $O(G^2)$) one was tackled in Refs. [39–42]). More recently, the investigation of classical PM gravity has been revived by showing how the EOB formalism [43–45] was able to provide a much simplified description of PM gravity, based on the gauge-invariant information contained in the scattering function $\frac{1}{2}\chi(E, J)$. In particular: (i) Ref. [46] has shown how the 1PM-accurate classical scattering of two non-spinning bodies could be transcribed, within the EOB formalism into the geodesic dynamics of a particle of mass

$\mu = m_1 m_2 / (m_1 + m_2)$ in a (linearized) Schwarzschild background of mass $M = m_1 + m_2$. [This EOB formulation of the 1PM dynamics is much simpler than the previously obtained Arnowitt-Deser-Misner one [38].]; (ii) Ref. [47] has shown how to transcribe within the EOB formalism the 1PM gravitational interaction of spinning bodies at all orders in the spins (see also [48]); (iii) Ref. [20] derived, for the first time, a next-to-leading-order, $O(G^2)$ (second-post-Minkowskian, 2PM) Hamiltonian EOB description of the (nonspinning) two-body dynamics from the classical 2PM scattering angle [41] [This EOB description of the 2PM dynamics is equivalent, but simpler, than the one later derived in [22], using a potential of the form of Eq. (1.1)]; (iv) Ref. [49] derived (by using the 2PM-accurate metric of Ref. [40]) a 2PM-accurate Hamiltonian EOB description of the gravitational interaction of two spinning bodies at linear order in spins; and (v) a conjectural 2PM-level generalization of the 1PM result of Ref. [47] concerning the nonlinear-in-spin dynamics of aligned-spin bodies was proposed in Ref. [50]. In addition, the 5PN-level truncation of the classical 3PM dynamics extracted from the two-loop result of Bern *et al.* [24,25] (see also Ref. [51]) has been confirmed by an independent, purely classical computation [52]. See below for the discussion of more recent, classical and quantum, 6PN-level confirmations.

The main aim of the present work is to derive some structural properties of the classical scattering angle, χ , considered as a function of the various arguments in which it can be expressed: energy, angular momentum, impact parameter, and masses. This will allow us to derive several new results of direct importance for improving our current knowledge of the dynamics of two-body systems. In particular, we shall derive a property of the dependence of χ on the masses which was crucially used in Ref. [52] for determining most of the mass dependence of the 5PN-level dynamics. We shall also discuss a constraint on the high-energy behavior of χ that follows from the SF result of Ref. [53]. The latter high-energy constraint seems to be discrepant with the high-energy (or massless) limit of the 3PM-level results of Bern *et al.* [24,25]. We will suggest two types of possible resolutions of this apparent discrepancy. One resolution consists in conjecturing that the 3PM dynamics is described by another classical Hamiltonian, yielding the same 5PN-level $O(G^3)$ scattering angle (which was recently independently obtained [52]), but a softer high-energy behavior than that of Refs. [24,25]. Another resolution consists in conjecturing a special structure of the 4PM ($O(G^4)$) dynamics, such that its high-energy behavior modifies the consequences drawn from considering the high-energy behavior of the 3PM-level-only result of Bern *et al.*. Both types of resolutions will be shown to lead to a classical massless scattering angle that disagrees with the one derived from the eikonal-approximated quantum two-loop massless amplitude [26,28].

²Let us note that Corinaldesi [6] incorrectly concluded that the full 1PN Einstein-Infeld-Hoffmann equations of motion could be derived from the one-graviton-exchange amplitude. The first formally correct and complete derivation of the 1PN Hamiltonian from the *one-loop* scattering amplitude of two scalar particles is due to Iwasaki [10].

A secondary aim of the present work is to clarify the various links between the physical quantities involved in the maps that have been recently used to relate classical and quantum dynamics. These three quantities are: the classical scattering angle χ , the quantum scattering amplitude \mathcal{M} (considered in a limit formally corresponding to classical scattering), and the two different potentials (EOB-type [20] or EFT-type [22]) used to transcribe (classical or quantum) scattering observables into an Hamiltonian description. In this connection, we will explicitly derive below the map going from the 3PM-level classical Hamiltonian to the corresponding piece of the two-loop amplitude. [Some of the results derived below (which have been presented in various talks [54]), have been recently discussed from quite different (non-EOB-based) perspectives in two papers [55,56].] Our 3PM-level map will be found to be fully compatible with the corresponding results in Sec. 10 of Ref. [25], but are more complete in that they detail the IR-divergent contributions coming from iterating the 1PM and 2PM levels, which also contribute IR-finite terms.

In addition, we will point out various subtleties in the relation between perturbative amplitudes and classical dynamics. Several tools, concerning the link between the classical PM dynamics and the quantum amplitude \mathcal{M} , have been presented in the recent literature [13–26,34,35]. These tools have been checked to give a correct result at the 2PM (one-loop) level [19–21]. As the 3PM-level classical dynamics of Refs. [24,25] has not yet been confirmed by an independent *classical* derivation, it might be useful to point out the existence of conceptual subtleties in the map going from the quantum \mathcal{M} toward the classical dynamics (which is the inverse of the classical-to-quantum map that we shall discuss below). We shall recall in this respect a classic result of Niels Bohr [57] highlighting the lack of overlap between the domains of validity of classical and quantum (perturbative) scattering theory.

Technically speaking, we will be dealing below with the 3PM-accurate expansions (i.e., the expansions in powers of the gravitational constant G up to G^3 included) of various physical quantities: the classical (half) scattering angle expressed as a function of (center-of-mass) energy ($E = \sqrt{s}$) and angular momentum (J),

$$\frac{1}{2}\chi(E, J) = \frac{\chi_1(\hat{\mathcal{E}}_{\text{eff}}, \nu)}{j} + \frac{\chi_2(\hat{\mathcal{E}}_{\text{eff}}, \nu)}{j^2} + \frac{\chi_3(\hat{\mathcal{E}}_{\text{eff}}, \nu)}{j^3} + O(G^4), \quad (1.4)$$

(see below the definitions of the dimensionless variables $\hat{\mathcal{E}}_{\text{eff}}$, j and ν); the (relativistic) quantum scattering amplitude expressed as a function of Mandelstam invariants $s = -(p_1 + p_2)^2$ and $t = -(p'_1 - p_1)^2$ (in the mostly plus signature we use),

$$\begin{aligned} \mathcal{M}(s, t) = & G\mathcal{M}_1(s, t) + G^2\mathcal{M}_2(s, t) \\ & + G^3\mathcal{M}_3(s, t) + O(G^4); \end{aligned} \quad (1.5)$$

and the PM expansions of the two (closely connected) types of EOB potentials describing the gravitational interaction of two classical masses. Namely, with $u \equiv GM/R_{\text{EOB}}$, and now including the 4PM, $O(G^4)$, contribution,

$$\hat{Q}(p, u) = u^2 q_2(p) + u^3 q_3(p) + u^4 q_4(p) + \dots, \quad (1.6)$$

and (with $\bar{u} \equiv GM/\bar{R}_{\text{EOB}}$; in isotropic coordinates)

$$w(\gamma, \bar{u}) = w_1(\gamma)\bar{u} + w_2(\gamma)\bar{u}^2 + w_3(\gamma)\bar{u}^3 + w_4(\gamma)\bar{u}^4 + \dots \quad (1.7)$$

As we will explicitly discuss, these EOB potentials are equivalent (and simpler) than the more traditional type of potential $V(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$ entering Eq. (1.1), and used in the EFT-type formalism of Refs. [22,24,25]. We briefly discuss in the Appendix A the link between the EOB potentials and the PM expansion of the isotropic-gauge EFT-type potential [22] in the center of mass (c.m.) frame,

$$V(\mathbf{P}, \mathbf{X}) = G \frac{c_1(\mathbf{P}^2)}{|\mathbf{X}|} + G^2 \frac{c_2(\mathbf{P}^2)}{|\mathbf{X}|^2} + G^3 \frac{c_3(\mathbf{P}^2)}{|\mathbf{X}|^3} + \dots \quad (1.8)$$

The precise technical meaning of the EOB potentials, $\hat{Q}(p, u)$ and $w(\gamma, \bar{u})$, will be presented below. On the right-hand side of Eq. (1.4) we have replaced the total c.m. energy of the two-body system, $E = E_{\text{real}} = \sqrt{s}$, by the corresponding dimensionless EOB “effective energy” [43–46],

$$\hat{\mathcal{E}}_{\text{eff}} \equiv \frac{\mathcal{E}_{\text{eff}}}{\mu} \equiv \frac{(E_{\text{real}})^2 - m_1^2 - m_2^2}{2m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2m_1 m_2}. \quad (1.9)$$

Let us note in advance that, in scattering situations, $\hat{\mathcal{E}}_{\text{eff}}$ is equal to the relative Lorentz gamma factor of the incoming worldlines, denoted γ below (and σ in Refs. [24,25]). In addition, we have replaced the total (c.m.) angular momentum J by the dimensionless variable

$$j \equiv \frac{J}{Gm_1 m_2} = \frac{J}{G\mu M}, \quad (1.10)$$

with

$$M \equiv m_1 + m_2; \quad \mu \equiv \frac{m_1 m_2}{m_1 + m_2}; \quad \nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}. \quad (1.11)$$

As $1/j = Gm_1 m_2/J$, the perturbative expansion of the (classical) scattering function in powers of the gravitational

constant G (i.e., its PM expansion) is seen to be equivalent to an expansion in inverse powers of the angular momentum.

II. ON THE MASS DEPENDENCE OF THE CLASSICAL TWO-BODY SCATTERING FUNCTION

The aim of the present section is to extract from PM perturbation theory simple rules constraining the mass dependence of the scattering function at each PM order. Though their technical origin is rather simple, these rules turn out to give very useful constraints on the functional structure of the scattering function. The PM perturbation theory of interacting point masses has been worked out at the 2PM (one-loop) level long ago [39–41]. Recently, Refs. [20,46,49] have outlined a formal iteration scheme for computing the PM expansion of the scattering function to all PM orders, and showed how it could be naturally expressed as a sum of Feynman-like diagrams (see Fig. 1 in [46], and Figs. 1 and 2 in [20]). Let us recall this construction. The PM expansion of the classical momentum transfer (dubbed the “impulse” in Ref. [34]), i.e., the total change $\Delta p_{a\mu}$, between the infinite past and the infinite future, of the 4-momentum $p_{a\mu} = m_a u_{a\mu}$ of the particle labeled by $a = 1, 2$, is obtained by inserting on the right-hand side of the integral expression

$$\Delta p_{a\mu} = -\frac{m_a}{2} \int_{-\infty}^{+\infty} ds_a \partial_\mu g^{\alpha\beta}(x_a) u_{a\alpha} u_{a\beta}, \quad (2.1)$$

the iterative solutions (in successive powers of G) of the combined system of equations describing the coupled evolution of the two worldlines

$$\begin{aligned} \frac{dx_a^\mu}{ds_a} &= g^{\mu\nu}(x_a) u_{a\nu}, \\ \frac{du_{a\mu}}{ds_a} &= -\frac{1}{2} \partial_\mu g^{\alpha\beta}(x_a) u_{a\alpha} u_{a\beta}, \end{aligned} \quad (2.2)$$

and of the metric $g_{\mu\nu}$. The latter mediates the interaction between the two worldlines, and is generated by them via Einstein’s equations,

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 8\pi G T^{\mu\nu}, \quad (2.3)$$

with

$$T^{\mu\nu}(x) = \sum_{a=1,2} m_a \int ds_a u_a^\mu u_a^\nu \frac{\delta^4(x - x_a(s_a))}{\sqrt{g}}, \quad (2.4)$$

where $u_a^\mu \equiv g^{\mu\nu} u_{a\nu}$ and $g = -\det g_{\mu\nu}$.

Here we need to work in some gauge (say in harmonic gauge), and, as we are discussing the conservative dynamics of two particles, we iteratively solve Einstein

equations (2.3) by means of the time-symmetric classical graviton propagator (in Minkowski spacetime)

$$\mathcal{P}^{\alpha\beta;\alpha'\beta'}(x-y) = \left(\eta^{\alpha\alpha'} \eta^{\beta\beta'} - \frac{1}{2} \eta^{\alpha\beta} \eta^{\alpha'\beta'} \right) \mathcal{G}_{\text{sym}}(x-y), \quad (2.5)$$

with $\mathcal{G}_{\text{sym}}(x-y) = \delta[\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)]$.

The crucial point for our present purpose is that this iterative procedure, which involves expanding in powers of G both the worldlines, say

$$\begin{aligned} x_a^\mu(s_a) &= {}_0x_a^\mu(s_a) + G_1 x_a^\mu(s_a) + G^2 {}_2x_a^\mu(s_a) + \dots, \\ u_{a\mu}(s_a) &= {}_0u_{a\mu}(s_a) + G_1 u_{a\mu}(s_a) + G^2 {}_2u_{a\mu}(s_a) + \dots \end{aligned} \quad (2.6)$$

and the metric

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - G h_1^{\mu\nu}(x) - G^2 h_2^{\mu\nu}(x) - \dots, \quad (2.7)$$

yields, at each order G^n , expressions that are *homogeneous polynomials* of degree n in the masses m_a . E.g.,

$$\begin{aligned} h_1^{\mu\nu}(x) &= m_1 h_{m_1}^{\mu\nu}(x) + m_2 h_{m_2}^{\mu\nu}(x), \\ h_2^{\mu\nu}(x) &= m_1^2 h_{m_1^2}^{\mu\nu}(x) + m_2^2 h_{m_2^2}^{\mu\nu}(x) + m_1 m_2 h_{m_1 m_2}^{\mu\nu}(x). \end{aligned} \quad (2.8)$$

Here, we assume that the iterative solutions are systematically expressed in terms of the mass-independent data describing the two asymptotic incoming worldlines, say ${}_0x_a^\mu(s_a) = x_{a0}^\mu + u_{a0}^\mu s_a$. See, e.g., Sec. IV of Ref. [49] for an explicit example of the structure of the PM-expanded metric, and worldlines, expressed as explicit functionals of the incoming worldline data (and for a discussion of the logarithmic asymptotic corrections to the asymptotic free motions). From a geometric perspective, the latter incoming worldline data can be described by the two incoming 4-velocity vectors u_{10}^μ and u_{20}^μ , and by the vectorial impact parameter $b^\mu = x_{10}^\mu - x_{20}^\mu$ (chosen so as to be orthogonal to u_{10}^μ and u_{20}^μ).

At the end of the day, one gets a PM expansion for $\Delta p_{1\mu} = -\Delta p_{2\mu}$ (expressed in terms of b^μ/b , u_{10}^μ and u_{20}^μ) that is, at each order in G , a polynomial in the masses. It can be written as

$$\Delta p_{1\mu} = -2G m_1 m_2 \frac{2(u_{10} \cdot u_{20})^2 - 1}{\sqrt{(u_{10} \cdot u_{20})^2 - 1}} \frac{b_\mu}{b^2} + \frac{G m_1 m_2}{b} \Delta_\mu. \quad (2.9)$$

Here we displayed the leading-order term [37,41,46] and indicated that the higher PM contributions (described by the term $\frac{G m_1 m_2}{b} \Delta_\mu$ with $\Delta_\mu = G \Delta_\mu^{(1)} + G^2 \Delta_\mu^{(2)} + \dots$) all contain $m_1 m_2$ as a common factor. Each PM contribution $\Delta_\mu^{(n)}$ is a combination of the three vectors b^μ/b , u_{10}^μ and u_{20}^μ ,

with coefficients that are, at each order in G , homogeneous polynomials in Gm_1 and Gm_2 . By dimensional analysis, as the only length scale entering each order in the PM expansion³ is the impact parameter b , we can write the three vectorial coefficients of the dimensionless $\Delta_\mu^{(n)}$ as polynomials in Gm_1/b and Gm_2/b , with coefficients depending only on the dimensionless quantity

$$\gamma \equiv -u_{10} \cdot u_{20}. \quad (2.10)$$

The latter quantity (denoted σ in Refs. [24,25]), which is the relative Lorentz factor between the two incoming particles, will play a central role in the following. Let us immediately note that it is equal to the dimensionless effective EOB energy of the binary system:

$$\gamma = \hat{\mathcal{E}}_{\text{eff}}. \quad (2.11)$$

Indeed,

$$\begin{aligned} \mathbf{Q} = \frac{2Gm_1m_2}{b} & \left[\mathbf{Q}^{1\text{PM}}(\gamma) + \left(\mathbf{Q}_1^{2\text{PM}}(\gamma) \frac{Gm_1}{b} + \mathbf{Q}_2^{2\text{PM}}(\gamma) \frac{Gm_2}{b} \right) + \left(\mathbf{Q}_{11}^{3\text{PM}}(\gamma) \left(\frac{Gm_1}{b} \right)^2 + \mathbf{Q}_{22}^{3\text{PM}}(\gamma) \left(\frac{Gm_2}{b} \right)^2 \right. \right. \\ & + \mathbf{Q}_{12}^{3\text{PM}}(\gamma) \frac{Gm_1}{b} \frac{Gm_2}{b} \left. \left. + \left(\mathbf{Q}_{111}^{4\text{PM}}(\gamma) \left(\frac{Gm_1}{b} \right)^3 + \mathbf{Q}_{222}^{4\text{PM}}(\gamma) \left(\frac{Gm_2}{b} \right)^3 \right. \right. \right. \\ & \left. \left. + \mathbf{Q}_{112}^{4\text{PM}}(\gamma) \left(\frac{Gm_1}{b} \right)^2 \frac{Gm_2}{b} + \mathbf{Q}_{122}^{4\text{PM}}(\gamma) \frac{Gm_1}{b} \left(\frac{Gm_2}{b} \right)^2 \right) + \dots \right] \end{aligned} \quad (2.15)$$

where

$$\mathbf{Q}^{1\text{PM}}(\gamma) = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}. \quad (2.16)$$

Three apparently trivial, but quite useful, pieces of information controlling the structure of this PM expansion are: (i) the homogeneous polynomial dependence in m_1 and m_2 (and therefore, by dimensional analysis, in Gm_1/b and Gm_2/b) at each PM order; (ii) the exchange symmetry between the two masses; and (iii) the consideration of the test-particle limit where, say, $m_1 \ll m_2$. The exchange symmetry tells us that, for instance, $\mathbf{Q}_1^{2\text{PM}}(\gamma) = \mathbf{Q}_2^{2\text{PM}}(\gamma)$, $\mathbf{Q}_{11}^{3\text{PM}}(\gamma) = \mathbf{Q}_{22}^{3\text{PM}}(\gamma)$, $\mathbf{Q}_{111}^{4\text{PM}}(\gamma) = \mathbf{Q}_{222}^{4\text{PM}}(\gamma)$, $\mathbf{Q}_{112}^{4\text{PM}}(\gamma) = \mathbf{Q}_{122}^{4\text{PM}}(\gamma)$, etc. In other words, at each PM order, we will have a symmetric polynomial in m_1 and m_2 , with γ -dependent coefficients. In addition, the test-mass limit tells us that all the functions involving only one mass are equal to the corresponding function of γ appearing in the scattering of a test mass around a Schwarzschild black hole. Therefore, we have

$$\begin{aligned} \gamma &= -\frac{p_{10} \cdot p_{20}}{m_1 m_2} = -\frac{(p_{10} + p_{20})^2 - p_{10}^2 - p_{20}^2}{2m_1 m_2} \\ &= \frac{s - m_1^2 - m_2^2}{2m_1 m_2}, \end{aligned} \quad (2.12)$$

to be compared with the EOB definition (1.9).

Let us now consider the magnitude of the (classical) momentum transfer, namely

$$\mathbf{Q} \equiv \sqrt{-t} \equiv \sqrt{\eta^{\mu\nu} \Delta p_{1\mu} \Delta p_{1\nu}}, \quad (2.13)$$

which is related to the center-of-mass (c.m.) scattering angle χ , and the c.m. three-momentum $P_{\text{c.m.}}$, via

$$\mathbf{Q} = 2P_{\text{c.m.}} \sin \frac{\chi}{2}. \quad (2.14)$$

The structure of the PM expansion of the vectorial momentum transfer (2.9) is easily seen to imply that

$$\begin{aligned} \mathbf{Q}^{1\text{PM}}(\gamma) &= \mathbf{Q}_S^{1\text{PM}}(\gamma), \\ \mathbf{Q}_1^{2\text{PM}}(\gamma) &= \mathbf{Q}_2^{2\text{PM}}(\gamma) = \mathbf{Q}_S^{2\text{PM}}(\gamma), \\ \mathbf{Q}_{11}^{3\text{PM}}(\gamma) &= \mathbf{Q}_{22}^{3\text{PM}}(\gamma) = \mathbf{Q}_S^{3\text{PM}}(\gamma), \\ \mathbf{Q}_{111}^{4\text{PM}}(\gamma) &= \mathbf{Q}_{222}^{4\text{PM}}(\gamma) = \mathbf{Q}_S^{4\text{PM}}(\gamma), \end{aligned} \quad (2.17)$$

where the subscript S refers to the Schwarzschild limit.

The 1PM-level result [first line of Eq. (2.17)] was already used in [46] to show that the 1PM dynamics is equivalent (after using the EOB energy map) to geodesic motion in a linearized Schwarzschild metric of mass $M = m_1 + m_2$. Let us emphasize that the 2PM-level result [second line of (2.17)] gives a one-line proof that the 2PM fractional contribution to the momentum transfer (considered as a function of the impact parameter) of a two-body system is simply given by the formula,

$$\mathbf{Q}_S^{2\text{PM}}(\gamma) \frac{G(m_1 + m_2)}{b}, \quad (2.18)$$

where $\mathbf{Q}_S^{2\text{PM}}(\gamma)$ denotes the function of γ obtained by computing the 2PM-accurate scattering of a test particle around a Schwarzschild black hole, namely (see, e.g., [20])

³This contrasts with the PN expansion where one has two different length scales: b and the characteristic wavelength of the gravitational radiation $\lambda \sim cb/v$.

$$\mathbf{Q}_S^{2\text{PM}}(\gamma) = \frac{3\pi}{8} \frac{5\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}. \quad (2.19)$$

The test-mass computation yielding (2.19) (equivalent to Eq. (3.19) in [20]) is much simpler than the full, two-body 2PM scattering computation (involving complicated nonlinear terms and recoil effects) first done by Westpfahl [41] (and recently redone in [49]). The simple link between the 2PM test-mass result and the two-body one was also recently discussed in Ref. [50], but in a different context, and arguing from the structure of the so-called classical part of the one-loop amplitude [19,21], instead of our purely classical analysis above. Note that the mass-dependence we are talking about here has taken an especially simple form because we focused on the variable \mathbf{Q} as a function of γ and b . As we shall see next, the mass-dependence of the scattering angle χ as a function of γ and either b or $j \equiv \frac{J}{Gm_1m_2}$ is more involved.

Summarizing so far, we conclude that both the 1PM and 2PM two-body scattering can be deduced (without any extra calculation) from the 1PM and 2PM test-mass scattering.

Let us now consider what happens at higher PM orders. At the 3PM order, $O(G^3)$, we conclude from the above results that the scattering depends not only on the test-mass-derivable function $\mathbf{Q}_{11}^{3\text{PM}}(\gamma) = \mathbf{Q}_{22}^{3\text{PM}}(\gamma) = \mathbf{Q}_S^{3\text{PM}}(\gamma)$, but also on a *single* further function of γ , namely $\mathbf{Q}_{12}^{3\text{PM}}(\gamma)$. Similarly, at the 4PM order, the full two-body scattering depends, besides the test-mass-derivable function $\mathbf{Q}_{111}^{4\text{PM}}(\gamma) = \mathbf{Q}_{222}^{4\text{PM}}(\gamma) = \mathbf{Q}_S^{4\text{PM}}(\gamma)$, on a *single* further function of γ , namely $\mathbf{Q}_{112}^{4\text{PM}}(\gamma) = \mathbf{Q}_{122}^{4\text{PM}}(\gamma)$.

It is easy to generalize this result to higher PM orders. E.g., at 5PM, modulo the $1 \leftrightarrow 2$ symmetrization, there will be terms $\propto m_1^4$, $m_1^3m_2$ and $m_1^2m_2^2$. The first one of these is deducible from the test-mass limit, so that the full two-body 5PM scattering depends on only *two* nontrivial extra functions of γ . The same counting applies at the 6PM level where there will be (modulo $1 \leftrightarrow 2$ symmetrization) terms $\propto m_1^5$ (test-mass-deducible), $m_1^4m_2$ and $m_1^3m_2^2$. The general rule is that, at the n PM order, there will appear only (using $[\dots]$ to denote the integer part)

$$d(n) \equiv \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (2.20)$$

nontest-mass-deducible functions of γ .

The latter result can be translated into a dependence on the symmetric mass ratio $\nu \equiv m_1m_2/(m_1+m_2)^2$ if one expresses m_1 and m_2 (with, say, $m_1 \leq m_2$) in terms of the total mass $M = m_1 + m_2$, and of the two dimensionless mass ratios

$$\begin{aligned} X_1 &\equiv \frac{m_1}{m_1+m_2} = \frac{1-\sqrt{1-4\nu}}{2}, \\ X_2 &\equiv \frac{m_2}{m_1+m_2} = 1-X_1 = \frac{1+\sqrt{1-4\nu}}{2}, \end{aligned} \quad (2.21)$$

such that $\nu \equiv X_1X_2$. Indeed, an homogeneous, symmetric polynomial of degree n in the masses yields (after division by M^n) a sum $\sum_k c_k X_1^k X_2^{n-k}$. Using $X_2 \equiv 1 - X_1$ and symmetrizing over $1 \leftrightarrow 2$ yields a sum $\sum_k c'_k (X_1^k + X_2^k)$ over $0 \leq k \leq n$. What will be important here is the maximum power of ν entering such symmetric polynomials in the mass ratios. We note the following results

$$\begin{aligned} X_1^2 + X_2^2 &= 1 - 2\nu, \\ X_1^3 + X_2^3 &= 1 - 3\nu, \\ X_1^4 + X_2^4 &= 1 - 4\nu + 2\nu^2, \\ X_1^5 + X_2^5 &= 1 - 5\nu + 5\nu^2. \end{aligned} \quad (2.22)$$

More generally, $X_1^k + X_2^k$ is a polynomial in ν of degree $\lfloor \frac{k}{2} \rfloor$. At the n PM order, after having factored the prefactor,

$$\frac{2Gm_1m_2}{b} \left(\frac{GM}{b} \right)^{n-1}, \quad (2.23)$$

there appears such an homogeneous, symmetric polynomial of degree $n-1$ in X_1 and X_2 .

Finally, the PM expansion of the momentum transfer can be written as:

$$\mathbf{Q} = \frac{2Gm_1m_2}{b} \sum_{n \geq 1} \left(\frac{GM}{b} \right)^{n-1} \mathbf{Q}^{n\text{PM}}(\gamma, \nu), \quad (2.24)$$

where $\mathbf{Q}^{n\text{PM}}(\gamma, \nu)$ is a polynomial in ν of degree $d(n) \equiv \lfloor \frac{n-1}{2} \rfloor$:

$$\mathbf{Q}^{n\text{PM}}(\gamma, \nu) = Q_0^{n\text{PM}}(\gamma) + \nu Q_1^{n\text{PM}}(\gamma) + \dots + \nu^{d(n)} Q_{d(n)}^{n\text{PM}}(\gamma). \quad (2.25)$$

For instance, at the 3PM level, we have explicitly

$$\begin{aligned} \mathbf{Q}^{3\text{PM}}(\gamma, \nu) &= \mathbf{Q}_{11}^{3\text{PM}}(\gamma)(X_1^2 + X_2^2) + \mathbf{Q}_{12}^{3\text{PM}}(\gamma)X_1X_2 \\ &= \mathbf{Q}_S^{3\text{PM}}(\gamma)(1-2\nu) + \mathbf{Q}_{12}^{3\text{PM}}(\gamma)\nu. \end{aligned} \quad (2.26)$$

It is easily seen that, at all PM orders, the coefficient of ν^0 is simply the result given by the test-mass computation:

$$Q_0^{n\text{PM}}(\gamma) = \mathbf{Q}_S^{n\text{PM}}(\gamma). \quad (2.27)$$

Let us now translate the above structural information into an information about the classical scattering function itself, i.e., the half scattering angle $\chi/2$ considered as a function of

the energy and angular momentum of the system. As indicated in Eq. (1.4), it is convenient to measure the total c.m. energy of the system by means of the dimensionless effective energy $\hat{\mathcal{E}}_{\text{eff}} = \gamma$ given by Eq. (1.9), and to measure the total c.m. angular momentum by means of the dimensionless variable $j = J/(Gm_1m_2)$, Eq. (1.10). We also need the relations connecting the c.m. linear momentum $P_{\text{c.m.}}$ both to b , to J and to γ . These are (see Eqs. (7.6) and (10.27) in [20])

$$\begin{aligned} bP_{\text{c.m.}} &= J = Gm_1m_2j, \\ E_{\text{real}}P_{\text{c.m.}} &= \sqrt{(p_{10} \cdot p_{20})^2 - p_{10}^2 p_{20}^2} = m_1m_2\sqrt{\gamma^2 - 1}. \end{aligned} \quad (2.28)$$

From these links follows the relation

$$\frac{GM}{b} = \frac{\sqrt{\gamma^2 - 1}}{h(\gamma, \nu)j} = \frac{p_{\text{eob}}}{h(\gamma, \nu)j}. \quad (2.29)$$

Here we introduced some abbreviated notation for two dimensionless quantities crucially entering many equations, namely

$$\begin{aligned} h(\gamma, \nu) &\equiv \frac{E_{\text{real}}}{M} = \frac{\sqrt{s}}{M} = \sqrt{1 + 2\nu(\gamma - 1)}, \\ p_{\text{eob}} &\equiv \sqrt{\gamma^2 - 1} \equiv p_{\infty}. \end{aligned} \quad (2.30)$$

[We will indifferently use the notation p_{eob} or p_{∞} .] Inserting these relations in the above expression of the momentum transfer \mathbf{Q} , and computing

$$\sin \frac{\chi}{2} = \frac{\mathbf{Q}}{2P_{\text{c.m.}}}, \quad (2.31)$$

yields

$$\sin \frac{\chi}{2} = \frac{1}{j} \sum_{n \geq 1} \left(\frac{p_{\text{eob}}}{h(\gamma, \nu)j} \right)^{n-1} \mathbf{Q}^{n\text{PM}}(\gamma, \nu). \quad (2.32)$$

This reads more explicitly

$$\begin{aligned} \sin \frac{\chi}{2} &= \frac{\mathbf{Q}^{\text{1PM}}(\gamma)}{j} + \frac{p_{\text{eob}} \mathbf{Q}^{\text{2PM}}(\gamma)}{h(\gamma, \nu)j^2} \\ &+ \frac{p_{\text{eob}}^2 \mathbf{Q}^{\text{3PM}}(\gamma, \nu)}{h^2(\gamma, \nu)j^3} + \frac{p_{\text{eob}}^3 \mathbf{Q}^{\text{4PM}}(\gamma, \nu)}{h^3(\gamma, \nu)j^4} \\ &+ \dots \end{aligned} \quad (2.33)$$

Let us compare this expression to the usual way of writing the scattering function, namely (using $\gamma \equiv \hat{\mathcal{E}}_{\text{eff}}$ as energy variable and $j \equiv J/(Gm_1m_2)$ as angular momentum variable)

$$\begin{aligned} \frac{1}{2}\chi(E_{\text{real}}, J) &= \frac{\chi_1(\gamma, \nu)}{j} + \frac{\chi_2(\gamma, \nu)}{j^2} + \frac{\chi_3(\gamma, \nu)}{j^3} \\ &+ \frac{\chi_4(\gamma, \nu)}{j^4} + \dots, \end{aligned} \quad (2.34)$$

which implies

$$\begin{aligned} \sin \frac{1}{2}\chi(\gamma, j, \nu) &= \frac{\tilde{\chi}_1(\gamma, \nu)}{j} + \frac{\tilde{\chi}_2(\gamma, \nu)}{j^2} + \frac{\tilde{\chi}_3(\gamma, \nu)}{j^3} \\ &+ \frac{\tilde{\chi}_4(\gamma, \nu)}{j^4} + \dots \end{aligned} \quad (2.35)$$

where

$$\begin{aligned} \tilde{\chi}_1 &= \chi_1, \\ \tilde{\chi}_2 &= \chi_2, \\ \tilde{\chi}_3 &= \chi_3 - \frac{1}{6}\chi_1^3, \\ \tilde{\chi}_4 &= \chi_4 - \frac{1}{2}\chi_1^2\chi_2. \end{aligned} \quad (2.36)$$

When comparing the definitions of the expansion coefficients χ_n and $\tilde{\chi}_n$ to the structural result (2.33) we find

$$\tilde{\chi}_n(\gamma, \nu) = \frac{p_{\text{eob}}^{n-1} \mathbf{Q}^{n\text{PM}}(\gamma, \nu)}{h^{n-1}(\gamma, \nu)}. \quad (2.37)$$

Remember the fact that $\mathbf{Q}^{n\text{PM}}(\gamma, \nu)$ was proven above to be a polynomial in ν of degree $d(n)$ (with γ -dependent coefficients). We then get the rule that

$$h^{n-1}(\gamma, \nu)\tilde{\chi}_n(\gamma, \nu) = \tilde{P}_{d(n)}^\gamma(\nu), \quad (2.38)$$

where $\tilde{P}_{d(n)}^\gamma(\nu)$ denotes a polynomial in ν of degree $d(n)$ with γ -dependent coefficients. When transferring this information into a corresponding information for the expansion coefficients $\chi_n(\gamma, \nu)$ of $\frac{1}{2}\chi(\gamma, j)$, using Eqs. (2.36), it is easily seen that we have the same structure for them, namely

$$h^{n-1}(\gamma, \nu)\chi_n(\gamma, \nu) = P_{d(n)}^\gamma(\nu), \quad (2.39)$$

where $P_{d(n)}^\gamma(\nu)$ denotes another degree- $d(n)$ polynomial in ν with γ -dependent coefficients.

We can combine this structural information with the knowledge of the test-mass limit of the $\chi_n(\gamma, \nu)$'s. In the context of the functions $\chi_n(\gamma, \nu)$, the test-mass limit is simply the $\nu \rightarrow 0$ limit. Therefore, the $\nu \rightarrow 0$ limit of the various $\chi_n(\gamma, \nu)$'s must coincide with the values $\chi_n^{\text{Schw}}(\gamma)$ of the scattering coefficients for a test particle in a Schwarzschild background. The latter values were computed in [20] with the results

$$\chi_1^{\text{Schw}}(p_{\text{eob}}) = \frac{2p_{\text{eob}}^2 + 1}{p_{\text{eob}}} = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}, \quad (2.40)$$

$$\chi_2^{\text{Schw}}(p_{\text{eob}}) = \frac{3\pi}{8}(5p_{\text{eob}}^2 + 4) = \frac{3\pi}{8}(5\gamma^2 - 1), \quad (2.41)$$

$$\chi_3^{\text{Schw}}(p_{\text{eob}}) = \frac{64p_{\text{eob}}^6 + 72p_{\text{eob}}^4 + 12p_{\text{eob}}^2 - 1}{3p_{\text{eob}}^3}, \quad (2.42)$$

$$\chi_4^{\text{Schw}}(p_{\text{eob}}) = \frac{105\pi}{128}(33p_{\text{eob}}^4 + 48p_{\text{eob}}^2 + 16). \quad (2.43)$$

We then get the information that

$$P_{d(n)}^\gamma(0) = \chi_n^{\text{Schw}}(p_{\text{eob}}). \quad (2.44)$$

As already implied by the discussion above, this fully determines the 1PM [37,46] and 2PM [41,49] scattering coefficients, namely

$$\chi_1(\gamma, \nu) = \chi_1^{\text{Schw}}(\gamma) = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}, \quad (2.45)$$

and

$$\chi_2(\gamma, \nu) = \frac{\chi_2^{\text{Schw}}(\gamma)}{h(\gamma, \nu)} = \frac{3\pi(5\gamma^2 - 1)}{8h(\gamma, \nu)}. \quad (2.46)$$

Note in passing that it is crucial, in order to find the ν -independence of $\chi_1(\gamma, \nu)$, to measure the energy by means of γ (i.e., the EOB effective energy), and not by means of the total c.m. energy $E_{\text{real}} = \sqrt{s} = Mh(\gamma, \nu)$.

Concerning the higher-order expansion coefficients, using the fact that $h^2(\gamma, \nu) = 1 + 2\nu(\gamma - 1)$ is a linear function of ν [so that a polynomial in ν can be reexpressed as a polynomial in $h^2(\gamma, \nu)$] they can be written in the following form

$$\begin{aligned} \chi_3(\gamma, \nu) &= \hat{\chi}_3^{(0)}(\gamma) + \frac{\hat{\chi}_3^{(2)}(\gamma)}{h^2(\gamma, \nu)}, \\ \chi_4(\gamma, \nu) &= \frac{\hat{\chi}_4^{(1)}(\gamma)}{h(\gamma, \nu)} + \frac{\hat{\chi}_4^{(3)}(\gamma)}{h^3(\gamma, \nu)}, \\ \chi_5(\gamma, \nu) &= \hat{\chi}_5^{(0)}(\gamma) + \frac{\hat{\chi}_5^{(2)}(\gamma)}{h^2(\gamma, \nu)} + \frac{\hat{\chi}_5^{(4)}(\gamma)}{h^4(\gamma, \nu)}, \\ \chi_6(\gamma, \nu) &= \frac{\hat{\chi}_6^{(1)}(\gamma)}{h(\gamma, \nu)} + \frac{\hat{\chi}_6^{(3)}(\gamma)}{h^3(\gamma, \nu)} + \frac{\hat{\chi}_6^{(5)}(\gamma)}{h^5(\gamma, \nu)}, \end{aligned} \quad (2.47)$$

with the information that, at each PM order, the sum over k of the various numerators $\hat{\chi}_n^{(k)}(\gamma)$ is equal to the Schwarzschild limit $\chi_n^{\text{Schw}}(\gamma)$. This implies, for instance, that at the 3PM level we can also write

$$\chi_3(\gamma, \nu) = \chi_3^{\text{Schw}}(\gamma) + \hat{\chi}_3^{(2)}(\gamma) \left(\frac{1}{h^2(\gamma, \nu)} - 1 \right), \quad (2.48)$$

where the last term vanishes when $\nu \rightarrow 0$. A similar structure describes the 4PM-level scattering, namely

$$\chi_4(\gamma, \nu) = \frac{\chi_4^{\text{Schw}}(\gamma)}{h(\gamma, \nu)} + \frac{\hat{\chi}_4^{(3)}(\gamma)}{h(\gamma, \nu)} \left(\frac{1}{h^2(\gamma, \nu)} - 1 \right). \quad (2.49)$$

In both cases, we see that the full 3PM and 4PM dynamical information is encapsulated in a single function of γ , namely $\hat{\chi}_3^{(2)}(\gamma)$ and $\hat{\chi}_4^{(3)}(\gamma)$, respectively.

Let us note that in the high-energy (HE) limit ($\gamma \rightarrow \infty$, i.e., $p_{\text{eob}} \rightarrow \infty$) we have the following asymptotic behavior of the test-mass-limit scattering coefficients

$$\chi_n^{\text{Schw}}(p_{\text{eob}}) \stackrel{\text{HE}}{=} c_n^{\chi^{\text{Schw}}} p_{\text{eob}}^n \stackrel{\text{HE}}{=} c_n^{\chi^{\text{Schw}}} \gamma^n, \quad (2.50)$$

where $c_n^{\chi^{\text{Schw}}}$ is a numerical constant. It was suggested in Ref. [20] that the same asymptotic behavior (though with different numerical constants c_n^{χ}) holds for the building blocks $\hat{\chi}_n^{(k)}(\gamma)$ introduced above. We shall rediscuss this suggestion below.

III. PM-EXPANDED EOB HAMILTONIAN AND EOB RADIAL POTENTIAL

A. EOB Hamiltonian in PM gravity

References [20,46] introduced a new, PM-based, approach to the conservative dynamics of two-body systems based on the EOB formalism. This led to simple EOB descriptions of the 1PM [46], 2PM [20], and 3PM [51] Hamiltonians. Here, we will reconsider the 3PM EOB Hamiltonian derived from the quantum-amplitude approach of Refs. [24,25]. Let us start by recalling the PM-EOB formalism of Refs. [20,46].

The basic feature of the EOB formalism [43–45] is to describe the two-body dynamics in terms of a one-body Hamiltonian, which describes the dynamics of the *relative* two-body motion within the c.m. frame of the two-body system. The simplest way to define the EOB Hamiltonian is to say that: (i) the (“real”) c.m. Hamiltonian of the two-body system is related to the conserved energy \mathcal{E}_{eff} of the “effective” dynamics by Eq. (1.9), i.e.,

$$H_{\text{real}}(\mathbf{R}, \mathbf{P}) = M \sqrt{1 + 2\nu \left(\frac{\mathcal{E}_{\text{eff}}}{\mu} - 1 \right)}; \quad (3.1)$$

and, (ii) the effective energy \mathcal{E}_{eff} is related to the dynamical variables \mathbf{R}, \mathbf{P} describing the relative c.m. dynamics via a mass-shell condition of the form

$$0 = g_{\text{eff}}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q(X^\mu, P_\mu), \quad (3.2)$$

where $g_{\text{eff}}^{\mu\nu}$ is (the inverse of) an effective metric of the form

$$g_{\mu\nu}^{\text{eff}} dx^\mu dx^\nu = -A(R)dT^2 + B(R)dR^2 + C(R)(d\theta^2 + \sin^2\theta d\varphi^2), \quad (3.3)$$

and where $Q(X^\mu, P_\mu)$ is a Finsler-type additional contribution, which contains higher-than-quadratic momenta contributions. The time-invariance, and spherical symmetry, of the effective metric (and of Q), implies (for equatorial motions) the existence of the two conserved quantities P_0 and P_φ , which are respectively identified with

$$P_0 = -\mathcal{E}_{\text{eff}}, \quad P_\varphi = J. \quad (3.4)$$

For any given additional mass-shell contribution Q expressed as a function of \mathbf{R} , \mathbf{P} , and \mathcal{E}_{eff} , say $Q = Q(\mathbf{R}, \mathbf{P}, \mathcal{E}_{\text{eff}})$, the effective Hamiltonian $\mathcal{E}_{\text{eff}} = H_{\text{eff}}(\mathbf{R}, \mathbf{P})$ is then obtained by solving

$$0 = -\frac{\mathcal{E}_{\text{eff}}^2}{A} + \frac{P_R^2}{B} + \frac{P_\varphi^2}{C} + \mu^2 + Q(\mathbf{R}, \mathbf{P}, \mathcal{E}_{\text{eff}}), \quad (3.5)$$

with respect to \mathcal{E}_{eff} , and then inserting the result in the real, two-body Hamiltonian (3.1).

In a PM framework, i.e., when working perturbatively in G , it was shown in [20,46] that: (i) the effective metric can be taken to be a Schwarzschild metric of mass $M = m_1 + m_2$; (ii) the Q term starts at order G^2 ; and (iii) one can (by using some gauge freedom) construct Q so that it depends only on $R = |\mathbf{R}|$ and some energy-like variable (“energy gauge”). There are two simple choices for defining such an energy-gauge. Using the shorthand notation

$$u \equiv \frac{GM}{R}, \quad (3.6)$$

one can either write Q as a function of u and \mathcal{E}_{eff} ,

$$Q^E(u, \mathcal{E}_{\text{eff}}) = u^2 Q_2(\mathcal{E}_{\text{eff}}) + u^3 Q_3(\mathcal{E}_{\text{eff}}) + u^4 Q_4^E(\mathcal{E}_{\text{eff}}) + O(G^5), \quad (3.7)$$

or, one can express Q as a function of position and momenta by writing

$$Q^H(u, H_S) = u^2 Q_2(H_S) + u^3 Q_3(H_S) + u^4 Q_4^H(H_S) + O(G^5), \quad (3.8)$$

where H_S denotes the Schwarzschild Hamiltonian, i.e.,

$$H_S(u, P_R, P_\varphi) = \sqrt{A(R) \left(\frac{P_R^2}{B(R)} + \frac{P_\varphi^2}{C(R)} + \mu^2 \right)}. \quad (3.9)$$

The second form was initially advocated in [20] because it allows one to explicitly solve the mass shell condition (3.5) for \mathcal{E}_{eff} as a function of position and momenta, namely

$$\begin{aligned} \mathcal{E}_{\text{eff}} &= H_{\text{eff}}(\mathbf{R}, \mathbf{P}) \\ &= \sqrt{A \left(\frac{P_R^2}{B} + \frac{P_\varphi^2}{C} + \mu^2 + Q^H[u, H_S(u, P_R, P_\varphi)] \right)}. \end{aligned} \quad (3.10)$$

However, Ref. [20] also used the first form (3.7) because of its usefulness in getting an explicit energy-dependent potential that can be easily quantized. As indicated by the notation used in Eqs. (3.7), (3.8), the difference between the expansion coefficients Q_n entering these two perturbative expansions starts at order G^4 . This follows from the fact that Q itself starts at order G^2 .

In the following we will mostly work with the first, E-form of the energy gauge. It will also be convenient to work with dimensionless, rescaled quantities, say

$$\hat{Q} \equiv \frac{Q}{\mu^2}, \quad \mathbf{p} \equiv \frac{\mathbf{P}}{\mu}, \quad \hat{H}_{\text{eff}} \equiv \frac{H_{\text{eff}}}{\mu}, \quad (3.11)$$

and to denote the PM expansion coefficients of \hat{Q} simply as $q_n \equiv Q_n/\mu^2$, e.g.,

$$\hat{Q}^E(u, \gamma) = u^2 q_2(\gamma) + u^3 q_3(\gamma) + u^4 q_4^E(\gamma) + O(G^5), \quad (3.12)$$

where we used Eq. (2.11) to write $\hat{\mathcal{E}}_{\text{eff}} \equiv \mathcal{E}_{\text{eff}}/\mu$ simply as γ .

B. Energy-dependent, radial scattering potential within the EOB framework

In the previous subsection we recalled how PM gravity can be encoded, within the EOB formalism, by means of a PM-expanded mass-shell function $Q(\mathbf{R}, \mathbf{P}, \mathcal{E}_{\text{eff}})$. When discussing the quantum scattering amplitude corresponding to a given PM-expanded Q , it was found convenient in [20] to transform Q into an equivalent PM-expanded, energy-dependent radial potential $W(\bar{R}, \mathcal{E}_{\text{eff}})$. Let us recall this transformation.

Most of the past work in EOB dynamics has found it convenient to represent the EOB effective metric (3.3) by using a Schwarzschild-like radial coordinate, i.e., by choosing a coordinate R such that the coefficient $C(R)$ of $d\theta^2 + \sin^2\theta d\varphi^2$ is equal to R^2 . In keeping with the latter usage, we shall denote simply by R such a Schwarzschild-like radial coordinate, and by u the corresponding quantity GM/R . On the other hand, when discussing the effective potential describing the scattering dynamics, it is convenient [following the 2PM-level treatment of Sec. X of Ref. [20]] to use isotropic coordinates, i.e., a new radial coordinate, say \bar{R} , such that $C(\bar{R}) = \bar{R}^2 B(\bar{R})$ for the

Schwarzschild metric entering the EOB mass shell condition (3.5). The link between R and \bar{R} is

$$R = \bar{R} \left(1 + \frac{GM}{2\bar{R}} \right)^2, \quad (3.13)$$

or

$$u = \bar{u} \left(1 + \frac{\bar{u}}{2} \right)^{-2}. \quad (3.14)$$

In these coordinates, the usual formulas $A(u) = 1 - 2u = 1/B(u)$ transform into

$$\bar{A}(\bar{u}) = \left(\frac{1 - \frac{1}{2}\bar{u}}{1 + \frac{1}{2}\bar{u}} \right)^2; \quad \bar{B}(\bar{u}) = \left(1 + \frac{1}{2}\bar{u} \right)^4, \quad (3.15)$$

where we added a bar on A , and B (and on the argument u), to recall the use of isotropic coordinates.

We shall denote the Cartesian coordinates linked in the usual way to \bar{R}, θ, φ as $X^i = \mathbf{X}$, and the corresponding (covariant) momenta P_i as \mathbf{P} (for simplicity we do not put bars on \mathbf{X} and \mathbf{P}). The E-type mass shell condition then directly leads to an energy-dependent quadratic constraint on the momenta of the form

$$\mathbf{P}^2 = P_\infty^2 + W(\bar{u}, P_\infty), \quad (3.16)$$

where

$$P_\infty^2 \equiv \mathcal{E}_{\text{eff}}^2 - \mu^2 = \mu^2(\gamma^2 - 1), \quad (3.17)$$

and where the energy-dependent ‘‘potential’’ W is defined by

$$P_\infty^2 + W(\bar{u}, P_\infty) \equiv \bar{B}(\bar{u}) \left(\frac{\mathcal{E}_{\text{eff}}^2}{\bar{A}(\bar{u})} - \mu^2 - Q(\bar{u}, \mathcal{E}_{\text{eff}}) \right). \quad (3.18)$$

The radial potential $W(\bar{u}, P_\infty)$ tends to zero at large distances (i.e., when $\bar{u} = GM/\bar{R} \rightarrow 0$) and can be rewritten as

$$W(\bar{u}, P_\infty) = \mathcal{E}_{\text{eff}}^2 \left(\frac{\bar{B}(\bar{u})}{\bar{A}(\bar{u})} - 1 \right) - \mu^2 (\bar{B}(\bar{u}) - 1) - \bar{B}(\bar{u}) Q^E(\bar{u}, \mathcal{E}_{\text{eff}}). \quad (3.19)$$

Its PM expansion directly follows by combining the \bar{u} expansion of the metric functions $\bar{A}(\bar{u}), \bar{B}(\bar{u})$, with the PM expansion of $Q^E(\bar{u}, \mathcal{E}_{\text{eff}})$. It reads

$$\begin{aligned} W(\bar{u}, P_\infty) &= W_1 \bar{u} + W_2 \bar{u}^2 + W_3 \bar{u}^3 + W_4 \bar{u}^4 + \dots \\ &= \frac{GMW_1}{\bar{R}} + \frac{G^2 M^2 W_2}{\bar{R}^2} + \frac{G^3 M^3 W_3}{\bar{R}^3} + \frac{G^4 M^4 W_4}{\bar{R}^4} \\ &\quad + \dots \end{aligned} \quad (3.20)$$

It is often more convenient to work with a rescaled version of these results in which one uses the dimensionless variables

$$\bar{r} = \frac{\bar{R}}{GM}, \quad \mathbf{p} = \frac{\mathbf{P}}{\mu}, \quad p_\infty = \frac{P_\infty}{\mu} = \sqrt{\gamma^2 - 1}. \quad (3.21)$$

One then has

$$\mathbf{p}^2 = p_\infty^2 + w(\bar{u}, p_\infty), \quad (3.22)$$

where

$$w(\bar{u}, p_\infty) = \frac{W(\bar{u}, P_\infty)}{\mu^2}, \quad (3.23)$$

i.e.,

$$\begin{aligned} w(\bar{u}, p_\infty) &= \gamma^2 \left(\frac{\bar{B}(\bar{u})}{\bar{A}(\bar{u})} - 1 \right) \\ &\quad - (\bar{B}(\bar{u}) - 1) - \bar{B}(\bar{u}) \hat{Q}^E(\bar{u}, \gamma). \end{aligned} \quad (3.24)$$

The rescaled potential $w(\bar{u}, p_\infty)$ has the following PM expansion

$$\begin{aligned} w(\bar{u}, p_\infty) &= w_1(\gamma) \bar{u} + w_2(\gamma) \bar{u}^2 + w_3(\gamma) \bar{u}^3 + w_4(\gamma) \bar{u}^4 + \dots \\ &= \frac{w_1(\gamma)}{\bar{r}} + \frac{w_2(\gamma)}{\bar{r}^2} + \frac{w_3(\gamma)}{\bar{r}^3} + \frac{w_4(\gamma)}{\bar{r}^4} + \dots \end{aligned} \quad (3.25)$$

where

$$w_n(\gamma) = \frac{W_n(\gamma)}{\mu^2}. \quad (3.26)$$

Note that these results mean that the *relativistic* (scattering) dynamics of a two-body system can be mapped (by using the EOB framework) onto the *nonrelativistic* dynamics of one particle of mass μ in an energy-dependent radial potential.

We can now use Eq. (3.25) to compute the link between the (rescaled) coefficients $w_n(\gamma)$ entering the PM expansion of the (rescaled) potential $w(\bar{u}, \gamma)$, and the coefficients $q_n^E(\gamma)$ entering the PM expansion of the energy-gauge Q function entering the EOB mass shell condition (3.2). The Q term is numerically independent of the radial gauge used in the EOB effective metric (3.3), but we must distinguish the functions $u \rightarrow \hat{Q}^E(u, \gamma)$ and $\bar{u} \rightarrow \hat{Q}^E(\bar{u}, \gamma)$. We shall denote their respective PM expansion coefficients as

$$\hat{Q}^E(u, \gamma) = u^2 q_2(\gamma) + u^3 q_3(\gamma) + u^4 q_4^E(\gamma) + O(G^5), \quad (3.27)$$

and

$$\hat{Q}^E(\bar{u}, \mathcal{E}_{\text{eff}}) = \bar{u}^2 \bar{q}_2(\gamma) + \bar{u}^3 \bar{q}_3(\gamma) + \bar{u}^4 \bar{q}_4^E(\gamma) + O(G^5), \quad (3.28)$$

with similar equations for $\hat{Q}^H(u, \hat{H}_S)$ and $\hat{Q}^H(\bar{u}, \hat{H}_S)$.

The relations between the q_n 's and the \bar{q}_n 's is easily obtained from Eq. (3.14). For instance, we have

$$\begin{aligned} \bar{q}_2(\gamma) &= q_2(\gamma), \\ \bar{q}_3(\gamma) &= q_3(\gamma) - 2q_2(\gamma), \\ \bar{q}_4^E(\gamma) &= q_4^E(\gamma) - 3q_3(\gamma) + \frac{5}{2}q_2(\gamma). \end{aligned} \quad (3.29)$$

We can then express the expansion coefficients $w_n(\gamma)$ of the EOB potential either in terms of the q_n 's or the \bar{q}_n 's. More precisely, the coefficient of $1/\bar{r}$ entirely comes from the linearized Schwarzschild metric and reads [20]

$$w_1(\gamma) = 2(2\gamma^2 - 1), \quad (3.30)$$

while the coefficients of higher powers of $1/\bar{r}$ are related to the \bar{q}_n 's via

$$\begin{aligned} w_2(\gamma) &= \frac{15}{2}\gamma^2 - \frac{3}{2} - \bar{q}_2(\gamma), \\ w_3(\gamma) &= 9\gamma^2 - \frac{1}{2} - \bar{q}_3(\gamma) - 2\bar{q}_2(\gamma), \\ w_4(\gamma) &= \frac{129}{16}\gamma^2 - \frac{1}{16} - \bar{q}_4^E(\gamma) - 2\bar{q}_3(\gamma) - \frac{3}{2}\bar{q}_2(\gamma), \end{aligned} \quad (3.31)$$

i.e.,

$$\begin{aligned} w_2(\gamma) &= \frac{15}{2}\gamma^2 - \frac{3}{2} - q_2(\gamma), \\ w_3(\gamma) &= 9\gamma^2 - \frac{1}{2} - q_3(\gamma), \\ w_4(\gamma) &= \frac{129}{16}\gamma^2 - \frac{1}{16} - q_4^E(\gamma) + q_3(\gamma). \end{aligned} \quad (3.32)$$

At the 2PM level, it was shown in [20] that

$$q_2(\gamma, \nu) = \frac{3}{2}(5\gamma^2 - 1) \left[1 - \frac{1}{h(\gamma, \nu)} \right], \quad (3.33)$$

where we recall that $h(\gamma, \nu) = \sqrt{1 + 2\nu(\gamma - 1)}$, so that

$$w_2(\gamma, \nu) = \frac{3}{2}(5\gamma^2 - 1) \frac{1}{h(\gamma, \nu)}. \quad (3.34)$$

The current knowledge of the values of the 3PM coefficients $q_3(\gamma, \nu)$ and $w_3(\gamma, \nu)$ will be assessed below.

C. Scattering function and scattering invariants of an energy-dependent radial potential

References [20,46] showed how to derive the scattering function $\chi(\mathcal{E}_{\text{eff}}, J)$ directly from the Q -form of the EOB PM dynamics. An equivalent, alternative procedure is to derive $\chi(\mathcal{E}_{\text{eff}}, J)$ from the EOB radial potential $W(\bar{u}, P_\infty)$ corresponding to the Schwarzschild-metric-plus- Q formulation. Actually this link is very general and applies to any dynamical formulation involving a radial potential.

The usual formulas of nonrelativistic potential scattering (recalled, e.g., in [46]) yield

$$\pi + \chi(\mathcal{E}_{\text{eff}}, J) = - \int_{-\infty}^{+\infty} d\bar{R} \frac{\partial P_R(\bar{R}; \mathcal{E}_{\text{eff}}, J)}{\partial J}, \quad (3.35)$$

where the radial momentum $P_R(\bar{R}; \mathcal{E}_{\text{eff}}, J)$ is obtained by solving the mass-shell condition with respect to P_R . When using an energy gauge, the mass-shell condition reads,

$$\mathbf{P}^2 = P_R^2 + \frac{J^2}{\bar{R}^2} = P_\infty^2 + W(\bar{u}, P_\infty), \quad (3.36)$$

so that

$$P_R(\bar{R}; \mathcal{E}_{\text{eff}}, J) = \pm \sqrt{P_\infty^2 + W(\bar{u}, P_\infty) - \frac{J^2}{\bar{R}^2}}. \quad (3.37)$$

Here the (energy-gauge) potential $W(\bar{u}, P_\infty)$ (where we recall that $\bar{u} = GM/\bar{R}$ and $P_\infty = \sqrt{\mathcal{E}_{\text{eff}}^2 - \mu^2}$) does not depend on the angular momentum J . We can then write (as in usual nonrelativistic potential theory)

$$\frac{\pi}{2} + \frac{1}{2}\chi(\mathcal{E}_{\text{eff}}, J) = + \int_{\bar{R}_{\text{min}}}^{+\infty} J \frac{d\bar{R}}{\bar{R}^2} \frac{1}{P_R(\bar{R}; \mathcal{E}_{\text{eff}}, J)}, \quad (3.38)$$

where $R_{\text{min}} = R_{\text{min}}(\mathcal{E}_{\text{eff}}, J)$ is the radial turning point defined by the vanishing of P_R .

In terms of rescaled variables [including $j = J/(GM\mu)$], this reads

$$\frac{\pi}{2} + \frac{1}{2}\chi(\gamma, j) = + \int_{\bar{r}_{\text{min}}}^{+\infty} j \frac{d\bar{r}}{\bar{r}^2} \frac{1}{p_r(\bar{r}; \gamma, j)}, \quad (3.39)$$

where

$$p_r(\bar{r}; \gamma, j) = + \sqrt{p_\infty^2 + w(\bar{u}, p_\infty) - \frac{j^2}{\bar{r}^2}}. \quad (3.40)$$

Indeed, one must use the positive square roots in the integrals above that have been written from the radial turning points (\bar{R}_{min} or \bar{r}_{min}) to infinity.

In terms of the variable $\bar{u} = 1/\bar{r} = GM/\bar{R}$, the above integral reads (with $\bar{u}_{\text{max}} \equiv 1/\bar{r}_{\text{min}}$)

$$\frac{\pi}{2} + \frac{1}{2}\chi(\gamma, j) = + \int_0^{u_{\max}(\gamma, j)} \frac{j d\bar{u}}{\sqrt{p_\infty^2 + w(\bar{u}, p_\infty) - j^2 \bar{u}^2}}. \quad (3.41)$$

Introducing the integration variable

$$x \equiv \frac{j\bar{u}}{p_\infty}, \quad (3.42)$$

this reads

$$\frac{\pi}{2} + \frac{1}{2}\chi(\gamma, j) = \int_0^{x_{\max}(\gamma, j)} \frac{dx}{\sqrt{1 - x^2 + \tilde{w}(x/j, p_\infty)}}, \quad (3.43)$$

where

$$\tilde{w}\left(\frac{x}{j}, p_\infty\right) \equiv \frac{1}{p_\infty^2} [w(\bar{u}, p_\infty)]_{\bar{u} \rightarrow x p_\infty / j}. \quad (3.44)$$

The PM expansion of $w(\bar{u})$ yields the following large- j expansion of $\tilde{w}(x/j, p_\infty)$:

$$\tilde{w}\left(\frac{x}{j}, p_\infty\right) = \tilde{w}_1 \frac{x}{j} + \tilde{w}_2 \frac{x^2}{j^2} + \tilde{w}_3 \frac{x^3}{j^3} + \tilde{w}_4 \frac{x^4}{j^4} + \dots \quad (3.45)$$

where we introduced

$$\begin{aligned} \tilde{w}_1(p_\infty) &= \frac{w_1(p_\infty)}{p_\infty}, \\ \tilde{w}_2(p_\infty) &= w_2(p_\infty), \\ \tilde{w}_3(p_\infty) &= p_\infty w_3(p_\infty), \\ \tilde{w}_4(p_\infty) &= p_\infty^2 w_4(p_\infty). \end{aligned} \quad (3.46)$$

Before doing any calculation, we see from the integral expression (3.43), with the expansion (3.45), that the scattering function $\chi(\gamma, j)$ will only depend on the coefficients

$$\hat{w}_n \equiv \frac{\tilde{w}_n(p_\infty)}{j^n}, \quad (3.47)$$

entering

$$\tilde{w}\left(\frac{x}{j}, p_\infty\right) = \sum_n \hat{w}_n x^n. \quad (3.48)$$

Moreover, as $1/j = O(G)$, the n th order term, $\propto G^n$, in the PM expansion of $\frac{1}{2}\chi(\gamma, j) = \sum_n \chi_n / j^n$ must be a polynomial in the \hat{w}_m 's of total degree $\sum m_i = n$. In other words, the coefficient χ_n of $1/j^n$ must be a polynomial in the \tilde{w}_m 's of total degree $\sum m_i = n$. This trivial remark suffices to prove that all the coefficients $\tilde{w}_n(\gamma)$

are gauge-invariant functions, independent of any canonical transformation (reducing to the identity when $G \rightarrow 0$) acting on the rescaled dynamical variables \mathbf{x} and \mathbf{p} (or on their unrescaled versions \mathbf{X}, \mathbf{P}).

To have more information on the physical meaning of the various gauge-invariant coefficients $\tilde{w}_n(\gamma)$, one needs to explicitly compute the PM (or $1/j$) expansion of the integral expression (3.43). One *a priori* technical difficulty is that if one straightforwardly expands the integral on the right-hand side (rhs) of Eq. (3.43) in powers of G , i.e., in powers of $\tilde{w}(x/j, p_\infty) = O(G)$, one generates formally divergent integrals. In addition, the upper limit of integration (where the expanded integral diverges) depends also on G : $x_{\max}(\gamma, j) = 1 + O(G)$. However, there is a simple way out. It was indeed shown in Ref. [58], that the correct result for such an expanded integral is simply obtained by ignoring the expansion of the upper limit, and by taking the *Hadamard partie finie* (Pf) of the divergent integrals. This yields the expansion

$$\frac{\chi(\gamma, j)}{2} = \sum_{n \geq 1} \text{Pf} \int_0^1 dx \binom{-\frac{1}{2}}{n} (1-x^2)^{-\frac{1}{2}-n} \left[\tilde{w}\left(\frac{x}{j}\right) \right]^n. \quad (3.49)$$

Each integral in this expansion (after reexpanding the n th power of $\tilde{w}(x/j) = \tilde{w}_1 x/j + \tilde{w}_2 x^2/j^2 + \dots$ in powers of $1/j = O(G)$) is an integral of the type

$$\text{Pf} \int_0^1 dx (1-x^2)^{-\frac{1}{2}-n} x^m. \quad (3.50)$$

Replacing, e.g., x by $z^{\frac{1}{2}}$, the latter integral becomes an Euler Beta function (and its Hadamard partie finie is trivially obtained by analytical continuation in the original power $-\frac{1}{2} \rightarrow -\frac{1}{2} + \epsilon$, taking finally $\epsilon \rightarrow 0$). This yields for the coefficients χ_n of the expansion of $\chi/2$ in powers of $1/j$

$$\begin{aligned} \chi_1 &= \frac{1}{2} \tilde{w}_1, \\ \chi_2 &= \frac{\pi}{4} \tilde{w}_2, \\ \chi_3 &= -\frac{1}{24} \tilde{w}_1^3 + \frac{1}{2} \tilde{w}_1 \tilde{w}_2 + \tilde{w}_3, \\ \chi_4 &= \frac{3\pi}{8} \left(\frac{1}{2} \tilde{w}_2^2 + \tilde{w}_1 \tilde{w}_3 + \tilde{w}_4 \right). \end{aligned} \quad (3.51)$$

By inserting in Eqs. (3.51) the definitions (3.46) of the \tilde{w}_n 's one gets the expressions of the χ_n 's in terms of the coefficients w_n of the potential $W(\bar{u})$. Relations equivalent to the latter relations have been also written down to 4PM order in Eq. (11.25) of [25], and to all orders in [55,56].

Then, by inserting in the latter expressions the expressions (3.32) of the w_n 's in terms of the q_n 's, we get the χ_n 's in terms of the coefficients q_n of the EOB Q function. For instance, we get at the 2PM, 3PM and 4PM levels

$$\begin{aligned}
 \chi_2 &= \frac{\pi}{4} \left(\frac{3}{2} (5\gamma^2 - 1) - q_2(\gamma) \right), \\
 \chi_3 &= \frac{-1 + 12p_\infty^2 + 72p_\infty^4 + 64p_\infty^6}{3p_\infty^3} \\
 &\quad - p_\infty \left(q_3(\gamma) + \frac{2\gamma^2 - 1}{\gamma^2 - 1} q_2(\gamma) \right), \\
 \chi_4 &= \frac{105\pi}{128} (16 + 48p_\infty^2 + 33p_\infty^4) \\
 &\quad - \frac{3\pi}{16} [3(4 + 5p_\infty^2)q_2(\gamma) - q_2(\gamma)^2 \\
 &\quad + (4 + 6p_\infty^2)q_3(\gamma) + 2p_\infty^2 q_4(\gamma)], \quad (3.52)
 \end{aligned}$$

where we mixed the use of γ and $p_\infty \equiv \sqrt{\gamma^2 - 1}$. The first two links (at the 2PM and 3PM levels) have already been obtained (by using the Q route) in [20], see Eqs. (5.5), (5.6) and (5.8) there.

We recall that the q_n 's are functions *both* of γ and of the symmetric mass ratio ν , and that $q_n \rightarrow 0$ as $\nu \rightarrow 0$. This implies in particular that the $q_n \rightarrow 0$ limits of the rhs's of the above equations are simply the values χ_n^{Schw} of the χ_n 's for a test particle moving in a Schwarzschild background (as given in Eqs. (3.18)–(3.21) of [20]). Let us also note in passing that, despite the appearance of denominators blowing up at low velocities (when $p_{\text{eob}}^2 \rightarrow 0$, i.e., $\gamma^2 \rightarrow 1$) in some of the expressions we will give below for them, the functions $q_n(\gamma, \nu)$ are all regular as $p_{\text{eob}}^2 \rightarrow 0$.

D. Summary of the current knowledge of the PM-expanded dynamics

The above-derived links between χ_n , q_n and w_n can be used in various ways. In particular, if one has derived the scattering coefficients χ_n up to some PM level, one can directly deduce from them the values of the corresponding q_n 's and w_n 's. This the way Refs. [20,46] derived the values of the q_n 's and w_n 's at the 1PM and 2PM levels. Let us summarize these results here.

$$\chi_1(\gamma, \nu) = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}} = \chi_1^{\text{Schw}}(\gamma), \quad (3.53)$$

$$\chi_2(\gamma, \nu) = \frac{3\pi(5\gamma^2 - 1)}{8h(\gamma, \nu)} = \frac{\chi_2^{\text{Schw}}(\gamma)}{h(\gamma, \nu)}, \quad (3.54)$$

$$q_1(\gamma, \nu) = 0, \quad (3.55)$$

$$q_2(\gamma, \nu) = \frac{3}{2}(5\gamma^2 - 1) \left[1 - \frac{1}{h(\gamma, \nu)} \right], \quad (3.56)$$

$$w_1(\gamma, \nu) = 2(2\gamma^2 - 1), \quad (3.57)$$

$$w_2(\gamma, \nu) = \frac{3(5\gamma^2 - 1)}{2h(\gamma, \nu)}. \quad (3.58)$$

Concerning the 3PM level, we have seen above that it depends on the knowledge of a *single* function of γ , entering as the coefficient of $1/h^2(\gamma, \nu)$ in $\chi_3(\gamma, \nu) - \chi_3(\gamma, 0)$. Let us define the auxiliary function $B(\gamma)$ as

$$B(\gamma) \equiv \frac{3(2\gamma^2 - 1)(5\gamma^2 - 1)}{2(\gamma^2 - 1)}, \quad (3.59)$$

and introduce two other functions of γ , $A(\gamma)$ and $C(\gamma)$, constrained to identically satisfy

$$A(\gamma) + B(\gamma) + C(\gamma) \equiv 0. \quad (3.60)$$

With this notation (and $p_\infty \equiv p_{\text{eob}} \equiv \sqrt{\gamma^2 - 1}$), our results above give the following structural information at the 3PM level

$$\chi_3(\gamma, \nu) = \chi_3^{\text{Schw}}(\gamma) - p_\infty(A(\gamma) + B(\gamma)) \left(1 - \frac{1}{h^2(\gamma, \nu)} \right), \quad (3.61)$$

$$q_3(\gamma, \nu) = A(\gamma) + \frac{B(\gamma)}{h(\gamma, \nu)} + \frac{C(\gamma)}{h^2(\gamma, \nu)}, \quad (3.62)$$

$$w_3(\gamma, \nu) = 9\gamma^2 - \frac{1}{2} - q_3(\gamma, \nu). \quad (3.63)$$

If we further introduce the notation

$$\bar{C}(\gamma) \equiv (\gamma - 1)(A(\gamma) + B(\gamma)) = -(\gamma - 1)C(\gamma), \quad (3.64)$$

we can rewrite Eq. (3.61) as

$$\chi_3(\gamma, \nu) = \chi_3^{\text{Schw}}(\gamma) - \frac{2\nu p_\infty}{h^2(\gamma, \nu)} \bar{C}(\gamma), \quad (3.65)$$

and Eq. (3.62) as

$$q_3(\gamma, \nu) = B(\gamma) \left(\frac{1}{h(\gamma, \nu)} - 1 \right) + \frac{2\nu \bar{C}(\gamma)}{h^2(\gamma, \nu)}. \quad (3.66)$$

This shows that the univariate function $\bar{C}(\gamma)$ directly parametrizes the bivariate 3PM scattering coefficient $\chi_3(\gamma, \nu)$ via the expression

$$2\nu p_\infty \bar{C}(\gamma) = -h^2(\gamma, \nu) (\chi_3(\gamma, \nu) - \chi_3^{\text{Schw}}(\gamma)). \quad (3.67)$$

Let us now discuss what is our current secure (i.e., cross-checked by at least two independent calculations) knowledge of $\chi_3(\gamma, \nu)$, and therefore of the function $\bar{C}(\gamma)$. From the $O(G^3)$ term in the 4PN-accurate expression of the scattering angle derived in Ref. [59], one can straightforwardly derive the following 4PN-accurate value of the function $\bar{C}(\gamma)$ (expanded in powers of $p_\infty = p_{\text{eob}}$):

$$\bar{C}^{4\text{PN}}(p_{\text{eob}}) = 4 + 18p_{\infty}^2 + \frac{91}{10}p_{\infty}^4 + O(p_{\infty}^6). \quad (3.68)$$

Recently, a new (purely classical) method [52] allowed one to compute the 5PN-level term in the $O(G^3)$ scattering angle, with the result

$$\bar{C}^{5\text{PN}}(p_{\text{eob}}) = 4 + 18p_{\infty}^2 + \frac{91}{10}p_{\infty}^4 - \frac{69}{140}p_{\infty}^6 + O(p_{\infty}^8). \quad (3.69)$$

On the other hand, the quantum-amplitude approach of Refs. [24,25] resulted in the computation of a classical value for $\chi_3(\gamma, \nu)$ (see Eq. (11.32) of Ref. [25], and Ref. [51]), from which one can derive the following value of the function $\bar{C}(\gamma)$:

$$\begin{aligned} \bar{C}^B(\gamma) &= \frac{2}{3}\gamma(14\gamma^2 + 25) \\ &+ 4(4\gamma^4 - 12\gamma^2 - 3) \frac{\text{as}(\gamma)}{\sqrt{\gamma^2 - 1}}, \end{aligned} \quad (3.70)$$

where we used the shorthand notation

$$\text{as}(\gamma) \equiv \text{arcsinh} \sqrt{\frac{\gamma - 1}{2}}. \quad (3.71)$$

Note in passing that the expression obtained by inserting Eq. (3.70) in the above formula for χ_3 is simpler than (though equivalent to) Eq. (11.32) of Ref. [25]. In particular, the $a + b/h^2$ structure of χ_3 is present (though somewhat hidden) in their Eq. (11.32).

Let us also note, for future use, other (simpler) forms of the arcsinh function, namely

$$\text{as}(\gamma) = \frac{1}{2} \ln(\gamma + p_{\infty}) = -\frac{1}{2} \ln(\gamma - p_{\infty}), \quad (3.72)$$

where we recall that $p_{\infty} \equiv \sqrt{\gamma^2 - 1}$, and

$$\text{as}(\gamma) = \frac{1}{4} \ln \frac{\gamma + p_{\infty}}{\gamma - p_{\infty}} = \frac{1}{4} \ln \frac{1 + v_{\infty}}{1 - v_{\infty}}. \quad (3.73)$$

Here v_{∞} denotes the (Lorentz-invariant) asymptotic relative velocity between the two bodies

$$v_{\infty} \equiv \frac{p_{\infty}}{\gamma} \equiv \sqrt{1 - \frac{1}{\gamma^2}} \text{ such that } \gamma = \frac{1}{\sqrt{1 - v_{\infty}^2}}. \quad (3.74)$$

Note that in the slow-velocity limit ($\gamma \rightarrow 1$, or $p_{\infty} \rightarrow 0$)

$$\text{as}(\gamma) = \frac{1}{2}p_{\infty} - \frac{1}{12}p_{\infty}^3 + \frac{3}{80}p_{\infty}^5 - \frac{5}{224}p_{\infty}^7 + \dots \quad (3.75)$$

so that the ratio $\text{as}(\gamma)/\sqrt{\gamma^2 - 1} = \text{as}(\gamma)/p_{\infty}$ entering $\bar{C}^B(\gamma)$ has a smooth slow-velocity limit

$$\frac{\text{as}(\gamma)}{p_{\infty}} = \frac{1}{2} - \frac{1}{12}p_{\infty}^2 + \frac{3}{80}p_{\infty}^4 - \frac{5}{224}p_{\infty}^6 + \dots \quad (3.76)$$

and is an even function of p_{∞} .

As we shall discuss below, the high-energy ($\gamma \rightarrow \infty$) behavior of the expression (3.70) seems, at face value, to be in contradiction with the high-energy behavior found in the SF computation of Ref. [53]. The origin of this tension lies in the fact that the high-energy (HE) behavior of the $\text{as}(\gamma)$ function is

$$\text{as}(\gamma) \stackrel{\text{HE}}{\equiv} \frac{1}{2} \ln(2\gamma), \quad (3.77)$$

so that the leading-order term in the high-energy behavior of the corresponding q_3 potential is

$$q_3^B(\gamma, \nu) \stackrel{\text{HE}}{\equiv} 8\gamma^2 \ln(2\gamma). \quad (3.78)$$

By contrast, Ref. [20] (see Eq. (6.8) there) had suggested that all EOB coefficients $q_n(\gamma, \nu)$ should have a logarithm-free high-energy behavior of the type

$$q_n(\gamma, \nu) \stackrel{\text{HE}}{\equiv} c_n^{(q)} \gamma^2, \quad (3.79)$$

with a ν -independent coefficient $c_n^{(q)}$. The latter high-energy behavior was suggested by several independent arguments, and notably because of its direct compatibility with the high-energy behavior of the SF-expanded EOB Hamiltonian found in Ref. [53]. We shall further discuss below the relation between the high-energy behavior of $q_3^B(\gamma, \nu)$ and that of the SF-expanded EOB Hamiltonian and suggest several ways of relieving the tension between the result (3.78), derived from Refs. [24,25], and the result of Ref. [53]. We shall also emphasize the importance of 6PN-accurate $O(G^3)$ computations to discriminate between various possible ways of relieving the latter tension.

IV. MAP BETWEEN THE 3PM EOB POTENTIAL AND THE QUANTUM SCATTERING AMPLITUDE

A. Prelude: Quasiclassical scattering amplitude associated with the classical scattering function

As a prelude to our discussion of the link between the quantum scattering amplitude and the classical dynamics, let us mention a direct way of using the scattering function $\frac{1}{2}\chi(\hat{\mathcal{E}}_{\text{eff}}, j)$ for constructing the quasiclassical (Wentzel-Kramers-Brillouin) approximation to the quantum scattering amplitude.

Let us start by clarifying the notation we shall use for the scattering amplitude \mathcal{M} . The Lorentz-invariant amplitude \mathcal{M} is defined from the two-body scattering matrix by

$$\begin{aligned} & \langle p'_1 p'_2 | S | p_1 p_2 \rangle \\ & = \text{Identity} + i(2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \frac{\mathcal{M}}{N}, \end{aligned} \quad (4.1)$$

with the normalization factor $N = (2E_1)^{1/2}(2E_2)^{1/2} \times (2E'_1)^{1/2}(2E'_2)^{1/2}$ when using the state normalization $\langle p' | p \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')$. With this definition, \mathcal{M} is dimensionless.

Starting from the dimensionless Lorentz-invariant amplitude $\mathcal{M}(s, t)$, it is convenient to introduce the associated amplitude $f_R(\theta)$ defined as

$$\mathcal{M} \equiv 8\pi \frac{s^{1/2}}{\hbar} f_R(\theta). \quad (4.2)$$

The amplitude $f_R(\theta)$ has the dimension of a length, and is related to the differential c.m. cross-section via $d\sigma = |f_R(\theta)|^2 d\Omega_{\text{c.m.}}$. Let us then consider the partial-wave expansion of the amplitude, written as

$$f_R(\theta) = \frac{\hbar}{P_{\text{c.m.}}} \sum_{l=0}^{\infty} (2l+1) \frac{e^{2i\delta_l} - 1}{2i} P_l(\cos\theta). \quad (4.3)$$

Here θ denotes the c.m. scattering angle, and $P_{\text{c.m.}}$ the c.m. momentum, related to the Mandelstam invariant $s = (E_{\text{c.m.}}^{\text{tot}})^2 = (E_1^{\text{c.m.}} + E_2^{\text{c.m.}})^2$, with $E_1^{\text{c.m.}} = \sqrt{m_1^2 + P_{\text{c.m.}}^2}$, $E_2^{\text{c.m.}} = \sqrt{m_2^2 + P_{\text{c.m.}}^2}$. The angle θ is related to the second Mandelstam invariant $t = -Q_{\text{c.m.}}^2$ via

$$\sqrt{-t} = Q_{\text{c.m.}} = 2 \sin \frac{\theta}{2} P_{\text{c.m.}}. \quad (4.4)$$

In the expansion (4.3), δ_l denotes the (dimensionless) phase shift of the partial wave corresponding to the c.m. angular momentum $L = \hbar l$, where $l = 0, 1, 2, \dots$. In the classical limit we can identify the quantized total c.m. angular momentum $L = \hbar l$ with J . In terms of the dimensionless quantities l and δ_l entering the expansion (4.3), a quasiclassical description of the dynamics *a priori* corresponds to a case where both of them are large: $l \gg 1$ and $\delta_l \gg 1$. This is formally clear because $l = L/\hbar$, while, for potential scattering, the quasiclassical (Wentzel-Kramers-Brillouin) approximation to the phase shift is $\delta_l \approx \Delta S_L/\hbar$ where ΔS_L is the (subtracted) half-radial action along a classical motion with angular momentum L [60,61]. Most useful for our present purpose is the fact that the phase-shift δ_l is linked, in the classical limit, to the scattering angle χ by

$$\frac{1}{2}\chi = -\frac{\partial \delta_l}{\partial l}. \quad (4.5)$$

When expressing $l \equiv L/\hbar \equiv J/\hbar$ in terms of the classical dimensionless angular momentum $j \equiv J/(Gm_1 m_2)$, the latter result reads

$$\frac{1}{2}\chi(\hat{\mathcal{E}}_{\text{eff}}, j) = -\hat{\hbar} \frac{\partial \delta_l}{\partial j}, \quad (4.6)$$

where we defined (as in [20]) the following dimensionless version of \hbar

$$\hat{\hbar} \equiv \frac{\hbar}{Gm_1 m_2} = \frac{\hbar}{GM\mu}. \quad (4.7)$$

Equation (4.6) shows that δ_l can be obtained (in the classical limit) by integrating over j the classical scattering function $\frac{1}{2}\chi(\hat{\mathcal{E}}_{\text{eff}}, j)$. Using the PM-expansion (1.4) of $\frac{1}{2}\chi(j)$ (and $\hat{\mathcal{E}}_{\text{eff}} = \gamma$), then yields the following expansion for δ_l

$$\delta_l = \frac{1}{\hat{\hbar}} \left(\chi_1(\gamma, \nu) \ln \left(\frac{j_0}{j} \right) + \frac{\chi_2(\gamma, \nu)}{j} + \frac{1}{2} \frac{\chi_3(\gamma, \nu)}{j^2} + \dots \right), \quad (4.8)$$

where j_0 is linked to the IR cutoff needed when evaluating the corresponding IR-divergent Coulomb phase.

B. Computation of the quantum scattering amplitude derived from the 3PM EOB potential

Reference [20] had shown how to map the simple 2PM-accurate, energy-gauge EOB description of the two-body dynamics onto a corresponding quantum scattering amplitude, say $\mathcal{M}_{\text{eob}}^{2\text{PM}}$, and had checked that $\mathcal{M}_{\text{eob}}^{2\text{PM}}$ agreed with what Refs. [18,19] (later followed by Refs. [21,23]) had computed as being the ‘‘classical part’’ of the G^2 -accurate quantum scattering amplitude. In this section we extend this result to the 3PM level. More precisely, we shall show that the extension of the map defined in Ref. [20] leads to a 3PM-accurate amplitude, $\mathcal{M}_{\text{eob}}^{3\text{PM}}$, that coincides with what Refs. [24,25] computed as being the classical part of the G^3 -accurate quantum scattering amplitude.

Let us start by recalling that the approach of Ref. [20] is simply to quantize the classical, energy-gauge EOB mass-shell condition, i.e., to quantize the motion of a particle of mass μ moving in a nonrelativisticlike radial potential. Indeed, the energy-gauge EOB mass-shell condition has the form

$$\mathbf{P}^2 = P_\infty^2 + W(R, P_\infty), \quad (4.9)$$

where

$$P_\infty^2 \equiv \mathcal{E}_{\text{eff}}^2 - \mu^2 = \mu^2(\gamma^2 - 1), \quad (4.10)$$

and where, to ease the notation, we henceforth suppress the bar over the isotropic EOB radial coordinate $R = |\mathbf{X}|$ (and its rescaled avatar $r = R/(GM) = \bar{r}$).

The canonical quantization of \mathbf{X} and \mathbf{P} , i.e.,

$$[X^i, P_j] = i\hbar \delta_j^i, \quad (4.11)$$

is equivalent to solving the fixed-energy Schrödinger equation in the energy-dependent radial potential $W(R, P_\infty)$. As in the classical problem, it is convenient to replace the canonically conjugated variables \mathbf{X}, \mathbf{P} by their (dimensionless) rescaled avatars $\mathbf{x} \equiv \mathbf{X}/(GM)$ and $\mathbf{p} \equiv \mathbf{P}/\mu$ (with $r \equiv |\mathbf{x}|$), satisfying the following rescaled commutation relation:

$$[x^i, p_j] = i\hat{h}\delta_j^i. \quad (4.12)$$

Here (following [20]) \hat{h} denotes the (dimensionless) rescaled version of \hbar defined in Eq. (4.7). In terms of these rescaled variables the mass-shell condition determining \mathbf{p} reads

$$\mathbf{p}^2 = p_\infty^2 + w(r, p_\infty), \quad (4.13)$$

where, as we have seen, the PM-expansion of the rescaled radial potential $w \equiv W/\mu^2$ reads

$$w(r, p_\infty) = \frac{w_1(\gamma)}{r} + \frac{w_2(\gamma)}{r^2} + \frac{w_3(\gamma)}{r^3} + \frac{w_4(\gamma)}{r^4} + \dots \quad (4.14)$$

One should keep in mind that, as $\frac{1}{r} = \frac{GM}{R}$, a contribution to the potential $\propto 1/r^n$ is of order $O(G^n)$.

The quantization of the EOB mass-shell condition (4.13) yields the following time-independent Schrödinger equation (here truncated at the 3PM level)

$$-\hat{h}^2 \Delta_{\mathbf{x}} \psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{r} + \frac{w_2}{r^2} + \frac{w_3}{r^3} + O\left(\frac{1}{r^4}\right) \right] \psi(\mathbf{x}). \quad (4.15)$$

In other words (as was already pointed out in [20,54]), the quantization of the isotropic-coordinates formulation of the EOB dynamics of two spinless particles leads to a potential scattering, with an energy-dependent potential which is a deformation of a Coulomb potential $\frac{w_1}{r}$ by higher inverse powers of $r \equiv \bar{r}$: $\frac{w_2}{r^2} + \frac{w_3}{r^3} + \dots$.

Given an incoming state $|\mathbf{k}_a\rangle = \varphi_a = e^{i\mathbf{k}_a \cdot \mathbf{x}}$ in the infinite past, impinging on this EOB-potential w , the scattering amplitude $f_{\text{eob}}(\hat{\mathbf{k}}_b)$ (where $\hat{\mathbf{k}}_b = \mathbf{k}_b/|\mathbf{k}_b|$) from $|\mathbf{k}_a\rangle$ to some outgoing state $|\mathbf{k}_b\rangle = \varphi_b = e^{i\mathbf{k}_b \cdot \mathbf{x}}$ is given by

$$f_{\text{eob}}(\hat{\mathbf{k}}_b) = + \frac{1}{4\pi\hat{h}^2} \langle \varphi_b | w | \varphi_a^+ \rangle. \quad (4.16)$$

Here ψ_a^+ is the stationary retarded-type solution of the scattering equation (4.15) describing the incoming state $|\mathbf{k}_a\rangle = \varphi_a = e^{i\mathbf{k}_a \cdot \mathbf{x}}$ in the infinite past, and having the following asymptotic structure at large distances

$$\psi_a^+ \underset{r \rightarrow \infty}{\approx} e^{i\mathbf{k}_a \cdot \mathbf{x}} + f_{\text{eob}}(\Omega) \frac{e^{ikr}}{r}, \quad (4.17)$$

where Ω denotes the polar coordinates of \mathbf{x} on the sphere of scattering directions.

The crucial point of Ref. [20] was that, modulo a simple rescaling, namely (see below)

$$\mathcal{M}_{\text{eob}} = \frac{8\pi G}{\hbar} (E_{\text{real}}^{\text{c.m.}})^2 f_{\text{eob}} = \frac{8\pi Gs}{\hbar} f_{\text{eob}}, \quad (4.18)$$

the EOB scattering amplitude could be identified, at the then existing $O(G^2)$ approximation, with the so-called classical part [18,19] of the quantum gravity amplitude \mathcal{M} . When rewriting Eq. (4.18) in terms of the corresponding “nonrelativistically normalized” amplitude, say M^{NR} , as used in Refs. [22,24,25], we have

$$M_{\text{eob}}^{\text{NR}} \equiv \frac{\mathcal{M}_{\text{eob}}}{4E_1^\infty E_2^\infty} = \frac{2\pi G}{\hbar \xi_\infty} f_{\text{eob}}, \quad (4.19)$$

where $\xi_\infty = E_1^\infty E_2^\infty / (E_1^\infty + E_2^\infty)^2$ is the asymptotic value of the symmetric energy ratio ξ defined in [22] (see also Eq. (A14) below).

In the dictionary of Ref. [20], the EOB scattering angle θ between $\hat{\mathbf{k}}_a$ and $\hat{\mathbf{k}}_b$ is directly equal to the physical c.m. scattering angle, as it enters the physical c.m. momentum transfer

$$\sqrt{-t} = Q^{\text{c.m.}} = 2 \sin \frac{\theta}{2} P_{\text{c.m.}}. \quad (4.20)$$

This is the quantum version of the fact, proven in Ref. [46], that the classical EOB scattering angle coincides with the corresponding c.m. scattering angle. On the other hand, one must remember that the various momenta and wave vectors, $p_\infty = \sqrt{\gamma^2 - 1}$, \mathbf{k}_a , \mathbf{k}_b , \mathbf{q} , entering the EOB description differ by some rescaling factors from the corresponding physical c.m. ones. First, the link between $p_{\text{eob}} \equiv p_\infty = \sqrt{\gamma^2 - 1}$ and the physical c.m. momentum is

$$P_\infty^{\text{EOB}} \equiv \mu p_\infty = \frac{E_{\text{real}}}{M} P_{\text{c.m.}} = h(\gamma) P_{\text{c.m.}}. \quad (4.21)$$

In addition, the conserved norm of the (rescaled) wave vector, $k = |\mathbf{k}_a| = |\mathbf{k}_b|$, is related to $p_\infty = \sqrt{\gamma^2 - 1}$ via

$$p_\infty = \hat{h}k, \quad (4.22)$$

so that the rescaled momentum transfer reads

$$\mathbf{q} = \mathbf{k}_b - \mathbf{k}_a; \quad q = |\mathbf{q}| = 2k \sin \frac{\theta}{2}. \quad (4.23)$$

As a consequence of these relations, we have the link

$$\begin{aligned}
 q &= 2 \sin \frac{\theta}{2} \frac{p_\infty}{\hbar} = 2 \sin \frac{\theta}{2} \frac{h(\gamma) P_{\text{c.m.}}}{\mu \hat{\hbar}} \\
 &= \frac{GM}{\hbar} h(\gamma) Q^{\text{c.m.}}. \quad (4.24)
 \end{aligned}$$

Rewriting the link (4.18) in terms of the relativistic (partial-wave) amplitude f_R , defined by Eq. (4.2), leads to the following relation between f_R and f_{eob} :

$$f_R = G\sqrt{s}f_{\text{eob}}. \quad (4.25)$$

Note that while f_R has the dimension of a length, f_{eob} is dimensionless. The partial-wave expansion of f_{eob} is, in close parallel to Eq. (4.3),

$$f_{\text{eob}}(\theta) = \frac{\hat{\hbar}}{p_\infty} \sum_{l=0}^{\infty} (2l+1) \frac{e^{2i\delta_l} - 1}{2i} P_l(\cos \theta), \quad (4.26)$$

with the same phase shifts, but a prefactor $\frac{\hat{\hbar}}{p_\infty} = \frac{1}{k}$ which is dimensionless, because of our various rescalings. At the conceptual level, the relative normalization factor given in Eq. (4.18) is most clearly understood by saying that the pure phase-shift dimensionless factor of the real amplitude \mathcal{M} , say

$$\hat{f}(\theta) \equiv \sum_{l=0}^{\infty} (2l+1) \frac{e^{2i\delta_l} - 1}{2i} P_l(\cos \theta), \quad (4.27)$$

coincides with the corresponding EOB one. An alternative way [20] to derive the relative normalization between \mathcal{M} and f_{eob} is to compare the LO value, (4.64), of \mathcal{M} to the corresponding LO value, $w_1/(\hat{\hbar}^2 q^2)$, of f_{eob} , as given in Eq. (10.23) of [20], and below.

Let us now derive the 3PM-accurate value of the EOB scattering amplitude f_{eob} , and compare it to the result of Refs. [24,25]. It can be written as

$$\mathcal{M}_{\text{eob}}^{\text{3PM}} = \mathcal{M}'_{\text{eob}} + \mathcal{M}''_{\text{eob}}, \quad (4.28)$$

where

$$\mathcal{M}'_{\text{eob}} \equiv \frac{8\pi Gs}{\hbar} f_{\text{eob}}^w, \quad (4.29)$$

denotes the first Born approximation to f_{eob} (which is *linear* in the potential w), while

$$\mathcal{M}''_{\text{eob}} \equiv \frac{8\pi Gs}{\hbar} f_{\text{eob}}^{w^2+w^3+\dots}, \quad (4.30)$$

denotes the sum of the terms coming from higher order Born iterations (which are *nonlinear* in the potential w).

The explicit form of the first Born approximation to f_{eob} is defined by replacing in Eq. (4.16) ψ_a^+ by the unperturbed state $\varphi_a = e^{i\mathbf{k}_a \cdot \mathbf{x}}$:

$$\begin{aligned}
 f_{\text{eob}}^w(\mathbf{q}) &= + \frac{1}{4\pi \hat{\hbar}^2} \langle \varphi_b | w(r) | \varphi_a \rangle \\
 &= + \frac{1}{4\pi \hat{\hbar}^2} \int d^3 \mathbf{x} e^{-i\mathbf{q} \cdot \mathbf{x}} w(r). \quad (4.31)
 \end{aligned}$$

We recall that the EOB potential, $w(r)$, Eq. (4.14), is a sum of contributions $\sum_n w_n/r^n$ coming from successive PM approximations, i.e., $w_n/r^n = O(G^n)$. This generates a corresponding sum of contributions in the first Born approximation (4.31), namely

$$f_{\text{eob}}^w(\mathbf{q}) = \sum_n f_{\text{eob}}^{w_n}(\mathbf{q}), \quad (4.32)$$

with

$$\begin{aligned}
 f_{\text{eob}}^{w_n}(\mathbf{q}) &= + \frac{1}{4\pi \hat{\hbar}^2} \langle \varphi_b | \frac{w_n}{r^n} | \varphi_a \rangle \\
 &= + \frac{1}{4\pi \hat{\hbar}^2} \int d^3 \mathbf{x} e^{-i\mathbf{q} \cdot \mathbf{x}} \frac{w_n}{r^n}. \quad (4.33)
 \end{aligned}$$

This is easily computed from the value of the Fourier transform of $1/r^n$, which is (in space dimension d)

$$\mathcal{F}^{(d)} \left[\frac{1}{r^n} \right] \equiv \int d^d \mathbf{x} e^{-i\mathbf{q} \cdot \mathbf{x}} \frac{1}{r^n} = \frac{C_n^{(d)}}{q^{d-n}}, \quad (4.34)$$

where

$$C_n^{(d)} = \pi^{\frac{d}{2}} \frac{2^{\bar{n}} \Gamma(\frac{1}{2} \bar{n})}{\Gamma(\frac{1}{2} n)}; \quad \text{with } \bar{n} \equiv d - n. \quad (4.35)$$

The Fourier transforms of the $1/r$ (1PM) and $1/r^2$ (2PM) potentials are convergent in dimension $d = 3$,

$$\mathcal{F}^{(3)} \left[\frac{1}{r} \right] = \frac{4\pi}{q^2}; \quad \mathcal{F}^{(3)} \left[\frac{1}{r^2} \right] = \frac{2\pi^2}{q}, \quad (4.36)$$

while the 3PM-level $1/r^3$ potential leads to a UV ($r \rightarrow 0$) divergence whose dimensional regularization ($d = 3 + \epsilon$) yields the result:

$$\begin{aligned}
 \mathcal{F}^{(3+\epsilon)} \left[\frac{1}{r^3} \right] &= 4\pi \left[\frac{1}{\epsilon} - \ln q + \frac{1}{2} \ln(4\pi) - \frac{1}{2} \gamma_E + O(\epsilon) \right] \\
 &\equiv -4\pi \ln \frac{q}{\hat{\Lambda}}, \quad (4.37)
 \end{aligned}$$

where, in the last line, we denoted by $\hat{\Lambda}$ a UV cutoff (in its EOB-rescaled version). This yields

$$\hat{\hbar}^2 f_{\text{eob}}^w = \frac{w_1}{q^2} + \frac{\pi w_2}{2q} - w_3 \ln \frac{q}{\hat{\Lambda}}. \quad (4.38)$$

When inserting in this result the values of w_1 and w_2 derived in [20], and the value of w_3 obtained by inserting

(3.70) in Eqs. (3.62), (3.63), and using the above-defined rescalings, it is straightforwardly checked that this yields

$$\mathcal{M}'_{\text{eob}} \equiv \frac{8\pi Gs}{\hbar} f_{\text{eob}}^w = \mathcal{M}'_1 + \mathcal{M}'_2 + \mathcal{M}'_3, \quad (4.39)$$

where, following the notation used in Refs. [24,25], \mathcal{M}'_i , $i = 1, 2, 3$, denote the IR-finite parts of the classical part of the amplitude \mathcal{M} derived there (written in Eqs. (13) and the first three lines of Eq. (8) in [24]). We work here with the Lorentz-invariant amplitude \mathcal{M} , i.e., we do not include the factor $(4E_1^\infty E_2^\infty)^{-1}$. [At the technical level, Eq. (4.39) means that, at the 3PM level, the EOB potential coefficient w_3 can be simply identified with $-1/(6h^2(\gamma, \nu))$ times the bracket $[3 - 6\nu + 206\nu\sigma + \dots]$ multiplying $\log \mathbf{q}^2$ in Eq. (8) of [25].]

The latter simple link between the Fourier transform of the EOB energy potential and the IR-finite part of the classical part of the amplitude \mathcal{M} of Refs. [24,25] has also been pointed out in recent works [55,56], however, we wish to emphasize that it is in great part tautological (in the sense that it follows from definitions). Indeed, on the one hand (as clearly recognized in Ref. [56]) the EOB formulation [20] of the map between the classical dynamics and the amplitude \mathcal{M} trivially shows that the linear-in-potential part of \mathcal{M} is simply given by the Fourier transform of the EOB energy-gauge potential (as was explicitly explained in several talks [54]), and, on the other hand Refs. [24,25] are actually *defining* \mathcal{M}'_i by selecting the parts of the total two-loop amplitude which satisfy two criteria: (i) to correspond to the $\sim G/q^2$, $\sim G^2/q$ and $G^3 \ln q$ terms that are precisely corresponding to the classical dynamics; and, (ii) to have been amputated of the extra contributions coming from iterated Born approximations of the type denoted $\mathcal{M}''_{\text{eob}} \equiv \frac{8\pi Gs}{\hbar} f_{\text{eob}}^{w^2+w^3+\dots}$ above. Indeed, as is stated in Ref. [25], and as we shall now check, the latter terms are precisely the IR-divergent contributions left in the form of integrals in Eq. (9.3) of Ref. [25]. In other words, given the simple EOB map of Ref. [20], and given the methodology of extracting the so-called classical part of \mathcal{M} proposed in [22], and implemented in [24,25], the apparently striking result (4.39) is a tautology.

Let us now discuss the detailed structure of the iterated Born approximations $\mathcal{M}''_{\text{eob}} \equiv \frac{8\pi Gs}{\hbar} f_{\text{eob}}^{w^2+w^3+\dots}$ that must be added to the linear-in-potential contribution $\mathcal{M}'_{\text{eob}} \equiv \frac{8\pi Gs}{\hbar} f_{\text{eob}}^w$. As $w_n = O(G^n)$, the 3PM ($O(G^3)$) accuracy necessitates to consider both the second iteration (with contributions proportional to w_1^2 and $w_1 w_2$), and the third iteration (with contributions proportional to w_1^3). [The 3PM-level contribution coming from w_3/r^3 is included in the first Born approximation, and does not need to be iterated.] The iterations of the Coulomb-type w_1/r potential can actually be deduced from the known, exact Coulomb scattering amplitude [60]. Alternatively, one can extract both the first two

iterations of the w_1/r potential ($O(w_1^2) + O(w_1^3)$) and the mixed iteration of the w_1/r and w_2/r^2 potentials ($O(w_1 w_2)$) from an old result of Kang and Brown [62]. Indeed, the latter reference computed the higher-Born approximations for the Coulomb scattering amplitude of a Klein-Gordon particle, i.e., for the wave equation

$$-\hbar^2 \Delta \psi = \left[\left(E - \frac{Ze^2}{r} \right)^2 - \mu^2 \right] \psi, \quad (4.40)$$

whose potential involves both a $w_1^{KG}/r = -2EZe^2/r$ potential and a $w_2^{KG}/r^2 = +(Ze^2)^2/r^2$ one.

Transcribing the results of Ref. [62] in terms of our scattering equation (4.15) yields explicit forms for the various Born-iterated contributions. We introduce the notation

$$\delta_1 = \frac{i}{2} \frac{w_1}{\hbar^2 k} \ln \frac{q^2}{\hat{\lambda}^2} = \frac{i}{2} \frac{w_1}{\hbar p_\infty} \ln \frac{q^2}{\hat{\lambda}^2}, \quad (4.41)$$

for the IR-divergent Coulomb phase ($\hat{\lambda}$ being an IR cutoff introduced by the replacement $Ze^2/r \rightarrow Ze^2 e^{-\hat{\lambda}r}/r$ in the Klein-Gordon equation (4.40)). [Note the fact that δ_1 contains a factor $1/\hbar$. This crucial property of the Born expansion will be discussed at length in the following subsection, starting with Eq. (4.54).]

$$\hat{\hbar}^2 f_{\text{eob}}^{w_1^2} = \delta_1 \frac{w_1}{q^2}, \quad (4.42)$$

$$\hat{\hbar}^2 f_{\text{eob}}^{w_1^3} = \frac{1}{2} \delta_1^2 \frac{w_1}{q^2}, \quad (4.43)$$

$$\hat{\hbar}^2 f_{\text{eob}}^{w_1 w_2} = \delta_1 \frac{\pi w_2}{2 q} + \frac{w_1 w_2}{\hat{\hbar}^2 q^2} x B_{29}(x). \quad (4.44)$$

Here the variable x denotes

$$x \equiv \sin \frac{\theta}{2} = \frac{q}{2k}, \quad (4.45)$$

and the function $B_{29}(x)$ denotes (see the last bracket in Eq. (29) of [62])

$$B_{29}(x) = i\pi \ln \frac{4}{(1+x)} + \ln x \ln \frac{1-x}{1+x} + L_2(x) - L_2(-x), \quad (4.46)$$

where

$$L_2(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \dots \quad (4.47)$$

is the dilogarithm function. All the above iterated contributions are clearly IR divergent because they all contain a

term proportional to the IR-divergent Coulomblike phase δ_1 .

Adding all those iterated Born contributions to the first-Born approximation $\hat{h}^2 f_{\text{eob}}^w$, Eq. (4.38), yields the complete 3PM-accurate EOB amplitude

$$\begin{aligned} \hat{h}^2 f_{\text{eob}} = & \left(1 + \delta_1 + \frac{1}{2}\delta_1^2\right) \frac{w_1}{q^2} + (1 + \delta_1) \frac{\pi w_2}{2q} \\ & + \frac{w_1 w_2}{\hat{h}^2 q^2} x B_{29}(x) - w_3 \ln \frac{q}{\Lambda}. \end{aligned} \quad (4.48)$$

Let us note in passing that the 3PM-expanded amplitude (4.48) is compatible with the fact (proven by Weinberg [63]) that the (gravitational) IR-divergent Coulomb phase δ_1 exponentiates, i.e., that one can factorize f_{eob} as

$$\hat{h}^2 f_{\text{eob}} = e^{\delta_1} \left[\frac{w_1}{q^2} + \frac{\pi w_2}{2q} + \frac{w_1 w_2}{\hat{h}^2 q^2} x B_{29}(x) - w_3 \ln \frac{q}{\Lambda} \right] + O(G^4) \quad (4.49)$$

where the terms within the square brackets are IR-finite.

As already explained, the methodology used in [24,25] consists of setting aside the various IR-divergent (Born-iterated) contributions (4.42), (4.43), (4.44), in (4.48), thereby retaining only the linear-in- w ones. This means in particular that Refs. [24,25] set aside not only the IR-divergent term proportional to $\delta_1 w_2$, but also its Born-iterated partner $\propto w_1 w_2$ (recall that $\delta_1 \propto w_1$). They then considered as only IR-finite $O(G^3)$ contribution the last term (proportional to $\ln q$) in Eq. (4.48), namely

$$-w_3 \ln \frac{q}{\Lambda}. \quad (4.50)$$

As we shall discuss next, a different IR-finite result would have been obtained if one had (following Weinberg) first factored e^{δ_1} , and then taken the small- q limit.

Let us, indeed, discuss the small-angle limit, $q \rightarrow 0$, and therefore $x \rightarrow 0$, of the complete 3PM EOB amplitude (4.48). We have the expansion

$$\begin{aligned} x B_{29}(x) = & i\pi(x \ln 4 - x^2 + O(x^3)) \\ & + \ln x(-2x^2 + O(x^4)) + 2x^2 + O(x^4). \end{aligned} \quad (4.51)$$

Here, the leading term $O(x)$ in the imaginary part modifies the Coulomb phase factor $(1 + \delta_1)$ in front of the $w_2/q \propto w_2/x$ term. The terms $O(x^2)$ (both in the imaginary part and in the real part) yield (after division by the q^2 prefactor) contributions $\propto q^0$, which are the Fourier transforms of contact terms.

Of most interest for our discussion of the nonanalytic-in- q contributions in the $q \rightarrow 0$ limit, is the fact that the $O(x^2 \ln x)$ term in the small- x expansion of the function $x B_{29}(x)$ yields the following additional contribution to the amplitude

$$\hat{h}^2 f_{\text{eob}}^{w_1 w_2} = -\frac{1}{2} \frac{w_1 w_2}{p_\infty^2} \ln \frac{q}{2k}. \quad (4.52)$$

This contribution has the same $\ln q$ structure as the linear-in- w contribution coming from w_3/r^3 .

Summarizing: the real part of the 3PM, $O(G^3)$, amplitude contains the following contributions (where we recall that $k = p_\infty/\hat{h}$)

$$\begin{aligned} \hat{h}^2 \text{Re}[f_{\text{eob}}^{3PM}] = & -\frac{1}{2} \frac{w_1^3}{\hat{h}^2 p_\infty^2} \frac{1}{q^2} \left(\ln \frac{q}{\Lambda} \right)^2 \\ & - \frac{1}{2} \frac{w_1 w_2}{p_\infty^2} \ln \frac{q}{2k} - w_3 \ln \frac{q}{\Lambda}. \end{aligned} \quad (4.53)$$

C. General concern about the link between a quantum scattering amplitude and classical dynamics

Several recent works have discussed the issue of the relation between \mathcal{M} and classical dynamics, see Refs. [13–20,22,24,25,34,35,55]. In particular, some *one-way* maps between (EOB or EFT) Hamiltonians describing the classical dynamics and the scattering amplitude have been defined, and implemented at both the 2PM [20,22] and 3PM levels [24,25]. However, we wish here to express a general concern (which has been already raised in [54]) about the program of transferring information between a quantum scattering amplitude and classical dynamics. As far as we know, this concern has not been explicitly addressed in the recent literature.

The basic idea of extracting classical information from an amplitude is simply that a same theory (namely GR) is underlying both the classical and the quantum dynamics, so that there should exist some “classical limit” under which it should be possible to extract the classical dynamics from a quantum scattering amplitude. [This idea was already the one of Refs. [6–11].] It seems that many recent papers simply assumed the existence of a “precise demarcation between classical and quantum contributions to the scattering amplitude” (as formulated in the Introduction of [25]). We wish to stress that the existence of such a demarcation is *a priori* unclear to us for a variety of related issues.

First, let us recall the basic fact that the domain of validity of the standard quantum scattering perturbation expansion (Born-Feynman expansion) *does not overlap* with the domain of validity of the standard classical scattering perturbation expansion when considering a Coulomblike potential $V = Z_1 Z_2 e^2/r + O(1/r^2)$, or $V = -GE_1 E_2/r + O(1/r^2)$ in the gravitational case. Here, E_1 and E_2 denote, say, the c.m. energies of two colliding particles (we set $c = 1$). This fact was eloquently expressed in the classic 1948 paper of Niels Bohr on the penetration of charged quantum particles in matter [57], and is also stressed in the treatise of Landau and Lifshitz [12,60]. The basic point is that the quantum expansion is *a priori*

valid when the dimensionless ratios (v denoting the relative velocity)

$$\frac{Z_1 Z_2 e^2}{\hbar v} \ll 1 \quad \text{or} \quad \frac{G E_1 E_2}{\hbar v} \ll 1 \quad (\text{quantum}), \quad (4.54)$$

while the domain of validity for a quasiclassical description of the scattering is just the opposite, namely

$$\frac{Z_1 Z_2 e^2}{\hbar v} \gg 1 \quad \text{or} \quad \frac{G E_1 E_2}{\hbar v} \gg 1 \quad (\text{classical}). \quad (4.55)$$

When a precise definition of the relative velocity v is needed, we shall define it as

$$v_\infty \equiv \sqrt{1 - \frac{1}{\gamma^2}} \quad \text{such that} \quad \gamma = \frac{1}{\sqrt{1 - v_\infty^2}}. \quad (4.56)$$

At the formal level of considering limits for \hbar , the classical domain of validity (4.55) does correspond to the expected limit $\hbar \rightarrow 0$, while the quantum domain of validity (4.54) corresponds to the less usually considered formal limit $\hbar \rightarrow \infty$.

The necessity of the inequalities (4.55) and (4.54) can be seen in various ways. At the conceptual level, Bohr points out (see subsection 1.3 of Ref. [57]) that the condition (4.55) is *necessary and sufficient* for being able “to construct wave packets which, to a high degree of approximation, follow the classical orbits” during the entire scattering process. Bohr only discusses nonrelativistic Coulomb-like scattering. Let us show how it works in the relativistic case, and in the c.m. frame. Each particle is described by an incoming relativistic wave packet having a relatively small transversal size d , e.g., realized (says Bohr) by a hole of radius d in a screen. The quantum diffraction angle ϕ caused by the hole is of order $\phi \sim \lambda/d$ where $\lambda = \hbar/P_{\text{c.m.}}$ is the (reduced) de Broglie wavelength of each particle. In other words, $\phi \sim \hbar/(dP_{\text{c.m.}})$ measures the angular spreading of the quantum wave packets. To be able to measure the classical scattering angle χ in spite of the quantum spread, one must have the inequality $\phi \ll |\chi|$. In addition, the transverse size must be small compared to the impact parameter: $d \ll b$. The leading-order (half) scattering angle is of the form

$$\frac{1}{2}\chi = \frac{a_s}{b}, \quad (4.57)$$

where the length a_s depends on the spin of the (massless) exchanged particle (scalar, vector or tensor). More precisely, one has

$$a_s = G_s \frac{Q_1 Q_2}{\mu} \frac{\hbar(\gamma, \nu)}{p_\infty^2} f_s(u_1 \cdot u_2), \quad (4.58)$$

where G_s is a coupling constant, Q_a a (scalar, electric or gravitational) charge, and where the factor $f_s(u_1 \cdot u_2)$ comes from the current-current interaction between the two worldlines, so that, for the scalar, electromagnetic and gravitational cases, respectively, one has

$$f_0 = 1, \quad f_1 = u_1 \cdot u_2, \quad f_2 = 2(u_1 \cdot u_2)^2 - 1. \quad (4.59)$$

Combining the inequalities $\phi \ll |\chi|$ and $d \ll b$ then leads to the inequality

$$\frac{|a_s| P_{\text{c.m.}}}{\hbar} \gg 1, \quad (4.60)$$

where

$$\frac{a_s P_{\text{c.m.}}}{\hbar} = \frac{G_s Q_1 Q_2}{\hbar} \frac{f_s(\gamma)}{\sqrt{\gamma^2 - 1}}. \quad (4.61)$$

For instance, in the gravitational case, we have $G_2 = G$, $Q_a = m_a$, so that the necessary inequality for quasiclassicality reads

$$\frac{G m_1 m_2}{\hbar} \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}} = \frac{G}{\hbar} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}} \gg 1. \quad (4.62)$$

This is easily seen to be (approximately) equivalent to the second condition (4.55), for all values of the relative velocity.

An important point for our discussion is that this inequality must be satisfied even when considering very large impact parameters, corresponding to *a priori* quasiclassical very large angular momenta (and very small scattering angles).

Another way of seeing the necessity of the inequality (4.62) comes from considering the LO contribution to the phase shift δ_l , namely

$$\begin{aligned} \delta_l^{\text{LO}} &= \frac{G m_1 m_2}{\hbar} \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}} \ln\left(\frac{j_0}{j}\right) \\ &= \frac{G}{\hbar} \frac{2(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \ln\left(\frac{j_0}{j}\right). \end{aligned} \quad (4.63)$$

This directly confirms that the classicality condition, Eqs. (4.55), (4.62), corresponds to large phase shifts $\delta_l \gg 1$, which is one of the standard conditions for the validity of the classical limit [60].

In addition, let us recall the basic structure of the perturbative expansion of the quantum scattering amplitude \mathcal{M} . The LO ($O(G/\hbar)$) contribution to \mathcal{M} coming from a one-graviton exchange in the t -channel (discarding the u - and s -channel contributions), reads (see, e.g., Refs. [21,64])

$$\mathcal{M}^{(\frac{G}{\hbar})}(s, t) = 16\pi \frac{G 2(p_1 \cdot p_2)^2 - p_1^2 p_2^2 + (p_1 \cdot p_2) Q^2}{\hbar Q^2}, \quad (4.64)$$

where $Q = p'_1 - p_1 = -(p'_2 - p_2)$, so that $Q^2 = -t$. When considering, for orientation, a generic relativistic collision, with large velocities $v \sim 1$, and significant momentum transfers, $Q^2 = -t \sim s$, the order of magnitude of the LO contribution (4.64) is

$$\mathcal{M}^{(\frac{G}{\hbar})} \sim \frac{Gs}{\hbar} \sim \alpha_g. \quad (4.65)$$

Here, we introduced the gravitational analog of the quantum electrodynamics coupling constant $\alpha = e^2/\hbar$ (or, more generally, $Z_1 Z_2 e^2/\hbar$), say

$$\alpha_g \equiv \frac{GE_1 E_2}{\hbar}. \quad (4.66)$$

Dimensional analysis (in the simple one-scale regime where $s \sim -t \gtrsim m_1^2 \sim m_2^2$) then shows that the Born-Feynman expansion (or loop-expansion) of \mathcal{M} has the rough structure

$$\begin{aligned} \mathcal{M} &\sim \frac{Gs}{\hbar} + \left(\frac{Gs}{\hbar}\right)^2 + \left(\frac{Gs}{\hbar}\right)^3 + \dots \\ &\sim \alpha_g + \alpha_g^2 + \alpha_g^3 + \dots \end{aligned} \quad (4.67)$$

This exhibits the *a priori* necessity of the quantum condition (4.54) (which implies $\alpha_g \ll 1$) for a reliable use of the Born-Feynman expansion of \mathcal{M} . [Let us note in passing that the systematic use of the small-velocity limit $v \rightarrow 0$ in Refs. [17,22,24,25] might exacerbate the classical-quantum conflict by making the usual, nonrelativistic Coulomb coupling constant $\frac{GE_1 E_2}{\hbar v}$ parametrically larger than the natural dimensionless quantum coupling constant $\alpha_g = \frac{GE_1 E_2}{\hbar}$ entering the loop expansion of \mathcal{M} .]

How can one hope to bridge the gap between the classical domain (4.55), and the quantum one (4.54)? If we could control the exact dependence of the function $\mathcal{M}(s, t, \alpha_g)$ for all values of α_g (both small and large), it would be straightforward to read off the classical dynamics [say via the use of the quasiclassical phase shifts (4.8)]. However, we often have only knowledge of the first few terms in the Born-Feynman (small α_g) expansion of $\mathcal{M}(s, t, \alpha_g)$. Several suggestions have been made in the recent literature for extracting classical information from \mathcal{M} .

On the one hand, Refs. [17–19,22,24,25] have emphasized that a crucial tool for retrieving classical information from \mathcal{M} is to focus, at each order in the formal Born-Feynman expansion in powers of $\alpha_g = \frac{GE_1 E_2}{\hbar}$ on a secondary expansion in $Q^{\text{c.m.}}$. As the corresponding small

dimensionless parameter is $Q^{\text{c.m.}}/P_{\text{c.m.}} = 2 \sin \frac{\theta}{2}$, this corresponds to a small-scattering-angle expansion. The idea is here related to the fact that the classical PM expansion is a large-impact-parameter limit, corresponding to a small-scattering-angle limit. This intuitive idea is certainly appealing, but the point, recalled above, made by Bohr [57] that sufficiently slowly spreading wave packets can only be constructed when the classicality condition (4.55) (which implies $\alpha_g \gg 1$) is satisfied makes it unclear (at least to the author) that focusing on a secondary expansion in $Q^{\text{c.m.}}$ is sufficient for correctly extracting, at all orders, the classical dynamical information. It would be interesting to examine in detail whether this conflicts with the approach pursued in Refs. [34,35] for extracting classical results from \mathcal{M} . Indeed, it seems that the latter approach assumes the existence of wave packets staying well-localized during the entire scattering process, but also uses the Born-Feynman perturbative expansion of \mathcal{M} in powers of $\frac{Z_1 Z_2 e^2}{\hbar}$ or $\frac{GE_1 E_2}{\hbar}$.

On the other hand, Refs. [21,23,26,27,65–69] have emphasized the usefulness of focusing on the so-called eikonal approximation, under which one can hopefully prove that part of the perturbative expansion of \mathcal{M} can be resummed by exponentiating a suitably defined “eikonal phase.” The idea here is that perturbative theory can correctly compute some of the first few diagrams, and therefore their associated exponentiated version. However, this program can reliably give the (large) quasi-classical exponentiated phase [as in Eq. (4.8)] only if one *proves* which perturbative diagrams do exponentiate and which do not. This is a nontrivial task, as shown, for instance, at the one-loop level in Ref. [68]. [The first and second versions of Ref. [68] differed in their conclusion of which perturbative contributions do exponentiate.] For further discussion of the subtleties of the eikonal approach and of the exponentiating contributions, see Refs. [27,69,70].

Let us just mention a specific example suggesting (without, however, proving) that, even when focusing on the small $Q^{\text{c.m.}}$ limit, it is delicate to try to unambiguously extract from the perturbative expansion of the amplitude the corresponding classical PM-expanded information. At the one-loop level (second order in α_g), there appears, when considering the $t/s \ll 1$ limit (or $q \rightarrow 0$), a nonanalytic $\ln q$ term [13–16,23]. This term corresponds to a quantum modification of the LO gravitational potential $-Gm_1 m_2 (2\gamma^2 - 1)/R$ (in physical units) by an additional term of the type $(L_p^2 \equiv \hbar G$ denoting the squared Planck length)

$$-\frac{Gm_1 m_2 (2\gamma^2 - 1)}{R} \left[1 + A(\gamma, \nu) \frac{L_p^2}{R^2} \right], \quad (4.68)$$

which corresponds, in the rescaled EOB units, to a correction of the potential $w(r) = w_1/r + \dots$ of the type

$$\delta w(r) = \nu \hat{h} A(\gamma, \nu) \frac{w_1}{r^3}, \quad (4.69)$$

i.e., a modification of the 3PM coefficient w_3 of the type

$$\delta w_3 = \nu \hat{h} A(\gamma, \nu) w_1. \quad (4.70)$$

Here the dimensionless coefficient $A(\gamma, \nu)$ has a finite limit at low velocities ($\gamma \rightarrow 1$) [13–16], but was recently found [23] to grow logarithmically at high energies ($\gamma \rightarrow \infty$). More precisely, Ref. [23] (see Eq. (2.25) there) found that the logarithmically growing part of $A(\gamma, \nu)$ comes from a factor proportional to the same arcsinh function entering the result of Ref. [24], denoted as $\alpha(\gamma)$ above. We note that, in the domain of validity of the perturbative regime $\alpha_g \rightarrow 0$, i.e., $\hat{h} \rightarrow \infty$, the one-loop contribution (4.70) to w_3 is *parametrically larger* than the (3PM-level) value w_3^B derived from the two-loop amplitude of Ref. [24]. This makes it unclear to us that a formal analytic continuation (in α_g) of the perturbative two-loop computation to the classically-relevant domain where $\alpha_g \gg 1$, i.e., $\hat{h} \ll 1$ can unambiguously read off the needed classical contribution to w_3 . We hope that our remarks will prompt some clarification of these subtle issues.

V. SELF-FORCE (SF) THEORY AND PM DYNAMICS

Before explaining in detail why the result of Ref. [53] seems to be in conflict with the logarithmic growth (3.78), derived from Refs. [24,25], let us point out a potentially interesting new use of SF theory for deriving *exact* PM dynamical results.

A. On the use of self-force (SF) theory to derive *exact* PM dynamics

Let us start by recalling that the discussion in Sec. II above allowed one to give a stringent upper bound on the number of unknown functions of γ entering each PM order. In particular, we found that, both at the 3PM and the 4PM levels, there was *only one a priori* unknown function of γ . Namely, in the parametrization of Eqs. (2.48) and (2.49), the function $\hat{\chi}_3^{(2)}(\gamma)$ at the 3PM level, and the function $\hat{\chi}_4^{(3)}(\gamma)$ at the 4PM level. We wish to point out here the rather remarkable fact that SF theory (which, in the framework of EOB theory means expanding the EOB dynamics to linear order in ν), can, in principle, be used to derive in an *exact* manner the 3PM and 4PM dynamics. The main point is that the first-order SF (1SF) expansions of the 3PM and 4PM scattering functions $\chi_3(\gamma, \nu)$ and $\chi_4(\gamma, \nu)$, i.e their expansions in powers of ν , keeping only the term linear in ν , contain enough information to compute the exact functions $\chi_3(\gamma, \nu)$ and $\chi_4(\gamma, \nu)$. Indeed, using the fact that

$$h(\gamma, \nu) = \sqrt{1 + 2\nu(\gamma - 1)} = 1 + \nu(\gamma - 1) + O(\nu^2), \quad (5.1)$$

and considering first the 3PM level, the 1SF expansion of $\chi_3(\gamma, \nu)$ reads, from Eq. (2.48),

$$\chi_3(\gamma, \nu) = \chi_3^{\text{Schw}}(\gamma) - 2\nu(\gamma - 1)\hat{\chi}_3^{(2)}(\gamma) + O(\nu^2). \quad (5.2)$$

Therefore the linear-in- ν , or 1SF contribution, to $\chi_3(\gamma, \nu)$ is proportional to the function $(\gamma - 1)\hat{\chi}_3^{(2)}(\gamma)$, so that an analytical knowledge of χ_3^{1SF} yields enough knowledge to compute $\hat{\chi}_3^{(2)}(\gamma)$, and thereby the exact, non-SF-expanded value Eq. (2.48) of $\chi_3(\gamma, \nu)$.

The same result holds at the 4PM level. Namely, starting from Eq. (2.49), the 1SF expansion of $\chi_4(\gamma, \nu)$ reads

$$\chi_4(\gamma, \nu) = (1 - \nu(\gamma - 1))\chi_4^{\text{Schw}}(\gamma) - 2\nu(\gamma - 1)\hat{\chi}_4^{(3)}(\gamma) + O(\nu^2). \quad (5.3)$$

Using the exact value of $\chi_4^{\text{Schw}}(\gamma)$, Eqs. (2.41), we see that an analytical knowledge of χ_4^{1SF} yields enough information to compute $\hat{\chi}_4^{(3)}(\gamma)$, and thereby the exact, non-SF-expanded value Eq. (2.49) of $\chi_4(\gamma, \nu)$.

One does not have today general enough 1SF results allowing one to extract $\hat{\chi}_3^{(2)}(\gamma)$, $\hat{\chi}_4^{(3)}(\gamma)$, and their higher-order analogs. Actually, the SF theory of scattering motions is still in its developing stages. Some years ago Ref. [71] had pointed out the interest of extending the SF approach (which is usually applied only to circular, or near-circular, states) to scattering states, and showed what information it could give. Due to technical issues, it is only very recently [72] that a numerical implementation of one of the scattering-type SF computations proposed in Ref. [71] has been accomplished. Here, we are suggesting to develop an analytical, PM-expanded SF framework, e.g., based on the G -expansion of the Mano-Suzuki-Takasugi formalism, for computing the G -expansion of the scattering angle in large-mass-ratio binary systems. When a second-order SF formalism becomes available, the same idea will allow one to compute the exact 5PM and 6PM (conservative) dynamics. Indeed, a look at Eqs. (2.47) shows that, after using the test-mass knowledge (χ_5^{Schw} , χ_6^{Schw}), one has two unknown functions of γ at 5PM and at 6PM, so that it is enough to know the 1SF ($O(\nu)$) and the 2SF [$O(\nu^2)$] contributions to the SF expansions of $\chi_5(\gamma, \nu)$ and $\chi_6(\gamma, \nu)$ to reconstruct their exact expressions for any mass ratio.

In Appendix C we discuss the high-energy limit of SF scattering theory, and the information it could bring on the structure of the PM expansion.

B. Tension between the 3PM dynamics of Refs. [24,25] and the HE behavior of the SF Hamiltonian of an extreme mass-ratio two-body system

Let us show in what technical sense the (numerical) circular-orbit SF computation of Ref. [53] provides a direct handle on the high-energy (HE) limit of the 1SF-expanded⁴ two-body dynamics. To be concrete, and explicitly display how the 3PM-level result of [24,25] seems to conflict, in the HE limit, with the 1SF HE result of [53], let us consider the 1SF expansion of the 3PM-accurate EOB Hamiltonian derived in [51] from the results of [24,25]. We recall that the two-body Hamiltonian is expressed by the general formula (3.1) in terms of the effective Hamiltonian $\mathcal{E}_{\text{eff}} = H_{\text{eff}}(\mathbf{R}, \mathbf{P})$. In turn, the effective Hamiltonian is obtained by solving the EOB mass-shell condition (3.5) for \mathcal{E}_{eff} . In the H -type energy gauge this yields a squared effective Hamiltonian of the form (in rescaled variables)

$$\hat{H}_{\text{eff}}^2(\mathbf{r}, \mathbf{p}) = \hat{H}_S^2 + (1 - 2u)\hat{Q}^H(u, \hat{H}_S), \quad (5.4)$$

where

$$\hat{H}_S^2(\mathbf{r}, \mathbf{p}) = (1 - 2u)(1 + (1 - 2u)p_r^2 + u^2 p_\phi^2), \quad (5.5)$$

and

$$\hat{Q}^H(u, \gamma, \nu) = u^2 q_2(\gamma, \nu) + u^3 q_3(\gamma, \nu) + O(G^4). \quad (5.6)$$

The 2PM coefficient $q_2(\gamma, \nu)$ is given by [20]

$$q_2(\gamma, \nu) = \frac{3}{2}(5\gamma^2 - 1) \left[1 - \frac{1}{h(\gamma, \nu)} \right], \quad (5.7)$$

while the 3PM coefficient derived in [51] by combining the results of [20,24,25] reads

$$\begin{aligned} q_3^B(\gamma, \nu) &= B(\gamma) \left(\frac{1}{h(\gamma, \nu)} - 1 \right) + C^B(\gamma) \left(\frac{1}{h^2(\gamma, \nu)} - 1 \right) \\ &= B(\gamma) \left(\frac{1}{h(\gamma, \nu)} - 1 \right) + 2\nu \frac{\bar{C}^B(\gamma)}{h^2(\gamma, \nu)}, \end{aligned} \quad (5.8)$$

where

$$B(\gamma) \equiv \frac{3}{2} \frac{(2\gamma^2 - 1)(5\gamma^2 - 1)}{\gamma^2 - 1}, \quad (5.9)$$

and where

⁴We recall that ‘‘1SF’’ means ‘‘first order in the symmetric mass-ratio ν ’’.

$$C^B(\gamma) = -\frac{\bar{C}^B(\gamma)}{\gamma - 1}, \quad (5.10)$$

with the explicit value of $\bar{C}^B(\gamma)$ written in Eq. (3.70) above.

A crucial point is that the HE limit $\gamma \rightarrow \infty$ and the SF limit $\nu \rightarrow 0$ do not commute because of the denominators involving powers of $h(\gamma, \nu) = \sqrt{1 + 2\nu(\gamma - 1)}$. When discussing SF results we are interested in performing first a linear expansion in ν , and in then taking the HE limit of this linear expansion. Let us denote, for simplicity, by $F^{1\text{SF}}$ the coefficient of ν in the linear-in- ν , or 1SF, expansion of any EOB function, F , considered as a function of the EOB phase-space variables \mathbf{r} , \mathbf{p} , and of ν : $F(\mathbf{r}, \mathbf{p}, \nu) = F(\mathbf{r}, \mathbf{p}, 0) + \nu F^{1\text{SF}}(\mathbf{r}, \mathbf{p}) + O(\nu^2)$.

Applied to $q_2(\gamma, \nu)$ this yields first

$$q_2^{1\text{SF}} = \frac{3}{2}(\gamma - 1)(5\gamma^2 - 1), \quad (5.11)$$

which becomes in the HE limit $\gamma \rightarrow \infty$

$$q_2^{1\text{SFHE}} \equiv \frac{15}{2}\gamma^3. \quad (5.12)$$

Applying the same (noncommuting) successive limits to $q_3^B(\gamma, \nu)$ yields

$$q_3^{B1\text{SFHE}} \equiv \frac{11}{3}\gamma^3 + 16\gamma^3 \ln(2\gamma). \quad (5.13)$$

Let us consider

$$\hat{Q}^{1\text{SF}} = \frac{[\hat{H}_{\text{eff}}^2]^{1\text{SF}}}{1 - 2u}. \quad (5.14)$$

We have

$$\hat{Q}_B^{1\text{SF}} = u^2 q_2^{1\text{SF}} + u^3 q_3^{B1\text{SF}} + O(u^4). \quad (5.15)$$

Its HE limit reads

$$\hat{Q}_B^{1\text{SFHE}} \equiv \frac{15}{2}\gamma^3 u^2 + \frac{11}{3}\gamma^3 u^3 + 16\gamma^3 \ln(2\gamma) u^3 + O(u^4). \quad (5.16)$$

The crucial point to note here is that the $\ln(2\gamma)$ contribution coming from the arcsinh term implies that the ratio $\hat{Q}_B^{1\text{SF}}/\gamma^3$ does not have a finite HE limit, when considered at the 3PM level, namely

$$\frac{\hat{Q}_B^{1\text{SF}}}{\gamma^3} \stackrel{\text{HE}}{\equiv} \frac{15}{2} u^2 + \frac{11}{3} u^3 + 16 \ln(2\gamma) u^3 + O(u^4). \quad (5.17)$$

In other words, if we truncate the PM expansion at the 3PM level included, and use

$$\hat{Q}_B^{\leq 3\text{PM}} = u^2 q_2(\gamma, \nu) + q_3^B(\gamma, \nu) u^3, \quad (5.18)$$

to define some exact dynamics, the latter dynamics implies a logarithmic growth of the ratio $\hat{Q}^{\text{ISF}}/\gamma^3$ in the HE limit.

Such a logarithmic growth is in conflict with a result of Akcay *et al.* [53]. Indeed, Ref. [53] has (numerically) computed a 1SF-accurate gauge-invariant function which can be directly related to \hat{Q}^{ISF} . More precisely, Ref. [53] considered the sequence of circular orbits of a small black hole (of mass m_1) around a large black hole (of mass m_2) and computed a function $a_E^{\text{1SF}}(u)$ which (using results from Refs. [73–75]) can be related to \hat{Q}^{ISF} in the following (gauge-invariant) way (see [53] for details)

$$\frac{a_E^{\text{1SF}}(u)}{(1-2u)^2} = \left[\frac{\hat{Q}^{\text{ISF}}}{\hat{H}_S^3} \right]^{\text{circ}}. \quad (5.19)$$

The superscript circ on the right-hand side means that the arguments of the EOB function $\hat{Q}^{\text{ISF}}/\hat{H}_S^3$ must be evaluated along the sequence of circular orbits around a Schwarzschild black hole of mass M , i.e., that we have the relation

$$\gamma^{\text{circ}} = \hat{H}_S^{\text{circ}} = \frac{1-2u}{\sqrt{1-3u}}. \quad (5.20)$$

Rigorously speaking, only the part of the sequence of circular orbits describing the unstable orbits below $R = 4GM$, i.e., $\frac{1}{4} < u < \frac{1}{3}$, leads to a value of $\gamma^{\text{circ}} > 1$ that can be directly inserted in the formulas above. However, one can formally consider the analytic continuation of the formulas above for smaller values of u . In particular, we could satisfactorily check that, in the PN limit $u \rightarrow 0$, $[\hat{Q}^{\text{ISF}}/\hat{H}_S^3]^{\text{circ}} = 2u^3 + O(u^4)$, which agrees with the LO PN term in $a_E^{\text{1SF}}(u)/(1-2u)^2$.

The tension with the result above then comes when focusing on the limit $u \rightarrow (\frac{1}{3})^-$. This limit, which physically corresponds to considering HE circular orbits near the light ring of the large-mass black hole, realizes the above-considered HE limit $\gamma \rightarrow \infty$. The crucial point is that Ref. [53] could numerically study with high accuracy the behavior of the $a_E^{\text{1SF}}(u)$ in this limit, and found that it admitted a *finite* limit yielding

$$\lim_{\gamma \rightarrow \infty} \left[\frac{\hat{Q}^{\text{ISF}}}{\hat{H}_S^3} \right]^{\text{circ}} = \frac{27}{4} \zeta, \quad (5.21)$$

where ζ is a finite number equal to 1 to good accuracy. In particular, the study of the behavior of $a_E^{\text{1SF}}(u)$ in the close vicinity of $u = \frac{1}{3}$ definitely excluded the presence of a LO logarithmic singularity $\propto \ln(1-3u)$, i.e., $\propto \ln \gamma$. On the other hand, the numerical results of [53] were compatible with the additional presence of a *subleading* logarithmic singularity, i.e., a behavior of $\hat{Q}^{\text{ISF}}/\gamma^3 - 27\zeta/4$ of the form $\propto (1-3u) \ln(1-3u)$, i.e., $\propto \gamma^{-2} \ln \gamma$.

How can we reconcile the (apparently) conflicting HE behaviors (5.17) and (5.21)? Barring some hidden numerical flaw in the work of [53], several possibilities come to mind. We wish here to propose two different possibilities for relieving the tension between (5.17) and (5.21).

The first possibility was suggested to the author by a statement made in the second sentence below Eq. (9.5) of [25] to the effect that their general ansatz for their $O(G^3)$, 3PM amplitude \mathcal{M}_3 was uniquely fixed only by the knowledge of the PN expansion of \mathcal{M}_3 at the 6PN-level included. However, as we recalled above, at the time of writing of (the preprint version of) this paper (November 2019), there existed no classical computation having confirmed the 3PM dynamics of [25] at the 6PN level. The highest PN level which had been independently checked was the 5PN level, as obtained in Ref. [52]. This then suggested exploring the conjecture that some error might have crept at the 6PN level in the computations of Ref. [25] (which rely in great part on working with the PN-expansion of the two-loop integrand), and in looking for a modified version of the 3PM dynamics exhibiting a softer, logarithmic-free HE behavior. This possibility is briefly discussed in the following section.

A second possibility relies on the fact that there might exist correlations between the various PM contributions to $\hat{Q}(u, \gamma, \nu)$,

$$\hat{Q}^E(u, \gamma, \nu) = u^2 q_2(\gamma, \nu) + u^3 q_3(\gamma, \nu) + u^4 q_4(\gamma, \nu) + \dots \quad (5.22)$$

leading to a cancellation⁵ of the problematic logarithmic term in Eq. (5.17). This second possibility relies on making another type of conjecture on the structure of the EOB potential $\hat{Q}(u, \gamma, \nu)$. It is also explored in the following section.

VI. DIFFERENT CONJECTURES ON THE HE BEHAVIOR OF PM GRAVITY AND THEIR CONSEQUENCES

A. Conjecture on the HE behavior of PM gravity

A striking feature of 2PM-level gravity, which is especially clear in its EOB formulation [20], is that it has a remarkably simple HE limit. Specifically, the (energy-gauge) EOB mass-shell condition (3.2) (which, for general energies and momenta, is a complicated, nonlinear function of energies and momenta) drastically simplifies in the HE limit and becomes quadratic in P_μ . Moreover, in this limit the dependence on the mass ratio ν completely disappears. Indeed, when $\gamma \rightarrow \infty$, the $O(G^2)$ Q term $\hat{Q}^{2\text{PM}}(u, \gamma, \nu) = u^2 q_2(\gamma, \nu)$, where

⁵This possibility was briefly alluded to in the preprint version of this work, but not pursued there because of its apparently fine-tuned nature.

$$q_2(\gamma, \nu) = \frac{3}{2}(5\gamma^2 - 1) \left[1 - \frac{1}{h(\gamma, \nu)} \right], \quad (6.1)$$

reduces to

$$\hat{Q}^{2\text{PM}}(u, \gamma, \nu) \stackrel{\text{HE}}{=} \frac{15}{2} u^2 \gamma^2, \quad (6.2)$$

where we recall that $\gamma \equiv -P_0/\mu \equiv \mathcal{E}_{\text{eff}}/\mu$, so that the mass-shell condition (3.2) simplifies to the following quadratic constraint

$$0 = g_{\text{Schw}}^{\mu\nu} P_\mu P_\nu + \frac{15}{2} u^2 P_0^2, \quad (6.3)$$

or, explicitly,

$$0 = -\frac{\mathcal{E}_{\text{eff}}^2}{A_{\text{HE}2\text{PM}}} + \frac{P_R^2}{B_{\text{HE}2\text{PM}}} + \frac{P_\varphi^2}{C_{\text{HE}2\text{PM}}}, \quad (6.4)$$

where $\frac{1}{A_{\text{HE}2\text{PM}}} = \frac{1}{A_{\text{Schw}}} - \frac{15}{2} u^2$, i.e., inserting the values of the Schwarzschild-metric coefficients (with $u \equiv GM/R$),

$$\begin{aligned} A_{\text{HE}2\text{PM}}(u) &= \frac{A_{\text{Schw}}(u)}{1 - \frac{15}{2} u^2 A_{\text{Schw}}(u)} = \frac{1 - 2u}{1 - \frac{15}{2} u^2 (1 - 2u)}, \\ B_{\text{HE}2\text{PM}}(u) &= B_{\text{Schw}}(u) = \frac{1}{1 - 2u}, \\ C_{\text{HE}2\text{PM}}(u) &= C_{\text{Schw}} = R^2. \end{aligned} \quad (6.5)$$

In other words, at the 2PM-level, and in the HE limit $\gamma \rightarrow \infty$ (which is equivalent to taking the massless limit $m_1 \rightarrow 0$, $m_2 \rightarrow 0$), the c.m. scattering angle of a two-body system becomes blind to the values of the masses and can be obtained from the null geodesic motion in the effective HE metric

$$\begin{aligned} g_{\mu\nu}^{\text{HE}2\text{PM}} dx^\mu dx^\nu &= -A_{\text{HE}2\text{PM}}(R) dT^2 + B_{\text{HE}2\text{PM}}(R) dR^2 \\ &+ C_{\text{HE}2\text{PM}}(R) (d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \quad (6.6)$$

Such a simple conclusion (equivalent⁶ to the discussion in Sec. VII of [20]), seems to be physically quite satisfactory. Indeed, as (classical and quantum) gravity couples to energy, rather than to rest-mass, one would *a priori* expect that a limit where the two masses m_1, m_2 tend toward zero, while keeping fixed the energies $E_1 = \sqrt{m_1^2 + p_1^2}$, $E_2 = \sqrt{m_2^2 + p_2^2}$, should exist, and be describable by the interaction of two (classical or quantum) massless particles. We sketch in Appendix B how a classical PM scattering computation might prove that such a limit exists.

Reference [20] assumed that such a HE limit exists not only at the 2PM level, but also at any higher PM order. Let

⁶Modulo a different parametrization leading there to $A_{\text{HE}2\text{PM}}(u) = (1 - 2u)(1 + f(u))$.

us recall at this point that, contrary to the PN expansion which can (and does) involve logarithms of $\frac{1}{c}$, there seems to be no way in which the PM expansion (when considered at finite γ) can involve logarithms of the gravitational coupling constant G . Indeed, as was indicated in Sec. II, the PM expansion of the classical scattering angle (equivalent to the knowledge of \hat{Q}) must, at each PM order, be a polynomial in the masses, and therefore in Gm_1/b and Gm_2/b . Therefore, when considering the generic case of finite masses and arbitrary (but finite) values of γ , we must have an expansion in powers of $u = GM/R$ of the type

$$\hat{Q} = \sum_{n \geq 2} q_n(p_\lambda, \nu) u^n. \quad (6.7)$$

Reference [20] then assumed that the HE limit of each PM coefficient $q_n(p_\lambda, \nu)$ would become a ν -independent quadratic form in p_λ . [This is equivalent to saying that the corresponding coefficient in the unrescaled Q becomes a ν -independent quadratic form in P_λ .]

The precise expression for the limiting behavior of $q_n(p_\lambda, \nu)$ depends on the gauge chosen to write it. In the first form (3.7) of the energy gauge (where $q_n(p_\lambda, \nu)$ is only a function of $p_0 = -\mathcal{E}_{\text{eff}}/\mu = \hat{\mathcal{E}}_{\text{eff}}$), one would have

$$\lim_{\hat{\mathcal{E}}_{\text{eff}} \rightarrow \infty} q_n^E(\hat{\mathcal{E}}_{\text{eff}}, \nu) \approx c_n^{(qE)} \hat{\mathcal{E}}_{\text{eff}}^2. \quad (6.8)$$

This is what we used in our 2PM-level discussion above. In the second (Hamiltonian) form (3.8) of the energy gauge one would have

$$\lim_{\hat{H}_{\text{Schw}} \rightarrow \infty} q_n(\hat{H}_{\text{Schw}}, \nu) \approx c_n^{(qH)} \hat{H}_{\text{Schw}}^2. \quad (6.9)$$

One can easily see that the two conditions are equivalent to each other, with some transformation between the coefficients corresponding to rewriting the higher-PM version of $A_{\text{HE}2\text{PM}}(u)$ either as

$$A_{\text{HE}}(u) = (1 - 2u)(1 - c_2^{(qE)} u^2 - \dots - c_n^{(qE)} u^n - \dots)^{-1}, \quad (6.10)$$

or as

$$A_{\text{HE}}(u) = (1 - 2u)(1 + c_2^{(qH)} u^2 + \dots + c_n^{(qH)} u^n + \dots). \quad (6.11)$$

B. Uniqueness of a conjectured 3PM dynamics compatible with the simple HE behavior (6.8)

The 3PM-level EOB Q potential derived from the 3PM result of [24,25] is given by Eq. (5.8). We discussed above why its 1SF expansion is in tension with the SF result

(5.21). In addition, its HE limit (without doing any SF expansion) does not respect the expected HE behavior (6.8). Indeed, we have

$$\frac{\hat{Q}^{\leq 3\text{PNB}}(u, \gamma, \nu)}{\gamma^2} \stackrel{\text{HE}}{=} \frac{15}{2} u^2 + \left(8 \ln(2\gamma) - \frac{17}{3}\right) u^3, \quad (6.12)$$

where the logarithmically growing term $8u^3 \ln(2\gamma)$ comes from the term $16\gamma^4 \text{as}(\gamma) / \sqrt{\gamma^2 - 1}$ in the function $\bar{C}^B(\gamma)$, Eq. (3.70).

In the present subsection we propose a conjectured modification of the function $\bar{C}^B(\gamma)$ that has the property of being compatible at once with four different constraints: (i) the same restricted analytic structure as $\bar{C}^B(\gamma)$; (ii) the SF result (5.21); (iii) the independently confirmed 5PN-level expansion of the 3PM dynamics; and (iv) the simple HE behavior (6.8). Moreover, these properties uniquely determine our conjectured modified function $\bar{C}(\gamma)$.

The general ansatz⁷ made (and motivated by several arguments) in Ref. [25] is [when transcribed in terms of $\bar{C}(\gamma)$] that

$$\bar{C}^c(\gamma) = c_1 \gamma + c_3 \gamma^3 + (d_0 + d_2 \gamma^2 + d_4 \gamma^4) \frac{\text{arcsinh} \sqrt{\frac{\gamma-1}{2}}}{\sqrt{\gamma^2 - 1}}, \quad (6.13)$$

with some numerical coefficients c_1, c_3, d_0, d_2, d_4 . This general structure corresponds to the structure of the coefficients τ_1 and τ_3 in Eq. (9.5) of [25], as determined by the requirement indicated just below Eq. (9.5) there that (after completing it by the overall factor $m_1^3 m_2^3$) $\bar{C}(\gamma)$ must be a polynomial in⁸ m_1^2, m_2^2 and $(p_1 \cdot p_2)$.

When redoing the computation of the HE limit of the quantity $\frac{\hat{Q}^{\text{1SF}}}{\gamma^3}$ considered in Eq. (5.17) above for the general ansatz (6.13), one finds that this ratio now takes the general form

$$\frac{\hat{Q}^{\text{1SF}}}{\gamma^3} \stackrel{\text{HE}}{=} \frac{15}{2} u^2 + (c_3 - 15) u^3 + \frac{d_4}{2} \ln(2\gamma) u^3 + O(u^4). \quad (6.14)$$

Barring the possibility (separately explored below) that the $O(u^4)$ 4PM remainder term in this result cancels the $O(\ln(2\gamma)u^3)$ term, the compatibility with the SF result (5.21), together with the general requirement (6.8), then determines that the coefficient d_4 in Eq. (6.13) should vanish:

⁷If we knew sufficiently many terms in the PN expansion of $\bar{C}(\gamma)$ the method of Ref. [76] would allow us to derive its exact form without assuming such a restricted form.

⁸I thank Mikhail Solon for clarifying the precise meaning of the statement written below Eq. (9.5) of [25].

$$d_4 = 0. \quad (6.15)$$

We note in passing that the term proportional to d_2 in Eq. (6.13) will generate a subleading logarithmic term in the SF quantity computed in [53] that is compatible with the best fits obtained there.

This leaves only *four* unknown parameters in the so-restricted ansatz (6.13), namely c_1, c_3, d_0, d_2 . If we now use the independently derived (by using purely classical methods) 5PN-level value of $\bar{C}(\gamma)$ [52], as written in Eq. (3.69) above, we have in hands four equations for the four unknowns c_1, c_3, d_0, d_2 . By solving these four equations, we have found that they uniquely determine c_1, c_3, d_0, d_2 , thereby uniquely determining a 3PM dynamics with softer HE behavior from the sole use of 5PN-level information.⁹ The resulting unique value of $\bar{C}(\gamma)$ is found to be

$$\bar{C}^c(\gamma) = \gamma(35 + 26\gamma^2) - (18 + 96\gamma^2) \frac{\text{as}(\gamma)}{\sqrt{\gamma^2 - 1}}. \quad (6.16)$$

Let us briefly contrast the predictions following from the conjectured 3PM dynamics defined by Eq. (6.16) to those following from the result (3.70) of Refs. [24,25]. First, the corresponding 3PM-level contribution to the scattering angle, namely

$$\chi_3^c(\gamma, \nu) = \chi_3^{\text{Schw}}(\gamma) - p_\infty \frac{\bar{C}^c(\gamma)}{\gamma - 1} \left(1 - \frac{1}{h^2(\gamma, \nu)}\right), \quad (6.17)$$

has a HE limit (equivalent to the massless limit at fixed momenta) equal to¹⁰

$$\chi_3^c(\gamma, \nu) \stackrel{\text{HE}}{=} -\frac{14}{3} \gamma^3. \quad (6.18)$$

Using the notation (following Ref. [20])

$$\alpha \equiv \frac{\gamma}{j} \equiv \frac{GM\mathcal{E}_{\text{eff}}}{J} = \frac{Gs - m_1^2 - m_2^2}{2J}, \quad (6.19)$$

and adding the HE limits of the 1PM and 2PM scattering angles, we get as conjectured 3PM-accurate prediction for the HE-limit of the scattering angle the following *finite* result

$$\frac{1}{2} \chi^c \stackrel{\text{HE}}{=} 2\alpha - \frac{14}{3} \alpha^3. \quad (6.20)$$

⁹This result is compatible with the statement made in the second sentence below Eq. (9.5) of [25] to the effect that their more general ansatz (involving an extra term $d_4 \gamma^4$ in the coefficient of the arcsinh) is uniquely fixed by the 6PN-level $O(G^3)$ amplitude.

¹⁰The negative coefficient $-\frac{14}{3}$ comes from combining the positive Schwarzschild contribution $+\frac{64}{3}$ with the contribution $-c_3 = -26$ from the first term in $\bar{C}^c(\gamma)$.

By contrast, if one formally computes the HE limit of the scattering angle derived from the 3PM dynamics of Refs. [24,25] one gets a logarithmically divergent 3PM-level contribution, namely (at the leading-logarithm accuracy),

$$\frac{1}{2}\chi^{B\text{HE}}=2\alpha - 8\ln(2\gamma)\alpha^3. \quad (6.21)$$

We note in passing that the sign of the logarithmically divergent coefficient $-8\ln(2\gamma)$ of α^3 is *negative*. This agrees with the sign of the corresponding (finite) term in Eq. (6.20). By contrast, the eikonal-approximation two-loop result of Amati, Ciafaloni and Veneziano [26] (which has been recently checked to hold also in several supergravity theories [27,28], and confirmed in the pure gravity case [28]) gives the result (after using a correction suggested by Ciafaloni and Colferai [77] and confirmed in [28])

$$\frac{1}{2}\chi^{\text{eikonalHE}}=2\alpha + \frac{16}{3}\alpha^3, \quad (6.22)$$

where the sign of α^3 is *positive*. Independently of the consideration of the HE-softer conjecture Eq. (6.16), we note that the HE limit of the result of Refs. [24,25] disagrees with the HE eikonal result of Refs. [26–28,77].

As a second type of predictions from Eq. (6.16), let us note that it leads to a specific 3PM-accurate EOB Q potential of the form

$$\hat{Q}^{3\text{PM}c}(u, \gamma, \nu) = u^2 q_2(\gamma, \nu) + u^3 q_3^c(\gamma, \nu), \quad (6.23)$$

where $q_3^c(\gamma, \nu)$ is obtained by replacing in Eq. (3.62) the function $C^c(\gamma)$ given in Eq. (6.16) [using also Eqs. (3.60) and (3.59)]. Let us now consider the 1SF-accurate value of $\hat{Q}^{c\text{3PM}}(u, \gamma, \nu)$, i.e., the coefficient of ν in the ν -expansion of the full function $\hat{Q}^{c\text{3PM}}(u, \gamma, \nu)$. A straightforward calculation yields for the HE behavior of $\hat{Q}^{c\text{3PM}1\text{SF}}(u, \gamma)$, i.e., its asymptotic behavior as $\gamma \rightarrow \infty$, the value

$$\frac{\hat{Q}^{c\text{3PM}1\text{SF}}}{\gamma^3} \stackrel{\text{HE}}{=} \frac{15}{2}u^2 + 37u^3 + O(u^4). \quad (6.24)$$

Contrary to the corresponding result following from [25] that led to the logarithmically divergent result Eq. (5.16), we now get a finite limit when inserting the value $u = \frac{1}{3}$ corresponding to the light ring, namely

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} \left[\frac{\hat{Q}^{c\text{3PM}1\text{SF}}}{\gamma^3} \right]_{\text{lightring}} &= \frac{5}{6} + \frac{37}{27} \\ &= \frac{119}{54} \approx 2.2037. \end{aligned} \quad (6.25)$$

The corresponding numerical result of [53], Eq. (5.21), was $\approx \frac{27}{4} = 6.75$. We should not expect a close numerical

agreement because we have used in our analytical estimate only the first two terms (2PM and 3PM) in the (visibly badly convergent) infinite PM expansion of this ratio. However, the 3PM conjectural expression Eq. (6.16) is (contrary to the 3PM result of Bern *et al.*) qualitatively compatible in sign and in order of magnitude (and in its finiteness!) with the numerical SF result of [53].

On the other hand, the conjectured, HE-softer, 3PM dynamics starts differing from the result of Refs. [24,25] at the 6PN level. Indeed, the 6PN-accurate expansion of (3.70) reads

$$\begin{aligned} \bar{C}^{B6\text{PN}}(p_{\text{eob}}) &= 4 + 18p_\infty^2 + \frac{91}{10}p_\infty^4 - \frac{69}{140}p_\infty^6 \\ &\quad - \frac{1447}{10080}p_\infty^8 + O(p_\infty^{10}), \end{aligned} \quad (6.26)$$

while that of (6.16) reads

$$\begin{aligned} \bar{C}^{c6\text{PN}}(p_{\text{eob}}) &= 4 + 18p_\infty^2 + \frac{91}{10}p_\infty^4 - \frac{69}{140}p_\infty^6 \\ &\quad - \frac{233}{672}p_\infty^8 + O(p_\infty^{10}). \end{aligned} \quad (6.27)$$

Several independent groups have very recently performed 6PN-accurate $O(G^3)$ computations [78–80]. All those calculations agree among themselves and have directly confirmed the 6PN-accurate expansion (6.26), thereby disproving the (HE-softer) conjectured 3PM dynamics (6.16), leading to Eq. (6.27).

We must therefore discard the possibility, explored above, of relieving the tension between the high-energy behavior (3.78) derived from Eq. (3.70) and the high-energy behavior found in Ref. [53] by softening the HE behavior of the 3PM dynamics in the simple-minded way¹¹ (6.15). Let us, however, emphasize again that our search for some type of resolution of the tension between the result of Refs. [24,25] and the HE result of Ref. [53] should continue. In addition, we have emphasized the presence of another tension between the HE limit of the result of Refs. [24,25] and the (now confirmed) HE eikonal result of ACV. Before continuing our effort toward understanding how to reconcile these contrasting results, let us put forward what we consider to be minimal requirements concerning the HE behavior of PM gravity.

¹¹We shall not explore here the more farfetched possibility that the 3PM dynamics involve nonperturbative factors, say $\propto 1 - \exp(-\frac{1}{p_\infty})$, that would not be detectable at any finite PN approximation, but that would soften the HE behavior of the arcsinh term.

C. Minimal requirement on the HE behavior of PM gravity

We recalled above the arguments suggesting that the HE (or massless) limit of the EOB mass-shell constraint (3.2) should yield a (mass-independent) massless quadratic constraint of the type

$$0 = g_{\text{HE}}^{\mu\nu} P_\mu P_\nu. \quad (6.28)$$

This constraint is equivalent to requiring that the HE limit of the exact unrescaled $Q(u, P_\mu)$ term be quadratic in the unrescaled effective momentum P_μ , or that the exact rescaled $\hat{Q}(u, p_\mu) \equiv Q/\mu^2$ term be quadratic in the rescaled effective momentum $p_\lambda \equiv P_\lambda/\mu$:

$$\hat{Q}(u, p_\mu) \stackrel{\text{HE}}{=} q_{\text{HE}}^{\mu\nu}(u) p_\mu p_\nu, \quad (6.29)$$

with a mass-independent tensor $q^{\mu\nu}(u)$. In the energy gauge, this requirement reads

$$\hat{Q}(u, \gamma, \nu) \stackrel{\text{HE}}{=} q_{\text{HE}}(u) \gamma^2. \quad (6.30)$$

Above, we implicitly assumed that the limiting HE behavior of Eqs. (6.29), (6.30) *separately* applies at each PM order. In other words, we assumed that the two limits $G \rightarrow 0$ and $\gamma \rightarrow \infty$ commuted. However, another possibility is that these two limits do not commute, and that though individual PM contributions $q_n(\gamma, \nu) u^n$ in Eq. (6.7) do not separately exhibit the expected HE quadratic behavior, the sum of all the PM contributions does lead to a nice quadratic mass-shell condition (6.29), (6.30) in the HE limit. A structure allowing such a mechanism is presented in the next subsections.

D. Transmutation of post-Minkowskian order in the radiative corrections to the dynamics

We shall present below a mechanism able to reconcile the 3PM dynamics derived in [24,25], with the 1SF, HE behavior found in [53]. The basic idea of this mechanism is a particular type of noncommutativity of the two limits $G \rightarrow 0$ and $\gamma \rightarrow \infty$ by which the HE (or massless) limit of the $O(G^{\geq 4})$ dynamics trickles down to the $O(G^3 \ln G)$ level. Before presenting a specific conjecture exhibiting such an effect and thereby reconciling Refs. [24,25,53], let us show that such effects are indeed present in PM gravity, when considering the conservative part of classical radiative corrections.

We recall that it was pointed out long ago [81] that classical radiative effects start having a nonpurely dissipative dynamical effect at the 4PN level, via the so-called hereditary tail. At the 4PN level, a part of the near-zone gravitational field becomes a nonlocal functional of the two worldlines that cannot be simply obtained by a usual, small-retardation PN expansion. The conservative part of

the corresponding nonlocal-in-time dynamics can be described by a nonlocal action, either of the Schwinger-Keldysh type [82], or of the Fokker type [83]. The latter conservative radiative correction is the source of the first logarithm entering the PN expansion of the two-body dynamics. This logarithm¹² arises at the $O(G^4)$ (and 4PN) level [71,84]. This might suggest that delicate physical effects linked to time-nonlocality start occurring at the $O(G^4)$ level, and have no effect on the $O(G^3)$ level. This is likely to be true when considering nonzero masses and a finite value of γ . However, the following argument shows that this is not true when considering the HE limit where $\gamma \rightarrow \infty$ (with the masses $m_i \rightarrow 0$, keeping fixed the c.m. energy E).

Let us start from a simple formula obtained¹³ in Ref. [59] for the value of the scattering angle associated with the conservative effect of the radiative correction, namely

$$\chi_s^{\text{rad}}(E, J) = \frac{\partial}{\partial J} W_s^{\text{rad}}(E, J), \quad (6.31)$$

where

$$W_s^{\text{rad}} = 2GH \int d\omega \frac{dE^{\text{gw}}}{d\omega} \ln(2e^{\gamma\epsilon} |\omega| s). \quad (6.32)$$

Here, $E = H$ is the total c.m. energy of the binary system, $\frac{dE^{\text{gw}}}{d\omega}$ is the spectrum of the energy that would be emitted in gravitational waves if one would use a retarded Green's function (rather than a time-symmetric one), and s is a length scale to be chosen (after differentiation) of order of the size of the system. [The results of Ref. [59] show, for scattering motions, that taking $s = b$ allows one to capture all the relevant nonlocal effects.]

The crucial point is that $W_b^{\text{rad}} \equiv W_{s=b}^{\text{rad}}$, and the corresponding $\chi_{s=b}^{\text{rad}}$, is of order G^4 in the case of the scattering of massive particles at finite γ (see [59]), but becomes of order $G^3 \ln(1/G)$ in the case of the classical scattering of massless particles. Indeed, following the results of Gruzinov and Veneziano [88] on the gravitational radiation from *classical massless* particle collisions,¹⁴ we see that, while in the finite- γ (and finite masses) scattering case $\frac{dE^{\text{gw}}}{d\omega}$ (which is $O(G^3)$) starts decaying exponentially above a frequency of order v/b [59], in the massless case $\frac{dE^{\text{gw}}}{d\omega}$ decays only very slowly ($\propto \ln(1/(GE\omega))$) above $1/b$. Using the approximate expression of $\frac{dE^{\text{gw}}}{d\omega}$ [when $1/b \lesssim \omega \lesssim 1/(GE)$] derived in Ref. [88], and neglecting the

¹²As shown in Ref. [59] the 4PN, $O(G^4)$, logarithmic contribution to the scattering angle involves the logarithm of a dimensionless velocity $\ln p_\infty$, but *does not involve* the logarithm of G (e.g., through the form $\ln j$, with $j = J/(Gm_1 m_2)$).

¹³It was written down there at the leading PN order, but, in view of Refs. [81,82,85–87] it clearly has a general validity.

¹⁴The results of Ref. [88] have been confirmed by a quantum-amplitude derivation [89].

contribution¹⁵ from $\omega \gtrsim 1/(GE)$, yields, for W_b^{rad} , an integral proportional to

$$\int_{1/b}^{1/GE} d\omega \ln(\omega b) \ln\left(\frac{1}{GE\omega}\right) \approx \frac{1}{GE} \ln\left(\frac{b}{GE}\right). \quad (6.33)$$

The crucial point to note is that this integral generates a factor $\frac{1}{GE}$ due to the slow decay of the HE gravitational-wave spectrum between $1/b$ and $1/GE \gg 1/b$. [We are considering the small scattering angle case, $GE/b \ll 1$.] Adding the factor corresponding to the zero-frequency limit of $\frac{dE^{\text{sw}}}{d\omega}$ [63], and the characteristic tail prefactor $2GH = 2GE$, leads to the following estimate

$$W_b^{\text{rad}\gamma \rightarrow \infty} \sim + \frac{G^3 E^4}{b^2} \ln\left(\frac{b}{GE}\right). \quad (6.34)$$

Differentiating with respect to $J \approx Eb/2$, finally leads to a scattering angle for massless particles of order

$$\chi_b^{\text{rad}\gamma \rightarrow \infty} \sim - \left(\frac{GE}{b}\right)^3 \ln\left(\frac{b}{GE}\right) \sim -\chi^3 \ln \frac{1}{\chi}, \quad (6.35)$$

where $\chi \sim GE/b$, on the rhs, denotes the leading-order scattering angle. By contrast, the radiative contribution to χ in the finite-masses, finite- γ case is

$$\chi_b^{\text{rad}\gamma \text{ finite}} \sim \left(\frac{GE}{b}\right)^4 \sim \chi^4 = O(G^4). \quad (6.36)$$

As announced, we have here a conservative dynamical effect, the radiative contribution to the scattering angle of two classical particles, which is $O(G^4)$ when γ is finite, but becomes $O(G^3 \ln 1/G)$ in the $\gamma \rightarrow \infty$ limit. Note that our estimates only concern the nonlocal (tail-transported [81]) contribution to the conservative dynamics. However, this is a clear proof that the 4PM-level ($O(G^4)$) conservative dynamics undergoes a transmutation of PM order (down to the $O(G^3 \ln 1/G)$ level) in the $\gamma \rightarrow \infty$ limit.

We also note that our reasoning indicates that, at the leading-log approximation, the sign of $\chi_b^{\text{rad}\gamma \rightarrow \infty}$ is *negative*. Indeed, both $\ln(\omega b)$ and $\frac{dE^{\text{sw}}}{d\omega}$ are positive in the relevant interval; and the differentiation with respect to J , i.e., b , changes the sign. We will come back below to this point.

E. Second conjecture to reconcile the 3PM result of Refs. [24,25], with the 1SF, HE behavior of Ref. [53]

Let us recap the conundrum we are trying to solve. The two-loop result of Refs. [24,25] leads to the following 3PM-accurate EOB Q potential

¹⁵The $\sim 1/\omega$ decay of the latter contribution might generate an additional logarithmic factor.

$$\hat{Q}^{\leq 3\text{PM}B}(u, \gamma, \nu) = q_2(\gamma, \nu)u^2 + q_3^B(\gamma, \nu)u^3, \quad (6.37)$$

where the 3PM-level coefficient $q_3^B(\gamma, \nu)$ reads

$$q_3^B(\gamma, \nu) = B(\gamma) \left(\frac{1}{h(\gamma, \nu)} - 1 \right) + 2\nu \frac{\bar{C}^B(\gamma)}{h^2(\gamma, \nu)}, \quad (6.38)$$

with

$$B(\gamma) \equiv \frac{3(2\gamma^2 - 1)(5\gamma^2 - 1)}{2\gamma^2 - 1}, \quad (6.39)$$

and

$$\begin{aligned} \bar{C}^B(\gamma) &= \frac{2}{3}\gamma(14\gamma^2 + 25) \\ &+ 4(4\gamma^4 - 12\gamma^2 - 3) \frac{\text{as}(\gamma)}{\sqrt{\gamma^2 - 1}}. \end{aligned} \quad (6.40)$$

The crucial contribution in $\bar{C}^B(\gamma)$ is the term¹⁶ $16\gamma^4 \text{as}(\gamma)/\sqrt{\gamma^2 - 1}$, where we recall that the arcsinh function can be written as

$$\begin{aligned} \text{as}(\gamma) &= \frac{1}{2} \ln(\gamma + p_\infty) = -\frac{1}{2} \ln(\gamma - p_\infty) \\ &= \frac{1}{4} \ln \frac{\gamma + p_\infty}{\gamma - p_\infty} = \frac{1}{4} \ln \frac{1 + v_\infty}{1 - v_\infty}, \end{aligned} \quad (6.41)$$

where $v_\infty \equiv \frac{p_\infty}{\gamma} \equiv \sqrt{1 - \frac{1}{\gamma^2}}$.

Indeed, this contributes to $q_3^B(\gamma, \nu)$ the term

$$q_3^{\text{log}}(\gamma, \nu) = \frac{16\nu\gamma^4}{h^2(\gamma, \nu)p_\infty} \ln(\gamma + p_\infty). \quad (6.42)$$

The latter term is the source of the various logarithmic divergences entailed by the result of Refs. [24,25]. First, it causes the HE ($\gamma \rightarrow \infty$) behavior of $q_3^B(\gamma, \nu)$ to contain a $\ln \gamma$ enhancement of the $\sim \gamma^2$ behavior ensuring a well-defined massless limit (see Eq. (6.30) above), indeed

$$\hat{Q}^{\leq 3\text{PM}B}(u, \gamma, \nu) \stackrel{\text{HE}}{=} \frac{15}{2}u^2\gamma^2 + \left(8\ln(2\gamma) - \frac{17}{3}\right)u^3\gamma^2. \quad (6.43)$$

Second, it is the source of the tension with the result of Ref. [53]. Indeed, it generates a logarithmically divergent contribution to the 1SF quantity $\hat{Q}_B^{\text{1SF}}/\gamma^3$:

¹⁶As it is the large γ behavior that is of concern here, we could rephrase our discussion below by replacing everywhere the factor $16\gamma^4$ by its gravitational-coupling origin $w_1^2 = 4(2\gamma^2 - 1)^2$, as per the penultimate equation (9.2) of Ref. [25].

$$\frac{\hat{Q}_B^{\text{ISF}}}{\gamma^3} \stackrel{\text{HE}}{=} \frac{15}{2} u^2 + \frac{11}{3} u^3 + 16 \ln(2\gamma) u^3, \quad (6.44)$$

while $\hat{Q}^{\text{ISF}}/\gamma^3$ was found in Ref. [53] to have a finite HE limit [see Eq. (5.21)]. And third, it also leads to a logarithmic divergence when considering the HE limit $\gamma \rightarrow \infty$ (letting the masses $m_i \rightarrow 0$, and keeping fixed the c.m. energy E) of the two-particle scattering angle, namely (at the leading-logarithm accuracy),

$$\frac{1}{2} \chi^{B\text{HE}} = 2\alpha - 8 \ln(2\gamma) \alpha^3, \quad (6.45)$$

with $\alpha \stackrel{\text{HE}}{=} GE^2/(2J)$, as defined in Eq. (6.19). The latter result is in tension with the eikonal computations of the gravitational scattering angle of (quantum) massless particles [26,28].

These tensions have motivated us to propose above a modification (having a softer HE behavior) of the 3PM dynamics of Refs. [24,25]. However, the recently performed 6PN-accurate $O(G^3)$ computations [78–80] have disproved our softer-HE conjecture Eq. (6.16).

We wish now to present an alternative conjectural mechanism for canceling the three related logarithmic divergences, Eqs. (6.43), (6.44), and (6.45). We have seen in the previous subsection, that the $\gamma \rightarrow \infty$ limit of the radiative contribution entering the 4PM-level ($O(G^4)$) scattering angle χ had the remarkable property of descending from the $O(G^4)$ level to the $O(G^3 \ln G)$ one. In a similar manner, our proposed mechanism invokes the presence of a structure in (part of) the 4PM-level contribution to the EOB Q potential whose HE limit trickles down to the 3PM level and tames the three problematic 3PM-level logarithmic growths linked to the presence of the arcsinh function in Eq. (3.70). To motivate the possibility of this mechanism, let us start by noticing that one way to understand the technical origin of these various logarithmic growths is to view them (when considering the various rewritings of the arcsinh function exhibited in Eq. (6.41)) as due to the vanishing of the quantity $\gamma - p_\infty$, or, equivalently, of $1 - v_\infty$, as $\gamma \rightarrow \infty$. [In the HE limit, $\gamma \rightarrow \infty$, $\gamma - p_\infty = (\gamma + p_\infty)^{-1}$ tends to zero like $O(\frac{1}{\gamma})$, while $1 - v_\infty$ tends to zero like $O(\frac{1}{\gamma^2})$.] Both these quantities make use (in their construction) of the flat Minkowski metric, $\eta_{\mu\nu}$. E.g., $1 - v_\infty = (1 - v_\infty^2)/(1 + v_\infty)$ crucially involves $1 - v_\infty^2 = -\eta_{\mu\nu} dx^\mu dx^\nu / (dx^0)^2$. Now, the crucial contribution (6.40) comes from the $O(G^3)$ ‘‘H-diagram’’ 7 in Fig. 14 of Ref. [25]. At the next PM levels, $O(G^{\geq 4})$, there will appear (among other diagrams) modifications of the H-diagram comprising extra graviton exchanges between one of the external massive particle lines and, either the other massive particle, or one of the internal graviton lines. From the classical point of view, such modifications are related to some extra coupling to the metric field $h_{\mu\nu} = O(G)$, and can therefore be

viewed as modifying some of the occurrences of the flat metric $\eta_{\mu\nu}$ within the $O(G^3)$ diagrams. This intuitive argument suggests the possibility of an effective blurring of the light-cone-related quantity $1 - v_\infty^2 = -\eta_{\mu\nu} dx^\mu dx^\nu / (dx^0)^2$ that is at the root of the logarithmic blow-up of the $O(G^3)$ H-diagram. In other words, it is conceivable that some $O(G^{\geq 4})$ corrections will soften the $\gamma \rightarrow \infty$ logarithmic blow-up contained in the $O(G^3)$ arcsinh function. Such a possibility is connected with the known fact (discussed next) that the classical PM expansion is *not valid* for all Lorentz factors γ , but makes sense only if γ is *smaller* than some G -dependent upper limit.

The issue of the domain of physical validity of the PM expansion has been discussed in the literature on relativistic gravitational bremsstrahlung [90–93], though with unclear or conflicting answers. Peters [90] concludes (in the small mass-ratio case, $\nu \rightarrow 0$), that the PM expansion is valid only if

$$\gamma^2 \frac{GM}{b} \ll 1; \quad (6.46)$$

while D’Eath (see p. 1016 in [91]), cited by Kovács and Thorne [93], concludes that, for comparable masses [$\nu = O(1)$], the PM expansion is valid only if

$$h^2 \frac{GM}{b} \sim \gamma \frac{GM}{b} \ll 1. \quad (6.47)$$

To illustrate one of the technical origins of the limit (6.46), let us consider the scalar $h_{2\mu\nu} u_1^\mu u_1^\nu$, where (see, e.g., [48])

$$h_{2\mu\nu}(x) = 2 \frac{Gm_2}{R_2} (2u_{2\mu} u_{2\nu} + \eta_{\mu\nu}), \quad (6.48)$$

is the value, along the worldline of the first particle m_1 , of the harmonic-gauge linearized gravitational field generated by the second particle m_2 . During a small-angle hyperbolic encounter, the scalar (6.48) reaches the maximum value $(h_{2\mu\nu} u_1^\mu u_1^\nu)^{\text{max}} = 2 \frac{Gm_2}{b} (2\gamma^2 - 1) = w_1 \frac{Gm_2}{b}$. It seems then natural to require for the validity of the PM expansion that $w_1 \frac{Gm_2}{b} \sim \gamma^2 \frac{GM}{b} \ll 1$ (in agreement with Eq. (6.46)).

If we reexpress the various possible limits of validity of the PM expansion, Eqs. (6.46) or (6.47), in terms of the scattering angle

$$\frac{1}{2} \chi = \frac{GMh(\gamma, \nu)}{b} \frac{2\gamma^2 - 1}{p_\infty^2} + O(G^2), \quad (6.49)$$

we get limits of validity of the general type

$$\chi \gamma^n \ll 1, \quad (6.50)$$

with some (strictly) positive n ($n = \frac{3}{2}$ according to Peters, and our argument, and $n = \frac{1}{2}$ according to D’Eath).

Independently of the differences¹⁷ between these various validity constraints, the general requirement (6.50) (with any positive exponent n) is saying that one cannot trust taking the HE limit $\gamma \rightarrow \infty$ independently of the $\chi \rightarrow 0$ (or of the $G \rightarrow 0$) limit. This points out toward a possible noncommutativity of the two limits $\gamma \rightarrow \infty$ and $G \rightarrow 0$.

We are interested in transcribing the validity limit (6.50) in terms of the EOB gravitational potential $u = GM/R$, which enters the Q potential. When considering a small-angle scattering, the maximum value of u is defined by inserting $p_r = 0$ in the free-motion ($G \rightarrow 0$) EOB mass-shell condition, $\hat{\mathcal{E}}_{\text{eff}}^2 = 1 + p_r^2 + j^2 u^2$, so that

$$u_{\text{max}} = \frac{p_\infty}{j} = \frac{GMh(\gamma, \nu)}{b}. \quad (6.51)$$

We thereby see that in the HE limit $u_{\text{max}} \sim \chi$. Therefore the general limit (6.50) is equivalent to

$$\gamma^n u_{\text{max}} \ll 1 \quad \text{with} \quad n > 0 \quad \left(\text{and probably } \frac{1}{2} \leq n \leq \frac{3}{2} \right). \quad (6.52)$$

Combining this information about the limit of validity of the PM expansion, both with the reasoning above concerning $O(G^{\geq 4})$ corrections to the crucial $O(G^3)$ H-diagram, and with the proof given in the previous subsection of a $O(G^4) \mapsto O(G^3 \ln G)$ transmutation of PM order in the radiative part of the scattering angle, leads us to conjecture that the higher-PM contribution to the 3PM-accurate EOB Q potential will contain a term $\Delta\hat{Q}(\gamma, u, \nu)$ which is of order $u^4 = O(G^4)$ when $\gamma^n u \ll 1$, but which becomes of order $u^3 \ln(\gamma^n u)$ when $\gamma^n u \gg 1$, and which cancels the HE logarithmic blow-up, Eq. (6.43), of the 3PM potential $\hat{Q}^{\leq 3\text{PM}^B}(u, \gamma, \nu)$, Eq. (6.37).

Such a general requirement about the nature of the noncommutativity of the two limits $G \rightarrow 0$ and $\gamma \rightarrow \infty$ might be realized in many different ways. Let us illustrate the possibility of such a mechanism by a specific example of a $O(G^{\geq 4})$ term $\Delta\hat{Q}(\gamma, u, \nu)$. We are not claiming here that our example must be exactly the one that will enter the $O(G^{\geq 4})$ dynamics, but we propose it as an existence proof of a $O(G^{\geq 4})$ modification of the 3PM dynamics having interesting HE properties, and, in particular, reconciling the current 3PM dynamics, Eqs. (6.37), (6.38), with the SF result, Eq. (5.21).

Our proposed example consists in modifying the 3PM-accurate Q potential, Eqs. (6.37), (6.38), by an extra $O(G^{\geq 4})$ contribution of the form

$$\Delta\hat{Q}(\gamma, u, \nu) = -\frac{16\nu\gamma^3}{h^2(\gamma, \nu)} u^3 \frac{\ln(1 + \gamma^n u)}{n}, \quad (6.53)$$

where $n > 0$ refers to the exponent entering the general limit of validity, Eq. (6.52), of the PM expansion. [For simplicity, we did not include in the illustrative model (6.53) various possible modifications, such as a prefactor containing lower powers of γ (e.g., w_1^2/γ in lieu of $16\gamma^3$), and a numerical coefficient in front of $\gamma^n u$ in the argument of the logarithm.] The only crucial elements (for our discussion below) entering this illustrative definition of $\Delta\hat{Q}(\gamma, u, \nu)$ are the following: (i) the factor 16 in front of γ^3 ; (ii) the fact that the mass ratio ν only enters via the overall factor $\nu/h^2(\gamma, \nu)$; and (iii) the fact that the function $u^3 \ln(1 + \gamma^n u)/n$ is of order $O(u^4)$ as $u \rightarrow 0$, and $\approx u^3 (\ln \gamma + \frac{1}{n} \ln u)$ as $\gamma \rightarrow \infty$. [Evidently, many other functions could realize such requirements, or suitable variants of them.]

The dynamics defined by the *modified* Q potential

$$\hat{Q}^{\text{mod}}(u, \gamma, \nu) \equiv \hat{Q}^{\leq 3\text{PM}^B}(u, \gamma, \nu) + \Delta\hat{Q}(u, \gamma, \nu), \quad (6.54)$$

has the following properties.

First, it relieves the tension between Refs. [24,25] and Ref. [53]. Let us take the 1SF (linear in ν) contribution to the modified EOB Q potential (6.54), and then consider its HE limit. As our modification enters the EOB Hamiltonian multiplied by the overall factor $2\nu/h^2(\gamma, \nu)$, the 1SF piece in the new EOB Q potential (6.54) is given by:

$$\frac{\hat{Q}^{\text{mod 1SF}}}{\gamma^3} \stackrel{\text{HE}}{=} \frac{15}{2} u^2 + \left(\frac{11}{3} + 16 \ln(2) - \frac{16}{n} \ln(u) \right) u^3. \quad (6.55)$$

The major difference with the previous result, Eq. (5.17), is that the divergent logarithm $+\ln(\gamma)$ has been now replaced by $-\frac{1}{n} \ln(u)$. When evaluated at the light ring $u = \frac{1}{3}$, we thereby get a finite contribution involving $-\frac{1}{n} \ln(\frac{1}{3})$ instead of the divergent $\ln(\gamma)$, in qualitative agreement with the finite result found in [53].

Second, let us consider the HE limit, $\gamma \rightarrow \infty$, of the modified EOB Q potential (6.54). Contrary to the HE limit of $\hat{Q}^{\leq 3\text{PM}^B}(u, \gamma, \nu)$, displayed in Eq. (6.43) above, which did not define a good, quadratic-in- γ HE limit, the $\gamma \rightarrow \infty$ limit of \hat{Q}^{mod} now leads to a well-defined quadratic-in- γ HE limit, namely

$$\hat{Q}^{\text{mod}}(\gamma, \nu, u) \stackrel{\text{HE}}{=} q^{\text{HE new}}(u) \gamma^2, \quad (6.56)$$

where

$$q^{\text{HE new}}(u) \stackrel{\text{HE}}{=} \frac{15}{2} u^2 + \left(-\frac{17}{3} + 8 \ln(2) - \frac{8}{n} \ln(u) \right) u^3. \quad (6.57)$$

As announced, the latter HE limit has featured a phenomenon of transmutation of PM order. The HE limit of the

¹⁷These differences could due to a gauge dependence, and could also refer to the domains of validity of different observables.

$O(u^4) = O(G^4)$ additional contribution $\Delta\hat{Q}$ has been transmuted into a contribution of order $u^3 \ln u = O(G^3 \ln G)$. This property is intimately linked with the fact that the additional contribution $\Delta\hat{Q}$ was devised so as to cancel the $\gamma^2 \ln \gamma$ contribution present in the HE limit of $\hat{Q}^{\leq 3\text{PM B}}(u, \gamma, \nu)$.

Third, if we consider a fixed, finite value of γ , and take the PM expansion of $\Delta\hat{Q}$, i.e., its expansion in powers of G , we find that its 4PM-level, $O(u^4) = O(G^4)$, structure reads

$$\Delta\hat{Q}^{\text{PM-expanded}} = -\frac{16\nu\gamma^4}{nh^2(\gamma, \nu)}\gamma^{n-1}u^4 + O(u^5). \quad (6.58)$$

Taking the HE limit of this PM-expanded contribution yields

$$\Delta\hat{Q}^{\text{PM-expanded HE}} = -\frac{8}{n}\gamma^{2+n}u^4 + O(u^5). \quad (6.59)$$

As we had assumed $n > 0$ (and probably $n \geq \frac{1}{2}$), we see that this contribution violates (in a *power-law* fashion) the expected quadratic-in- γ HE behavior. This violation at the level of the Hamiltonian entails a corresponding power-law violation of the naively expected behavior of scattering observables (at the 4PM level). Namely, instead of having a 4PM-level contribution $\chi_4(\gamma, \nu)$ behaving, when $\gamma \rightarrow \infty$, $\propto \gamma^4$ (like the test-particle one), the term (6.59) would yield a contribution $\propto \gamma^{4+n}$. This apparent fast growth as $\gamma \rightarrow \infty$ would, however, be an effect of having PM-expanded the factor $\ln(1 + \gamma^n u)$ and is absent in the exact, non-PM-expanded scattering angle $\chi(\gamma, j, \nu)$.

Indeed, the real value of the HE scattering angle predicted by Eq. (6.54) is obtained from the HE limit of the modified Q potential, i.e., from the HE quadratic mass-shell constraint Eq. (6.56), with Eq. (6.57). Similarly to what happened for the 1SF-level contribution $\hat{Q}^{\text{1SF new}}$, one finds that this now predicts a *finite* 3PM-level massless scattering angle which differs from the previous logarithmically divergent one, Eq. (6.21), by the replacement of $\ln(\gamma)$ by $-\frac{1}{n}\ln(\alpha)$. Namely, at the leading-logarithm accuracy (i.e., modulo some $\propto \alpha^3$ contribution), one finds

$$\frac{1}{2}\chi^{\text{mod HE}} = 2\alpha - \frac{8}{n}\ln\left(\frac{1}{\alpha}\right)\alpha^3. \quad (6.60)$$

We have written it here in terms of $\ln(\frac{1}{\alpha})$ to emphasize that the sign of the finite logarithmic contribution is the same (namely negative) as the sign of the previously divergent contribution $-8\ln(2\gamma)\alpha^3$. On the one hand, this sign differs from the corresponding positive 3PM contribution $\sim +\alpha^3$ found by ACV, and recently confirmed in Ref. [28]. On the other hand, we note that the estimate (6.60) agrees in magnitude and sign with the contribution $O(\alpha^3 \ln 1/\alpha)$ in

Eq. (6.35) derived above from considering the radiative correction to the classical scattering angle.

Let us finally note that the structure we used in our illustrative model (6.54) is not, by itself, leading to a scattering angle satisfying the general mass-ratio-dependence properties discussed in Sec. II. There are, however, ways to design a modified version of $\Delta\hat{Q}(\gamma, u, \nu)$ that would incorporate the latter expected mass-ratio-dependence. We found that such better (but more complicated) models predict the same general features we just discussed. For simplicity, and in the absence of precise guidelines for choosing among such models, we do not feel it is useful to complicate our discussion by indicating the construction of such models.

In conclusion: our second (illustrative) ansatz (6.54) relieves the tension between Refs. [24,25] and Ref. [53], and leads to some generic predictions for the HE behavior of, both, the full dynamics, and its 4PM-truncated version. There remains [as was the case with the first conjecture, Eq. (6.16)] a tension between the massless limit of the *classical* scattering, and the *quantum*, eikonal-based massless scattering angle of Refs. [26,28]. As we already pointed out, the root of the latter discrepancy might reside in subtleties of the quantum-to-classical transition (with a possible noncommutativity of the two limits $\gamma \rightarrow \infty$, and $\hbar \rightarrow 0$), or in the use of the quantum-eikonal-approximation.

VII. SUMMARY

This paper has derived new general properties of post-Minkowskian (PM) gravity, notably in its effective one body (EOB) formulation. Our first result has been to prove general expressions for the dependence of the momentum transfer (during the classical scattering of two masses) on the two masses, and thereby on the symmetric mass ratio ν [see Eqs. (2.15), (2.24)]. This implies specific constraints on the ν dependence of the scattering angle considered as a function of the reduced angular momentum $j \equiv J/(Gm_1m_2)$ [see Eqs. (2.45), (2.46), (2.48)]. A useful consequence of these results is that the full knowledge of the 3PM dynamics is encoded in a single function of the single variable $\gamma = -(p_1 \cdot p_2)/(m_1m_2)$. Moreover the same property holds also at the 4PM level. We pointed out that these properties allow *first-order self-force* (linear in mass ratio) computation of scattering to give access to the *exact* 3PM and 4PM dynamics.

We then generalized our previous work [20] by deriving, up to the 4PM level included, the explicit links between the scattering angle and the two types of potentials entering the Hamiltonian description of PM dynamics within EOB theory. The first type of potential is the Q potential entering the mass-shell condition of EOB dynamics

$$0 = g_{\text{Schwarz}}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q(X, P), \quad (7.1)$$

while the second one is an ordinary, energy-dependent radial potential $W(E, \bar{R})$ entering a nonrelativistic-like quadratic constraint on the EOB momentum,

$$\mathbf{P}^2 = P_\infty^2 + W(\bar{u}, P_\infty). \quad (7.2)$$

The first formulation is usually expressed in terms of a Schwarzschild-like radial coordinate R (with $u = GM/R$), while the second one uses an isotropic-like radial coordinate \bar{R} (with $\bar{u} = GM/\bar{R}$). The links between the PM expansion coefficients of both types of formulations, as well as their links with the PM expansion coefficients of the scattering function, were given in Sec. III. (See Appendix A for the link of the EOB potential with the potential used in Refs. [24,25].) At the end of Sec. III, we summarized the current knowledge of the PM-expanded dynamics and emphasized the apparent incompatibility between the recent classical 3PM-level dynamics derived by Bern *et al.* [24,25] and the self-force computation of Ref. [53]. We then suggested two different types of resolution of this tension. The first resolution conjectures that the 3PM dynamics has a softer high-energy (HE) behavior than the one derived in Refs. [24,25]. Namely, we conjectured that the function $\bar{C}(\gamma)$ entering the 3PM dynamics might have a softer HE behavior than Eq. (3.70) (see Eq. (6.16)). However, several recent 6PN-accurate $O(G^3)$ computations [78–80] have disproved the (HE-softer) conjectured 3PM dynamics (6.16).

In subsection IV C we recalled a classic argument of Niels Bohr showing the lack of overlap between the domains of validity of classical and quantum scattering theory. This fact might entail subtleties in the quantum-to-classical maps used in several recent works.

We also presented a second type of possible resolution of the tension between Refs. [24,25] and Ref. [53]. This second resolution does not call for a modification of the 3PM dynamics of Refs. [24,25] when it is considered at a finite value of the Lorentz factor γ (denoted σ in Refs. [24,25]), but assumes a particular type of non-commutativity of the two limits $\gamma \rightarrow \infty$, and $G \rightarrow 0$. We emphasized that the PM expansion is expected to lose its validity when γ becomes larger than some inverse power of GM/b (or GM/r), see Eqs. (6.50), (6.52). We gave an illustrative model of higher PM ($O(G^{\geq 4})$) contributions to the currently known $O(G^{\leq 3})$ dynamics able to reconcile the results of Refs. [24,25] and Ref. [53]; see Eq. (6.54). This model makes some generic predictions (explained in the previous section) and exhibits an interesting phenomenon of HE transmutation of post-Minkowskian order. Namely the HE limit of a $O(G^{\geq 4})$ term becomes of order $O(G^3 \ln G)$ when $\gamma^n u \gg 1$, for some positive exponent n . Independently of the motivation for our conjecture, we showed (in Sec. VID) that such a HE transmutation of PM order [from ($O(G^4)$) down to $O(G^3 \ln G)$] does take place in

the radiative contribution to the scattering angle of classical massless particles.

Section IV presented the 3PM generalization of a result of Ref. [20], namely the computation of the scattering amplitude derived from quantizing the 3PM EOB potential. Our computation explicitly takes into account the IR-divergent contributions coming from the Born iterations of the EOB radial potential. The usual potential-scattering amplitude f_{eob} in the EOB radial potential is linked to a corresponding Lorentz-invariant amplitude \mathcal{M} via the simple rescaling

$$\mathcal{M}_{\text{eob}} = \frac{8\pi G_s}{\hbar} f_{\text{eob}}. \quad (7.3)$$

Section VI (as well as Appendices B and C) discusses various features of the high-energy (or massless) limit of the PM dynamics.

Note finally that a general theme of the present work has been to highlight some of the subtleties involved when considering several *a priori* noncommuting limits: $\hbar \rightarrow 0$ versus $\hbar \rightarrow \infty$; $G \rightarrow 0$; $\gamma \rightarrow \infty$; and $\nu \rightarrow 0$. The existing tension between: (i) the (logarithmically divergent) high-energy limit, Eq. (6.45), of the scattering angle of Refs. [24,25]; (ii) the quantum-eikonal-based computation [26,28] of the scattering angle of *quantum* massless particles, Eq. (6.22); and, (iii) the type of (finite) scattering angle of *classical* massless particles predicted by both our HE-softer conjectures (see notably Eq. (6.60)), needs further clarification.

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APPENDIX A: MAP BETWEEN THE EOB POTENTIAL AND THE POTENTIAL OF CHEUNG, ROTHSTEIN, AND SOLON

Cheung, Rothstein and Solon (CRS) [22] have proposed to describe the classical dynamics of a two-body system by the same type of Hamiltonian that was considered long ago by Corinaldesi and Iwasaki, namely

$$H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) = c^2 \sqrt{m_1^2 + \frac{\mathbf{p}_1^2}{c^2}} + c^2 \sqrt{m_2^2 + \frac{\mathbf{p}_2^2}{c^2}} + V(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2), \quad (A1)$$

except that they did not limit themselves to working with the PN-expanded form of such an Hamiltonian

(à la Eq. (1.2)). In addition, when working in the c.m. frame (with the c.m. Hamiltonian reduction $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{P}$), they required a specific isotropiclike gauge-fixing of the c.m. potential $V(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)^{\text{c.m.}} = V(\mathbf{X}, \mathbf{P})$ such that $V(\mathbf{X}, \mathbf{P})$ depends only on \mathbf{P}^2 and $R \equiv |\mathbf{X}|$:

$$V(\mathbf{P}, \mathbf{X}) = G \frac{c_1(\mathbf{P}^2)}{|\mathbf{X}|} + G^2 \frac{c_2(\mathbf{P}^2)}{|\mathbf{X}|^2} + G^3 \frac{c_3(\mathbf{P}^2)}{|\mathbf{X}|^3} + \dots \quad (\text{A2})$$

Reference [22] derived a 2PM-accurate potential (from the quasi-classical one-loop amplitude of Refs. [19,21]) without connecting this potential to the previously derived (simpler) 2PM-accurate EOB potential of Ref. [20]. To complete our study, let us sketch how the two types of potentials are related by using the tools we have introduced above. We will be brief because results essentially equivalent (and sometimes to higher-orders) to the results below (though formulated differently) have already been displayed in Refs. [25,55,56].

The gauge-invariant characterisation of the successive coefficients w_n entering the energy-dependent version of the EOB potential obtained in Sec. III C gives a simple algorithmic procedure for extracting the gauge-invariant information from the PM expansion (A2) of the CRS potential $V(\mathbf{X}, \mathbf{P})$. Let us sketch how this can be done.

Starting from

$$H(\mathbf{P}, \mathbf{X}) = \sqrt{m_1^2 + \mathbf{P}^2} + \sqrt{m_2^2 + \mathbf{P}^2} + V(R, \mathbf{P}^2), \quad (\text{A3})$$

with

$$V(R, \mathbf{P}^2) = G \frac{c_1(\mathbf{P}^2)}{R} + G^2 \frac{c_2(\mathbf{P}^2)}{R^2} + G^3 \frac{c_3(\mathbf{P}^2)}{R^3} + \dots \quad (\text{A4})$$

and denoting as P_∞ the (common) magnitude of the c.m. incoming (and outgoing) momenta, such that the total (conserved) energy $E_{\text{real}} = \sqrt{s}$ of the two-body system reads

$$E_{\text{real}}(P_\infty^2) = \sqrt{m_1^2 + P_\infty^2} + \sqrt{m_2^2 + P_\infty^2}, \quad (\text{A5})$$

we can perturbatively solve the energy conservation law $E_{\text{real}} = H(\mathbf{P}, \mathbf{X})$ for \mathbf{P}^2 . Beware that, in this Appendix, we will use the notation P_∞ (without extra label) to denote the magnitude of the asymptotic physical c.m. three-momentum. This quantity differs from the corresponding EOB incoming momentum, which was also denoted $P_\infty = \mu p_\infty$ in the main text. Here, we will denote the latter EOB incoming momentum as $P_\infty^{\text{EOB}} = \mu p_\infty^{\text{eob}}$. The relation between $P_\infty \equiv P_\infty^{\text{cm}}$ and P_∞^{EOB} will be recalled below.

We look for a PM expansion of the type

$$\mathbf{P}^2 = P_\infty^2 + \frac{W_1(P_\infty)}{R} + \frac{W_2(P_\infty)}{R^2} + \frac{W_3(P_\infty)}{R^3} + \dots \quad (\text{A6})$$

where $W_n \propto G^n$, such that the insertion of the expansion (A6) in Eq. (A3), with the PM-expanded potential (A4) solves the constraint $E_{\text{real}} = H(\mathbf{P}, \mathbf{X})$. At first order in G , this yields the constraint

$$\frac{dE_{\text{real}}(P_\infty^2)}{dP_\infty^2} \frac{W_1(P_\infty)}{R} + G \frac{c_1(P_\infty^2)}{R} = 0, \quad (\text{A7})$$

which uniquely determines $W_1(P_\infty)$ in terms of $c_1(P_\infty^2)$, namely

$$W_1(P_\infty) = - \left(\frac{dE_{\text{real}}(P_\infty^2)}{dP_\infty^2} \right)^{-1} G c_1(P_\infty^2). \quad (\text{A8})$$

At second order in G , we similarly get an equation uniquely determining $W_2(P_\infty)$ in terms of $c_2(P_\infty^2)$, of the P_∞^2 derivative of $c_1(P_\infty^2)$, and of the previously determined $W_1(P_\infty)$, namely

$$W_2(P_\infty) = - \left(\frac{dE_{\text{real}}}{dP_\infty^2} \right)^{-1} (G^2 c_2(P_\infty^2) + G \frac{dc_1(P_\infty^2)}{dP_\infty^2} W_1 + \frac{1}{2} \frac{d^2 E_{\text{real}}}{(dP_\infty^2)^2} W_1^2). \quad (\text{A9})$$

This algorithmic procedure successively determines the coefficients $W_n(P_\infty)$ entering the PM expansion (A6) in terms of the sequence of functions $c_n(\mathbf{P}^2)$. The results of this procedure agree with the corresponding results in section 11.3.1 of Ref. [25], but we will use them here to relate the EOB Q potential to the CRS V potential.

The next step is to transform the coefficients $W_n(P_\infty)$ into their corresponding gauge-invariant avatars $\tilde{W}_n(P_\infty)$, defined in the same way as in Eq. (3.46) above, namely

$$\begin{aligned} \tilde{W}_1(P_\infty) &= \frac{W_1(P_\infty)}{P_\infty}, \\ \tilde{W}_2(P_\infty) &= W_2(P_\infty), \\ \tilde{W}_3(P_\infty) &= P_\infty W_3(P_\infty), \\ \tilde{W}_4(P_\infty) &= P_\infty^2 W_4(P_\infty). \end{aligned} \quad (\text{A10})$$

Then, applying the reasoning made around Eq. (3.46) above, we conclude that the $\tilde{W}_n(P_\infty)$'s extracted from the sequence of functions $c_n(\mathbf{P}^2)$'s must be numerically identical to the $\tilde{w}_n(p_\infty)$'s entering the EOB potential. One must simply take care of the presence of a factor $(Gm_1 m_2)^n$ due to the rescaling factors, $P = \mu p$, $E = Mh$, $J = GM\mu j$, used above, and of the (crucial) fact that the CRS and EOB quantities are expressed as functions of different variables, namely $P_\infty \equiv P_\infty^{\text{cm}}$ versus $P_\infty^{\text{EOB}} = \mu p_{\text{eob}}$. At this stage, we need to recall that, according to, e.g., Eq. (10.27) of Ref. [20], the (rescaled) EOB incoming momentum $p_{\text{eob}} = p_\infty^{\text{eob}}$ is related to the real, c.m. incoming momentum P_∞ by

$$E_{\text{real}} P_{\infty}^{\text{real}} = m_1 m_2 \sqrt{\gamma^2 - 1} \equiv m_1 m_2 p_{\infty}^{\text{eob}}. \quad (\text{A11})$$

Finally, we have the simple relations

$$\begin{aligned} \tilde{W}_1(P_{\infty}) &= G m_1 m_2 \tilde{w}_1^{\text{eob}}(\gamma), \\ \tilde{W}_2(P_{\infty}) &= (G m_1 m_2)^2 \tilde{w}_2^{\text{eob}}(\gamma), \\ \tilde{W}_3(P_{\infty}) &= (G m_1 m_2)^3 \tilde{w}_3^{\text{eob}}(\gamma), \\ \tilde{W}_4(P_{\infty}) &= (G m_1 m_2)^4 \tilde{w}_4^{\text{eob}}(\gamma). \end{aligned} \quad (\text{A12})$$

The first two EOB PM levels have been computed in Ref. [20] and yielded the results

$$\begin{aligned} \tilde{w}_1^{\text{eob}}(\gamma) &= \frac{2(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}}, \\ \tilde{w}_2^{\text{eob}}(\gamma) &= \frac{3(5\gamma^2 - 1)}{2 h(\gamma, \nu)}. \end{aligned} \quad (\text{A13})$$

We have checked that by inserting the latter simple expressions in the relations written above gave the (much more intricate) expressions of c_1 and c_2 derived in [22]. Note, in particular, that the asymptotic value ξ_{∞} of the symmetric energy ratio defined in [22], namely

$$\xi(\mathbf{P}^2) \equiv \frac{\sqrt{m_1^2 + \mathbf{P}^2} \sqrt{m_2^2 + \mathbf{P}^2}}{(\sqrt{m_1^2 + \mathbf{P}^2} + \sqrt{m_2^2 + \mathbf{P}^2})^2}, \quad (\text{A14})$$

which does not appear in the EOB results, enters in c_1 via the derivative

$$\frac{dE_{\text{real}}(P_{\infty}^2)}{dP_{\infty}^2} = \frac{1}{2\xi_{\infty} E_{\text{real}}(P_{\infty}^2)}. \quad (\text{A15})$$

When working at the 3PM-level one can similarly relate the coefficients c_3 , $\tilde{W}_3(P_{\infty})$, $\tilde{w}_3^{\text{eob}}(\gamma)$ and $q_3(\gamma)$, and explicitly check that the value of c_3 given in the last Eq. (10.10) of [25] is equivalent to the (much simpler) expression of q_3 obtained in the main text (and also derived in Ref. [51] by using the formulas of [20]). Let us finally note that Refs. [55,56] derived all-order expressions for the links between the quantities c_n and w_n (without considering, however, the more basic EOB coefficients q_n).

APPENDIX B: ON THE STRUCTURE OF THE HE LIMIT OF PM GRAVITY

To complete our discussion of PM gravity, let us briefly discuss some of the structures that might arise in the HE limit of the classical momentum transfer \mathbf{Q} , considered as a function of the impact parameter, Eq. (2.24). We have discussed above two different possibilities for reconciling the current quantum-based computation of 3PM dynamics, and older HE SF results. The first possibility assumes that

the HE limit of classical scattering is as tame at the third (and higher) PM order(s) than it is at the first and second PM orders. The second possibility allows for violations of the latter tame HE behavior. We shall contrast the structures corresponding to these two possibilities.

To discuss the HE behavior, let us reformulate the classical time-symmetric Lorentz-invariant, PM perturbation-theory computation of the momentum change $\Delta p_{1\mu} = -\Delta p_{2\mu}$. Above we wrote this PM perturbation theory in terms of two worldlines parametrized by their proper times s_a , so that $u_a^{\mu} = dx_a^{\mu}/ds_a$ were two unit vectors, because we wanted to keep track of the dependence on the two rest masses m_a , entering the stress-energy tensor as multiplicative factors. But we could have, instead, as was actually done in [20,46], use worldline parameters $\sigma_a = s_a/m_a$ such that $dx_a^{\mu}/d\sigma_a = m_a u_a^{\mu} = p_a^{\mu}$. In this parametrization the stress-energy tensor does not involve the masses, but only the momenta, and reads

$$\begin{aligned} T^{\mu\nu}(x) &= \sum_{a=1,2} \int d\sigma_a p_a^{\mu} p_a^{\nu} \frac{\delta^4(x - x_a(\sigma_a))}{\sqrt{g}} \\ &= \sum_{a=1,2} \int p_a^{\mu} dx_a^{\nu} \frac{\delta^4(x - x_a)}{\sqrt{g}}. \end{aligned} \quad (\text{B1})$$

One then checks that the masses will never explicitly occur in this reformulation of PM perturbation theory. This reformulation is useful for treating the limiting case where $m_a \rightarrow 0$, $u_a^{\mu} \rightarrow \infty$, keeping fixed the values of the momenta $p_a^{\mu} = m_a u_a^{\mu}$. In this limit the two momenta, and the two worldlines, become lightlike: $p_a^2 = -m_a^2 \rightarrow 0$. The expressions written down in Refs. [20,46] then define a formal PM perturbation theory that applies when one or two of the particles are massless. Let us consider the case where both particles are massless. A difference with the massive case is that the convolution of the time-symmetric propagator $\propto \delta[(x-y)^2]$ with a $T^{\mu\nu}(y)$ localized along a null geodesic (which is straight at LO) selects a single (advanced or retarded) source point x_a on each worldline. [Indeed, the LO equation to be solved in σ_a , for a given field point x , namely $(x - x_a^0 - p_a^0 \sigma_a)^2 = 0$, is linear, rather than quadratic, in σ_a because $(p_a^0)^2 = 0$.] The corresponding linearized approximation for the metric (in harmonic gauge) reads

$$h_{\mu\nu}^{m_a=0}(x) = \sum_a 4G \frac{P_{a\mu} P_{a\nu}}{|(x - x_a) \cdot p_a|} + O(G^2). \quad (\text{B2})$$

In the presently considered case where the p_a 's are null, the expression (B2) represents a sum of Aichelburg-Sexl metrics [94] associated with each worldline. Each Aichelburg-Sexl metric is flat (zero curvature) outside of the null hyperplanes $(x - x_a) \cdot p_a = 0$, but has nonzero curvature concentrated (in a Dirac-delta manner) on these hyperplanes. Correspondingly, the decay at large distances

of $h_{\mu\nu}^{m_a=0}(x)$ (in harmonic gauge) is nonuniform, and weaker in some directions than for its finite-mass analog. This raises delicate issues about the convergence of the integrals appearing at each order of the PM expansion. Some of these issues have been discussed by D'Eath [91] (who works with large but finite γ), and by Gruzinov and Veneziano [88] (who argue that the massless limit, $\gamma \rightarrow \infty$, does exist). This issue might be alleviated by choosing a suitable (nonharmonic) gauge for representing the physical content of the metric (B2).

We shall assume here that the formal PM perturbation theory for the scattering of two massless particles leads to well-defined integral expressions for the vectorial momentum transfer $\Delta p_\mu \equiv \Delta p_{1\mu} = p'_{1\mu} - p_{1\mu} = -\Delta p_{2\mu}$.

The (incoming) vectorial impact parameter b_μ [such that $b \cdot p_1(-\infty) = 0 = b \cdot p_2(-\infty)$] is easily seen to be uniquely defined by the geometrical configuration made by the two incoming (null) worldlines. One can then write Δp_μ as a Poincaré-covariant function of b_μ and of the two incoming momenta. As before the corresponding scalar

$$\mathbf{Q}(p_1, p_2, b) \equiv \sqrt{-t} \equiv \sqrt{\eta^{\mu\nu} \Delta p_{1\mu} \Delta p_{1\nu}}, \quad (\text{B3})$$

must be a Lorentz scalar covariantly constructed from the vectors b_μ , and $p_{a\mu}$ (the latter denoting the incoming values of the momenta). As b_μ is (by definition) orthogonal to the two momenta, and as the momenta have vanishing Lorentz norms, the only nonzero scalar product [besides $b^2 \equiv (b \cdot b)$] that can be extracted from the geometrical configuration p_1, p_2, b is the scalar product $|(p_1 \cdot p_2)| = -(p_1 \cdot p_2)$. [We assume that p_1 and p_2 are both future-oriented so that $(p_1 \cdot p_2) < 0$.] This technical fact can be geometrically understood as follows. After fixing the vectorial impact parameter b_μ , the geometrical configuration defined by the two incoming null worldlines admits as symmetry group the subgroup of the Lorentz group made of boosts acting in the two-plane spanned by the two null vectors p_1 and p_2 . If we consider a null frame with two null vectors ℓ^μ, n^μ , respectively parallel to p_1 and p_2 , but normalized so that $\ell \cdot n = -1$, these boosts are parametrized by a scalar k (equal to $\sqrt{(1-v)/(1+v)}$ in terms of the usual boost velocity v) acting on the null frame ℓ, n as $\ell \rightarrow k\ell, n \rightarrow k^{-1}n$. These boosts change the components of p_1 and p_2 along the null basis vectors ℓ, n (say $p_1^\mu = p_{1\ell}\ell^\mu$ and $p_2^\mu = p_{2n}n^\mu$) by factors k^{-1} and k , respectively. The Lorentz scalar $\mathbf{Q}(p_1, p_2, b)$ must be invariant under these Lorentz frame transformations. [One could gauge-fix this residual Lorentz symmetry by going to the c.m. frame where the spatial components of p_1 and p_2 are opposite, but the idea here is, on the contrary, to use this symmetry to constrain the expression of $\mathbf{Q}(p_1, p_2, b)$.]

Summarizing: The (classical) scalar momentum transfer $\mathbf{Q}(p_1, p_2, b)$ can only be a function of the two scalars $|(p_1 \cdot p_2)| = -(p_1 \cdot p_2)$ and b .

The first term in the PM expansion of $\mathbf{Q}(p_1, p_2, b)$ is obtained by taking the massless limit $m_a \rightarrow 0, p_a^2 \rightarrow 0$ (equivalent to considering the HE limit) of the beginning of the finite-mass expression of $\mathbf{Q}(p_1, p_2, b; m_1, m_2)$:

$$\begin{aligned} & \frac{1}{2} \mathbf{Q}(p_1, p_2, b, m_1, m_2) \\ &= \frac{G}{b} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}} \\ & \quad + \frac{3\pi G^2 (m_1 + m_2) (5(p_1 \cdot p_2)^2 - p_1^2 p_2^2)}{8 b^2 \sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}} + O(G^3). \end{aligned} \quad (\text{B4})$$

This yields

$$\frac{1}{2} \mathbf{Q}(p_1, p_2, b, 0, 0) = \frac{2G|(p_1 \cdot p_2)|}{b} + O\left(\frac{G^3}{b^3}\right). \quad (\text{B5})$$

The structure of PM perturbation theory formally generates, at each PM order G^N , an expression for $\mathbf{Q}(p_1, p_2, b)$ that is a homogeneous polynomial of order $N + 1$ in $p_{1\ell}$ and p_{2n} , and that is proportional to $1/b^N$. Using now dimensional analysis, and looking at the dimension of $\mathbf{Q}(p_1, p_2, b) \sim \frac{G|(p_1 \cdot p_2)|}{b}$, it is easy to see that $N + 1$ must be an even integer, and that the $O(G^N)$ contribution to $\mathbf{Q}(p_1, p_2, b)$ must be a polynomial (of order $(N + 1)/2$) in the product of components $p_{1\ell} p_{2n}$, i.e., in the scalar product $|(p_1 \cdot p_2)| = -(p_1 \cdot p_2)$. This leads to a PM expansion for $\mathbf{Q}(p_1, p_2, b)$ of the form

$$\begin{aligned} \frac{1}{2} \mathbf{Q}^{\text{massless}}(p_1, p_2, b) &= 2 \frac{G|(p_1 \cdot p_2)|}{b} + \mathbf{Q}_3 \frac{G^3 |(p_1 \cdot p_2)|^2}{b^3} \\ & \quad + \mathbf{Q}_5 \frac{G^5 |(p_1 \cdot p_2)|^3}{b^5} + \dots \end{aligned} \quad (\text{B6})$$

with some *dimensionless* odd-order coefficients $\mathbf{Q}_3, \mathbf{Q}_5$, etc. The corresponding structure for the scattering angle, considered as a function of

$$\alpha \equiv \frac{G|(p_1 \cdot p_2)|}{J} \stackrel{\text{HE}}{=} \frac{\gamma}{j}, \quad (\text{B7})$$

is

$$\frac{\chi}{2} \stackrel{\text{HE}}{=} 2\alpha + c_3^\chi \alpha^3 + c_5^\chi \alpha^5 + c_7^\chi \alpha^7 + \dots \quad (\text{B8})$$

with some corresponding dimensionless coefficients c_3^χ, c_5^χ , etc.

We have thereby recovered, at the classical level, the structure that was deduced, in the case of the HE quantum scattering, by Amati, Ciafaloni and Veneziano [26] from analyticity requirements in s . We see that it follows from the classical symmetry discussed above.

Let us first emphasize that there are two possibilities concerning the dimensionless coefficients $\mathbf{Q}_3, \mathbf{Q}_5, \dots$, or $c_3^\chi, c_5^\chi, \dots$, which can be thought of corresponding to the two possible conjectures made in the text. The most conservative scenario is that the latter dimensionless coefficients are pure numbers. This would naturally correspond to our first conjecture (of a soft HE behavior). Indeed, the conjectured (HE-soft) 3PM dynamics (6.16) leads to a nonzero $O(G^3/b^3)$ contribution in the HE limit of the form $\mathbf{Q}_3 G^3 |(p_1 \cdot p_2)|^2/b^3 = \mathbf{Q}_3 G^3 (m_1 m_2)^2 \gamma^2/b^3$, with a finite numerical coefficient \mathbf{Q}_3 . Let us note that this term is the only term to survive, at $O(G^3)$, in the HE limit of the general finite-mass expression (2.15) because the corresponding coefficient $\mathbf{Q}_{12}^{3\text{PM}}(\gamma)$ grows like γ^2 when $\gamma \rightarrow \infty$, i.e., faster than $\mathbf{Q}_{11}^{3\text{PM}}(\gamma) = \mathbf{Q}_{22}^{3\text{PM}}(\gamma) \sim \gamma$. [One can check that, at any PM order, all the coefficients $\mathbf{Q}_{11\dots 1}$, or $\mathbf{Q}_{22\dots 2}$ of the terms involving only one of the two masses, grow like γ when $\gamma \rightarrow \infty$.] A similar HE dominance $\sim \gamma^{n+1}$ of the coefficient of $(m_1 m_2)^n$ at $(2n+1)$ -PM (e.g., $\mathbf{Q}_{1122}^{5\text{PM}}(\gamma) \sim \gamma^3$) would ensure that the HE limit of Eq. (2.15) yields the form (B6).¹⁸ Moreover, in that case, the vanishing of the even coefficients c_{2n}^χ implies a specific HE behavior for the corresponding coefficients $q_{2n}^E(\gamma, \nu)$ in the PM expansion of the energy-gauge EOB Q potential¹⁹;

$$\hat{Q}^E(u, \gamma, \nu) = u^2 q_2(\gamma, \nu) + u^3 q_3(\gamma, \nu) + u^4 q_4^E(\gamma, \nu) + O(G^5). \quad (\text{B9})$$

Under our present soft-HE-behavior assumption, we would have a *quadratic* HE behavior for $q_n^E(\gamma, \nu)$, namely relations of the type (6.8) or (6.9), with ν -independent numerical coefficients c_n^{qE} or c_n^{qH} . Then the vanishing of the even asymptotic coefficients c_{2n}^χ leads to the following links

$$c_2^{qE} = c_2^{qH} = \frac{15}{2}, \quad (\text{B10})$$

which we already knew, and the new links

$$c_3^{qE} = c_3^{qH} = -c_3^\chi + \frac{64}{3} - 2c_2^{qE} = -c_3^\chi + \frac{19}{3}, \quad (\text{B11})$$

and

$$c_4^{qE} = -3c_3^{qE} + \frac{705}{16} = 3c_3^\chi + \frac{401}{16}. \quad (\text{B12})$$

For instance, the first conjectured 3PM result (6.16) implies $c_3^\chi = -\frac{14}{3}$, which would, in turn, imply the following results

¹⁸I thank Gabriele Veneziano for a useful question concerning this issue.

¹⁹Beware of not confusing the EOB Q potential with the momentum transfer \mathbf{Q} .

$$c_3^{qE} = +11, \quad \text{and} \quad c_4^{qE} = \frac{177}{16}. \quad (\text{B13})$$

In other words, the corresponding HE mass-shell condition would read

$$-\hat{\mathcal{E}}_{\text{eff}}^2 \left(\frac{1}{1-2u} - \bar{f}(u) \right) + (1-2u)p_r^2 + j^2 u^2 = 0, \quad (\text{B14})$$

with $\bar{f}(u) \equiv \frac{15}{2}u^2 + 11u^3 + \frac{177}{16}u^4 + O(u^5)$. Though the specific soft-HE conjecture (6.16) is now disproved, we mention these facts here to emphasize that, under the present (soft HE) assumption, one can derive 4PM-level information from a 3PM-level one (similarly to Eq. (7.14) of [20] which used the ACV result as input information).

Let us now discuss the impact of our alternative conjecture, exemplified above by the additional piece (6.53). In such a scenario, the result (6.60) shows that the dimensionless coefficient \mathbf{Q}_3 is no longer a pure number but involves the logarithm of the dimensionless quantity

$$\delta = \frac{GE_{\text{c.m.}}}{b} = \frac{G\sqrt{s}_{\text{HE}}}{b} \frac{G\sqrt{2|(p_1 \cdot p_2)|}}{b}. \quad (\text{B15})$$

As we indicated in Sec. VI E above, the corresponding ($\propto G^3 \ln G$) $\ln \delta$ contribution to \mathbf{Q}_3 has descended from a $O(G^4)$ contribution in the usual finite-mass PM expansion. We can similarly expect that the higher odd-order coefficients \mathbf{Q}_{2n+1} will also involve the logarithm of δ . Note that we are talking here about logarithmic contributions that might occur in the scattering angle of *classical* massless particles. The analytic structure of the scattering angle of *quantum* massless particle might be different, notably if analyticity requirements forbid the presence of $\ln s$ (and therefore $\ln \alpha$) in χ .

APPENDIX C: ON THE INTERPLAY BETWEEN THE SF EXPANSION, THE HE BEHAVIOR, AND THE PM EXPANSION

Let us show how SF theory gives us access to some structural information about the HE limit of the scattering angle. We can use a reasoning which generalizes the one used in Ref. [53] to understand the HE behavior found there when considering 1SF expanded quantities near the light ring.

Let us imagine analytically computing the SF expansion for the total change of momentum of a small-mass particle (say of mass m_1) scattering (at some given impact parameter, or with some given angular momentum) on a large-mass black hole (say of mass $m_2 \gg m_1$). It can be formally obtained by replacing on the right-hand side of

$$\Delta u_{1\mu} = \frac{1}{2} \int_{-\infty}^{+\infty} \partial_\mu g_{\alpha\beta}(x_a) u_a^\alpha dx_a^\beta, \quad (\text{C1})$$

$g_{\alpha\beta}$ by $g_{\alpha\beta}^{(0)}(m_2) + h_{\alpha\beta}$ (and correlated $O(\nu)$ changes in u_1^α and the worldline). Here, the perturbation $h_{\alpha\beta}$ of the metric must be determined by solving the linearized perturbed Einstein equations [around $g_{\alpha\beta}^{(0)}(m_2)$], say

$$\frac{\delta G^{\mu\nu}}{\delta g_{\alpha\beta}}[h_{\alpha\beta}] = 8\pi G m_1 \int u_1^\mu dx_1^\nu \frac{\delta^4(x - x_1(s_1))}{\sqrt{g}}. \quad (\text{C2})$$

If we consider an *ultrarelativistic motion* ($u_1^\mu \gg 1$, keeping the product $m_1 u_1^\mu$ small) of the small particle, the perturbation $h_{\alpha\beta}$ of the metric (which is sourced by $m_1 u_1^\mu$) will be proportional to, say, the conserved energy $\mathcal{E}_1 = -m_1 u_1^\mu \xi_\mu$ [where ξ^μ is the time-translation Killing vector of the background $g_{\alpha\beta}^{(0)}(m_2)$]. A direct consequence of this simple remark is that the fractional ISF change in the scattering angle will be of order $O(\mathcal{E}_1/m_2)$, rather than the naive estimate $O(m_1/m_2)$ that holds for particles with velocities small or comparable to the velocity of light. In the EOB formalism, the ISF effects are described by the linear-in- ν piece in the mass-shell term Q . The previous reasoning shows that, when considering the small back-reaction ultrarelativistic double limit where $-u_1^\mu \xi_\mu \rightarrow \infty$, $m_1 \rightarrow 0$ with $\mathcal{E}_1 = -m_1 u_1^\mu \xi_\mu$ fixed but much smaller than m_2 , i.e., a limit where one *first* expands to linear order in ν , and *then* formally considers the limit where $\gamma = -(p_1 \cdot p_2)/(m_1 m_2) \approx \mathcal{E}_1/m_1 \rightarrow \infty$, one will have fractional corrections to χ of order $\nu\gamma$. In other words, if we define the ISF contribution to the scattering function $\chi(\gamma, j; \nu)$ by writing

$$\chi(\gamma, j, \nu) = \chi^{\text{Schw}}(\gamma, j) + \nu \chi^{\text{ISF}}(\gamma, j) + O(\nu^2), \quad (\text{C3})$$

we expect the ratio $\chi^{\text{ISF}}(\gamma, j)/\gamma$ to have a finite limit as $\gamma \rightarrow \infty$, when keeping fixed the impact parameter, and therefore the ratio $\alpha \equiv \frac{j}{j}$, say

$$\lim_{\gamma \rightarrow \infty} \frac{\chi^{\text{ISF}}(\gamma, \gamma/\alpha)}{2\gamma} = F(\alpha). \quad (\text{C4})$$

The leading order (LO) contribution to the so-defined function $F(\alpha)$ is $O(\alpha^2)$ and comes from the 2PM-level term $\chi_2(\gamma)/(h(\gamma, \nu)j^2)$ in the PM expansion of $\frac{1}{2}\chi(\gamma, j, \nu)$,

$$\frac{1}{2}\chi(\gamma, j, \nu) = \frac{\chi_1^{\text{Schw}}(\gamma)}{j} + \frac{\chi_2^{\text{Schw}}(\gamma)}{h(\gamma, \nu)j^2} + \dots \quad (\text{C5})$$

when expanding $1/h(\gamma, \nu) = 1 - \nu(\gamma - 1) + O(\nu^2)$.

The limiting behavior (C4) would directly follow from the first conjecture made above, namely a tame HE behavior. Indeed, we have proven above that the PM expansion coefficients $\chi_n(\gamma, \nu)$ and $q_n(\gamma, \nu)$ had a restricted dependence on the symmetric mass ratio ν described through the interplay of some γ -dependent building blocks

$\hat{\chi}_n^{(p)}(\gamma)$ and $\hat{q}_n^{(p)}(\gamma)$ with some powers of the function $h(\gamma, \nu)$. More precisely, we obtained formulas of the following form

$$\chi_n(\gamma, \nu) = \frac{\hat{\chi}_n^{(n-1)}(\gamma)}{h^{n-1}(\gamma, \nu)} + \frac{\hat{\chi}_n^{(n-3)}(\gamma)}{h^{n-3}(\gamma, \nu)} + \dots \quad (\text{C6})$$

or

$$\begin{aligned} \chi_n(\gamma, \nu) - \chi_n^{\text{Schw}}(\gamma) &= \hat{\chi}_n^{(n-1)}(\gamma) \left(\frac{1}{h^{n-1}(\gamma, \nu)} - 1 \right) \\ &+ \hat{\chi}_n^{(n-3)}(\gamma) \left(\frac{1}{h^{n-3}(\gamma, \nu)} - 1 \right) + \dots \end{aligned} \quad (\text{C7})$$

and

$$\begin{aligned} q_2(\gamma, \nu) &= \hat{q}_2^{(1)}(\gamma) \left(1 - \frac{1}{h(\gamma, \nu)} \right), \\ q_3(\gamma, \nu) &= \hat{q}_3^{(1)}(\gamma) \left(1 - \frac{1}{h(\gamma, \nu)} \right) + \hat{q}_3^{(2)}(\gamma) \left(1 - \frac{1}{h^2(\gamma, \nu)} \right), \\ q_4(\gamma, \nu) &= \hat{q}_4^{(1)}(\gamma) \left(1 - \frac{1}{h(\gamma, \nu)} \right) + \hat{q}_4^{(2)}(\gamma) \left(1 - \frac{1}{h^2(\gamma, \nu)} \right) \\ &+ \hat{q}_4^{(3)}(\gamma) \left(1 - \frac{1}{h^3(\gamma, \nu)} \right). \end{aligned} \quad (\text{C8})$$

The conjecture of a tame HE behavior would be related to assuming that the building blocks $\hat{q}_n^{(p)}(\gamma)$ of the EOB potentials have a uniform HE behavior of the type

$$\hat{q}_n^{(p)}(\gamma) \stackrel{\text{HE}}{\sim} \gamma^2. \quad (\text{C9})$$

This behavior holds for the building blocks $\hat{q}_2^{(1)}(\gamma)$, $\hat{q}_3^{(1)}(\gamma)$ entering the first two PM contributions, namely

$$\hat{q}_2^{(1)}(\gamma) = \frac{3}{2}(5\gamma^2 - 1), \quad (\text{C10})$$

$$\hat{q}_3^{(1)}(\gamma) = -\frac{2\gamma^2 - 1}{\gamma^2 - 1} \hat{q}_2^{(1)}(\gamma) = -\frac{3(2\gamma^2 - 1)(5\gamma^2 - 1)}{\gamma^2 - 1}. \quad (\text{C11})$$

In addition, our first conjecture, Eq. (6.16), for modifying the 3PM dynamics by softening its HE behavior, would imply that the same behavior holds for the other 3PM-level function $\hat{q}_3^{(2)}(\gamma)$ (which is essentially a different notation for the function denoted $C(\gamma)$ in Eq. (3.62)).

When transcribed in terms of the related building blocks $\hat{\chi}_n^{(p)}(\gamma)$, one finds that the general conjectural HE behavior (C9) would imply the following uniform HE behavior

$$\hat{\chi}_n^{(p)}(\gamma) \stackrel{\text{HE}}{\sim} \gamma^n. \quad (\text{C12})$$

In turn, when inserting the HE behavior (C12) in the SF expansion of Eq. (C7) (with $1/h^p(\gamma, \nu) = 1 - p\nu(\gamma - 1) + O(\nu^2)$), we find that the 1SF contribution to each coefficient $\chi_n(\gamma, \nu)$, defined as,

$$\chi_n(\gamma, \nu) = \chi_n^{\text{Schw}}(\gamma) + \nu \chi_n^{\text{1SF}}(\gamma) + O(\nu^2), \quad (\text{C13})$$

would then behave as

$$\chi_n^{\text{1SF}}(\gamma) \sim \gamma^{n+1} \quad \text{as } \gamma \rightarrow \infty. \quad (\text{C14})$$

Finally, the latter HE behavior would be consistent with the existence of the limiting function $F(\alpha)$, (C4), if we assume (as holds within our presently assumed soft HE behavior) that the HE limit ($\gamma \rightarrow \infty$) commutes with the PM expansion (i.e., the expansion in powers of $1/j$ defining the various coefficients $\chi_n(\gamma, \nu)$). Furthermore the HE behavior (C14) is directly related to (C9), which predicts that the 1SF expansion of the mass-shell potential Q would

be compatible, at each separate PM order, with the HE behavior found in Ref. [53], namely the existence of a finite limit for the ratio $\frac{\hat{Q}^{n\text{PM 1SF}}}{\gamma^3}$ when $\gamma \rightarrow \infty$.

Summarizing: the conjectural scalings, Eqs. (C9), (C12), (C14), (based on the assumption of a tame HE behavior at each PM order) have been presented here as a simple way to transcribe within PM gravity the (independently derived) SF results, Eqs. (C3), (C4). However, the recent disproof [78–80] of our (first conjectured) HE-soft 3PM dynamics, Eq. (6.16), shows that our search for a unified understanding of the interplay between the SF expansion, the HE behavior and the PM expansion must be done within a wider framework. We have exemplified above, in Eq. (6.54), that another type of conjecture might reconcile the SF result of Ref. [53] with the logarithmically untame HE behavior of the 3PM dynamics of Refs. [24,25]. We leave to future work a discussion of how the interplay between the various noncommuting limits $\nu \rightarrow 0$, $\gamma \rightarrow \infty$, and $G \rightarrow 0$ might work when using similar structures at higher PM orders.

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