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Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders

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Using the new methodology introduced in a recent paper [D. Bini, T. Damour, and A. Geralico, Phys. Rev. Lett. **123**, 231104 (2019)], we present the details of the computation of the conservative dynamics of gravitationally interacting binary systems at the fifth post-Newtonian (5PN) level, together with its extension at the fifth-and-a-half post-Newtonian level. We present also the sixth post-Newtonian (6PN) contribution to the third-post-Minkowskian (3PM) dynamics. Our strategy combines several theoretical formalisms: post-Newtonian, post-Minkowskian, multipolar-post-Minkowskian, gravitational self-force, effective one-body, and Delaunay averaging. We determine the full functional structure of the 5PN Hamiltonian (which involves 95 nonzero numerical coefficients), except for two undetermined coefficients proportional to the cube of the symmetric mass ratio, and to the fifth and sixth power of the gravitational constant, G. We present not only the 5PN-accurate, 3PM contribution to the scattering angle but also its 6PN-accurate generalization. Both results agree with the corresponding truncations of the recent 3PM result of Bern *et al.* [Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon, and M. Zeng, Phys. Rev. Lett. **122**, 201603 (2019)]. We also compute the 5PN-accurate, fourth-post-Minkowskian (4PM) contribution to the scattering angle, including its nonlocal contribution, thereby offering checks for future 4PM calculations. We point out a remarkable hidden simplicity of the gauge-invariant functional relation between the radial action and the effective-one-body energy and angular momentum.

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I. INTRODUCTION

The main tool used up to now for the theoretical description of the general relativistic dynamics of a two-body system is the post-Newtonian (PN) formalism [1,2]. It encodes the corrections to the Newtonian Hamiltonian due to the weak-field, slow-motion, and small-retardation interaction between the bodies, expressed as a power series in inverse powers of the speed of light c. The PN knowledge of the conservative dynamics must then be completed by an analytical description of the gravitational wave emission and backreaction. The main analytical technique currently used for the latter task is the (PN-matched [3–7]) multipolar-post-Minkowskian (MPM) formalism [8].

The present status of PN knowledge is the 4PN accuracy, corresponding to $O(1/c^8)$ fractional corrections to the Newtonian Hamiltonian. A conceptually (and technically) important new feature of the 4PN Hamiltonian is the presence of a nonlocal-in-time interaction due to tail-transported large-time-separation correlations [3]. The current direct perturbative computations of the 4PN-level reduced action [9–16] have succeeded in tackling this time-nonlocality issue in various ways. However, this variety of approaches, which included discrepant intermediate results [11] before complete agreement was reached, shows that straightforward perturbative PN computations have reached their limit of easily verifiable reliability. This clearly implies that any nPN computation, with $n \geq 5$, is significantly more challenging than lower-order ones. Let us note in this respect that the recent 5PN-level works [17,18] based on using the standard PN expansion have computed only the small, and non-gauge-invariant, subset of "static" contributions to the 5PN Hamiltonian.

The present status of complete MPM knowledge of the gravitational wave emission is the 3.5PN level (see, however, Ref. [19] for recent significant progress at the 4PN level). The MPM formalism led to the discovery of (tail-transported) nonlocal dynamical correlations at the 4PN level [3] (later discussed within a different perspective in Refs. [20–22]). When projected on the *conservative* (time-symmetric) dynamics, the 4PN tail effects lead to a nonlocal action [9–15]. Here, we shall make use of the 5PN-accurate generalization of the latter tail-related action, first obtained by using results of the MPM formalism in Sec. IXA of Ref. [23] and recently discussed within a different perspective in Ref. [24] (see also Ref. [25]). Note that the MPM formalism is used here both to discuss tail-transported correlations and to control the needed PN-corrected multipole moments.

In view of this situation, we have recently introduced [26] a new strategy for computing the conservative

two-body dynamics to higher PN orders. This strategy combines information from different formalisms besides the PN and MPM ones, namely, gravitational self-force (SF) theory (see, e.g., Ref. [27] for a recent review), post-Minkowskian (PM) theory (see, e.g., Ref. [28–31] for latest achievements), effective one-body (EOB) theory [32,33], and Delaunay averaging [34]. The SF formalism has previously allowed the computation of several gauge-invariant quantities (redshift factor, gyroscope precession angle, etc.) at very high PN orders, but its validity is limited to small values of the mass ratio between the bodies (and to the first order up to now). SF computations do not distinguish local from nonlocal parts of the various quantities and give results that include both parts. The PM formalism is a weak-field expansion in powers of the gravitational constant G, which does not make any slow-motion assumption. An explicit spacetime metric associated with a two-body system was computed at the 2PM approximation in the 1980s [35]. The corresponding 2PM-accurate equations of motion (and scattering angle) were computed at the time [35–37]. A corresponding 2PM-accurate Hamiltonian was computed recently [28] (see also Ref. [38]). A recent breakthrough work of Bern *et al.* [29,31] has deduced a 3PM-accurate $[O(G^3)]$ scattering angle (and Hamiltonian) from a two-loop quantum scattering amplitude computation. No other complete 3PM calculation exists at present. As we explain below, one consequence of our new strategy is to allow for a 3PM-complete computation of the scattering angle at the PN accuracy at which we implement our method. We give here the details of our 5PN-accurate implementation, include its 5.5PN generalization, and we will also mention the result of a recent 6PN extension of our method [39]. Our results provide a 6PN-level confirmation of the $O(G^3)$ scattering angle of Refs. [29,31]. A similar confirmation was independently recently obtained, within a different approach, in Ref. [40].

Combining PN, SF, and PM information is efficiently done within the EOB formalism, which condenses any available analytical information (including nonlocal information) into a few gauge-fixed potentials. See, for example, the EOB formulation of the full (nonlocal) 4PN dynamics in Ref. [23]. We shall use below the EOB formalism as a convenient common language for extracting and comparing the gauge-invariant information contained in various other formalisms.

Here, we detail the application of our new strategy to the 5PN level. Essentially, we complete the 5PN-accurate (tail-related) nonlocal part of the action by constructing a complementary 5PN-accurate local Hamiltonian. The latter local Hamiltonian is obtained, modulo two undetermined coefficients, by combining the result of a new SF computation to sixth order in eccentricity with a general result within EOB-PM theory concerning the mass-ratio dependence of the scattering angle [41]. The transcription of the SF result into dynamical information is obtained by combining the first law of binary dynamics [42–44] with the EOB formalism.

In principle, our method can be extended to higher PN orders. We have recently been able to extend it to the next two PN levels, namely, the 5.5PN and 6PN levels. We present below our computation of the 5.5PN Hamiltonian. Our results extend previous studies of 5.5PN effects [45–47] and do not rely on SF computations but on the 5.5PN conservative action obtained in Ref. [23]. We also cite below the $O(G^3)$ consequences of the recent extension of our strategy to the 6PN level [39].

Note that, at each PN order, our strategy leaves undetermined a relatively small number of coefficients multiplying the cube of the symmetric mass ratio ν [defined in Eq. (1.1) below]. (On the other hand, we can determine many other coefficients entering the Hamiltonian multiplied by higher powers of ν.) Computing these missing coefficients presents a challenge that must be tackled by a complementary method. However, we wish to stress that our present 5PN-accurate results (as well as their 5.5PN and 6PN extensions) are complete at the 3PM and 4PM levels. In other words, all the terms $O(G^3)$ and $O(G^4)$ in the Hamiltonian are fully derived by our method at the PN accuracy of its implementation. It is this property which allows us to probe the recent 3PM result of Refs. [29,31] at the 6PN level and to make predictions about the 4PM dynamics.

We denote the masses of the two bodies as m_1 and m_2. We then define the reduced mass of the system $\mu \equiv m_1 m_2/(m_1+m_2)$, the total mass $M = m_1 + m_2$, and the symmetric mass ratio

$$\nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1+m_2)^2}. \tag{1.1}$$

We use a mostly plus signature. Depending on the context, we shall sometimes keep all G's and c's and sometimes set them (especially c) to 1. Beware also that it is often convenient to work with dimensionless rescaled quantities, such as radial distance, momenta, Hamiltonian, orbital frequency, etc.

To help the reader to follow the logic of our strategy, let us sketch the plan of our paper. Working in harmonic coordinates, we first compute the 5PN-accurate nonlocal part of the action.[1] We then consider an ellipticlike bound-state motion and take the (Delaunay) time average of the associated nonlocal, harmonic-coordinates (labeled by an h) Hamiltonian

$$\langle \delta H^{\rm 4PN+5PN,h}_{\rm nonloc} \rangle = \frac{1}{\oint dt_h} \oint \delta H^{\rm 4PN+5PN,h}_{\rm nonloc}(t_h) dt_h. \tag{1.2}$$

[1]It will be convenient to introduce some additional flexibility in the definition of the nonlocal action. For simplicity, we do not mention this technical detail here.

In this way, we get a gauge-invariant function of two orbital parameters. When computing $\langle \delta H^{\rm 4PN+5PN,h}_{\rm nonloc} \rangle$, Eq. (1.2), it is convenient to use as orbital parameters some *harmonic-coordinates* semimajor axis a^h_r and eccentricity e^h_t. However, these are known functions (given below) of the energy and angular momentum.

Let us note from the start that, when working with some specific spacetime coordinates, we will indicate this coordinate choice by adding an extra label to all the quantities describing the corresponding two-body (relative) motion. For example, $(t^h, r^h, \theta^h, \phi^h)$ denote the harmonic spherical-coordinate system, whereas $(t^e, r^e, \theta^e, \phi^e)$ denote the effective-one-body spherical-coordinate system. Correspondingly, the quasi-Keplerian parametric representation of the motion is expressed, in each coordinate system, in terms of some (coordinate-dependent) parameters, such as eccentricity, semimajor axis, etc. We then add on such quantities an extra label specifying the chosen coordinate system; e.g., a^h_r and e^h_t are harmonic coordinate–based semimajor axis and eccentricity, and a^e_r and e^e_t are effective one body coordinate–based semimajor axis and eccentricity. However, when working within a section devoted to a particular coordinate choice, we will sometimes skip the coordinate-label notation when the context makes it clear what is the underlying coordinate choice. Anyway, let us emphasize that all our final results are coordinate invariant and that we provide the necessary relations to express them in terms of the (center-of-mass) energy and angular momentum of the binary system.

Next, parametrizing with unknown coefficients the nonlocal part of the Hamiltonian expressed in EOB coordinates (labeled with "e," instead of "h"), we compute the corresponding Delaunay average

$$\left\langle \delta H^{\rm 4PN+5PN,e}_{\rm eob,nonloc} \right\rangle = \frac{1}{\oint dt_e} \oint \delta H^{\rm 4PN+5PN,e}_{\rm eob,nonloc}(t_e) dt_e. \quad (1.3)$$

Identifying the two Delaunay averages (when using the 1PN-accurate relation between the harmonic-coordinates orbital parameters a^h_r, e^h_t and the corresponding EOB parameters a^e_r, e^e_t) then determines the unknown coefficients used to parametrize the 5PN nonlocal part of the EOB Hamiltonian.

Having in hand the latter 5PN-accurate nonlocal part of the EOB Hamiltonian, we then determine the complementary 5PN-accurate local part of the EOB Hamiltonian. This is done by using SF information about small-eccentricity ellipticlike motions. Namely, we first compute the averaged redshift factor [48] to the sixth order in eccentricity. We had to generalize to the sixth order previous results that extended only to the fourth order in eccentricity [49,50]. To relieve the tedium, we relegated some of our derivations and results to Appendixes. We notably list in Appendix A the result of our SF computation of the (averaged) redshift factor along eccentric orbits in the Schwarzschild spacetime (accurate to the 9.5PN level) and its conversion into the EOB potential q_6 through the first law of eccentric binaries [44].

This determines the sum of the local and the nonlocal EOB Hamiltonian, but only at the second order in the symmetric mass ratio ν. [Here, we are talking about the unrescaled Hamiltonian, such that the test-particle Hamiltonian is $O(\nu)$.] Subtracting the above-determined nonlocal EOB Hamiltonian determines the local part of the EOB Hamiltonian up to $O(\nu^2)$ included [corresponding to an $O(\nu)$ knowledge of the potentials entering the effective EOB Hamiltonian].

Reference [41] has recently uncovered a simple property of the ν-dependence of the scattering angle for hyperbolic encounters. This property plays a crucial role in allowing us to complete the previously discussed $O(\nu^2)$ SF-based knowledge of the Hamiltonian and to determine most of the $O(\nu^{n\geq 3})$ contributions to the Hamiltonian. To use the result of Ref. [41] (which concerns the structure of the total scattering angle $\chi^{\rm tot} = \chi^{\rm loc} + \chi^{\rm nonloc}$), two separate steps are needed. On the one hand, we need to compute the nonlocal contribution $\chi^{\rm nonloc}$ to the scattering angle by generalizing the technique used at 4PN in Ref. [51]. On the other hand, it is convenient, in order to separately compute the local contribution $\chi^{\rm loc}$ to the scattering angle, to convert the local EOB Hamiltonian, so far obtained in the standard p_r gauge [33], into the so-called energy gauge [28]. Indeed, the latter gauge is more convenient for discussing hyperboliclike scattering motions. The computation of the total scattering angle $\chi^{\rm tot} = \chi^{\rm loc} + \chi^{\rm nonloc}$, together with the knowledge of the exact 2PM EOB Hamiltonian, then allows us to fix most of the parametrizing coefficients of the EOB potentials [actually, all coefficients with two exceptions only: $\bar{d}^{\nu^2}_5$ and $a^{\nu^2}_6$, i.e., the $O(\nu^2)$ coefficients of the local potentials $\bar{D}$ and A at 5PN].

Besides the results just summarized (which constitute the core of the present work), let us highlight other new results obtained below as byproducts of our computations:

(1) We have evaluated the averaged value of the 5.5PN Hamiltonian. It is entirely given by the (scale-independent) second-order-in-tail nonlocal Hamiltonian $H_{\rm tail^2}$, from which we have computed the half-PN-order coefficients $A_{6.5}$, $\bar{D}_{5.5}$, $q_{4,4.5}$, $q_{6,3.5}$, and $q_{8,2.5}$. The last one, $q_{8,2.5}$, is new and a prediction for future SF calculations (see Sec. VI).

(2) We have shown how to use an (inverse) Abel transform to compute in closed form the standard p_r-gauge version of the 2PM energy-gauge EOB potential $q_{\rm 2EG}$ (see Appendix B).

(3) We have explicitly computed the local contribution to the 5PN radial action, as well as the corresponding local Delaunay Hamiltonian (i.e., the local Hamiltonian expressed in terms of action variables). We find that the radial action has a remarkably simple structure. See Sec. XIII.

II. 5PN-ACCURATE NONLOCAL ACTION AND ITS ASSOCIATED HAMILTONIAN

The complete, reduced two-body conservative action ($S_{\rm tot}$) can be decomposed, at any given PN accuracy, by using the PN-matched [3–7] MPM formalism [8], in two separate pieces: a nonlocal-in-time part ($S_{\rm nonloc}$) and a local-in-time part ($S_{\rm loc}$),

$$S_{\rm tot}^{\leq n{\rm PN}} = S_{\rm loc,f}^{\leq n{\rm PN}} + S_{\rm nonloc,f}^{\leq n{\rm PN}}. \tag{2.1}$$

Here, each action piece is a time-symmetric functional of the worldlines of the two bodies, say, $x_1(s_1)$ and $x_2(s_2)$. The original total action $S_{\rm tot}[x_1(s_1), x_2(s_2)]$ (before approximating it at some PN accuracy) is defined as a PM-expanded Fokker action [52]. The PN-truncated nonlocal action $S_{\rm nonloc,f}^{\leq n{\rm PN}}$ (which starts at the 4PN level [3,9]) is defined by using the MPM formalism. Its 5PN-accurate value was first obtained in Sec. IXA of Ref. [23] (based on the effective action used in Ref. [53]). It was recently derived in a different (though related) way in Ref. [24]. (See also Refs. [42,54] for the related 5PN logarithmic terms and Ref. [25] for higher-order tail-related logarithms.) From Eq. (9.12) of Ref. [23], it reads

$$S_{\rm nonloc,f}^{4+5{\rm PN}}[x_1(s_1), x_2(s_2)] = \frac{G\mathcal{M}}{c^3}\int dt {\rm Pf}_{2r_{12}^f(t)/c} \times \int \frac{dt'}{|t-t'|}\mathcal{F}_{\rm 1PN}^{\rm split}(t,t'). \tag{2.2}$$

Here, $\mathcal{M}$ denotes the total Arnowitt-Deser-Misner (ADM) conserved mass energy of the binary system, while $\mathcal{F}_{\rm 1PN}^{\rm split}(t,t')$ is the time-split version of the fractionally 1PN-accurate gravitational wave energy flux emitted by the system, namely,

$$\mathcal{F}_{\rm 1PN}^{\rm split}(t,t') = \frac{G}{c^5}\left(\frac{1}{5}I_{ab}^{(3)}(t)I_{ab}^{(3)}(t') + \frac{1}{189c^2}I_{abc}^{(4)}(t)I_{abc}^{(4)}(t') + \frac{16}{45c^2}J_{ab}^{(3)}(t)J_{ab}^{(3)}(t')\right), \tag{2.3}$$

where the superscript in parentheses denotes repeated time derivatives. The specific choice of the timescale $2r_{12}^f(t)/c$ entering the *partie finie* (Pf) operation used in the definition of the nonlocal action, Eq. (2.2) (whose integral over t' is logarithmically divergent when $t' \to t$), will be discussed below.

The quantities I_{ab}, I_{abc}, and J_{ab} entering Eq. (2.3) are the MPM-derived Blanchet-Damour-Iyer mass and spin multipole moments defined by suitable integrals over the stress-energy tensor of the source [4,5]. Their (center-of-mass, harmonic coordinates) expressions for a binary system read[2] (see Eqs. (3.32) and (3.33) of Ref. [55])

$$\begin{aligned} I_{ij} &= \mu r_{\langle ij\rangle}\left[1 + \frac{29}{42c^2}(1-3\nu)v^2 - \frac{(5-8\nu)}{7c^2}\frac{GM}{r}\right] \\ &\quad + \mu\frac{1-3\nu}{21c^2}[-12(\mathbf{v}\cdot\mathbf{r})r_{\langle i}v_{j\rangle} + 11r^2 v_{\langle ij\rangle}], \\ I_{ijk} &= \mu\sqrt{1-4\nu}r_{\langle ijk\rangle}, \\ J_{ij} &= \mu\sqrt{1-4\nu}\epsilon_{kl\langle i}r_{j\rangle k}v_l, \end{aligned} \tag{2.4}$$

with $M\sqrt{1-4\nu} = m_2 - m_1$ and

$$\begin{aligned} r_{\langle ijk\rangle} &= r_{ijk} - \frac{3}{5}r^2\delta_{(ij}r_{k)}, \\ \epsilon_{kl\langle i}r_{j\rangle k}v_l &= (\mathbf{r}\times\mathbf{v})_{(i}r_{j)}, \end{aligned} \tag{2.5}$$

where the standard notations $A^{ijk\cdots} = A^iA^jA^k\ldots$ for tensorial products, $S^{(ij)} = \frac{1}{2}(S^{ij} + S^{ji})$ for the symmetric part of a tensor, and $S^{\langle ij\rangle}$ for the symmetric and trace-free part of a tensor have been used.

As stated above, Eq. (2.2) defines (for any choice of r_{12}^f) an explicit functional of the two worldlines. Subtracting it from the (in principle PM-computable) total action $S_{\rm tot}$ defines the corresponding local-in-time contribution $S_{\rm loc,f}^{\leq 5{\rm PN}}$ to the two-body dynamics. There is some *flexibility* in the choice of the timescale $2r_{12}^f/c$ entering the Pf operation used in Eq. (2.2). Let us first point out that the meaning here of $S_{\rm loc}$ (and its corresponding $H_{\rm loc}$) differs from the one in Refs. [23,51], where the timescale entering the partie finie defining $S_{\rm nonloc}$ was taken to be a fixed scale $2s/c$. The explicit results of Ref. [23] show that, with such a choice, the 4PN-accurate local Hamiltonian $H_{\rm loc}$ then includes several terms proportional to the logarithm $\ln(r_{12}/s)$. Choosing as length scale s the radial distance r_{12} between the two bodies has therefore the technical advantage of simplifying the local part of the Hamiltonian by removing all logarithms from it. At the 4PN level, a Newtonian-accurate definition of the radial distance r_{12} is adequate. However, as we are now working at a higher PN accuracy, we need to define the timescale $2r_{12}^f/c$ with at least 1PN fractional accuracy. Let us emphasize that the choice of any precise definition of r_{12}^f is purely conventional, and will affect in no way the end results of our methodology. Indeed, the total action $S_{\rm loc,f}^{\leq n{\rm PN}} + S_{\rm nonloc,f}^{\leq n{\rm PN}}$ will always be defined so as to be independent of the flexibility in the definition of r_{12}^f. Only the separation between $S_{\rm loc,f}$ and $S_{\rm nonloc,f}$ depends on this flexibility. Though it would be perfectly acceptable (and would lead, when consistently used, to the same final results) to use everywhere the harmonic-coordinate radial distance r_{12}^h as the length scale, we shall show here that there are some technical advantages to employing a more general scale of the general form

$$r_{12}^f(t) = f(t)r_{12}^h(t), \tag{2.6}$$

[2]In the present work, the 1PN fractional accuracy is only needed for I_{ij}.

where $f(t) = 1 + O(\frac{1}{c^2})$ is a combination of dynamical variables of the type

$$f(t) = 1 + c_1 \left(\frac{p_r}{\mu c}\right)^2 + c_2 \left(\frac{\mathbf{p}}{\mu c}\right)^2 + c_3 \frac{GM}{rc^2} + \ldots \tag{2.7}$$

A convenient criterion for choosing the (5PN-level) flexibility parameter $f(t)$ will be discussed below. It will imply, in particular, the fact that the dimensionless coefficients $c_1, c_2, c_3, \ldots$ entering the definition of $f(t)$ are proportional to the symmetric mass ratio ν.

It is convenient to rewrite $S^{4+5\mathrm{PN}}_{\mathrm{nonloc}}$ as

$$S^{4+5\mathrm{PN}}_{\mathrm{nonloc,f}} = -\int dt \delta H^{4+5\mathrm{PN}}_{\mathrm{nonloc,f}}(t) \tag{2.8}$$

and to rewrite $\delta H^{4+5\mathrm{PN}}_{\mathrm{nonloc,f}}(t)$ as

$$\delta H^{4+5\mathrm{PN}}_{\mathrm{nonloc,f}}(t) = \delta H^{4+5\mathrm{PN}}_{\mathrm{nonloc,h}} + \Delta^{\mathrm{f-h}} H(t), \tag{2.9}$$

where

$$\delta H^{4+5\mathrm{PN}}_{\mathrm{nonloc,h}}(t) \equiv -\frac{G^2 H}{c^5} \mathrm{Pf}_{2s/c} \int \frac{d\tau}{|\tau|} \mathcal{F}^{\mathrm{split}}_{\mathrm{1PN}}(t, t+\tau) + 2\frac{G^2 H}{c^5} \mathcal{F}^{\mathrm{split}}_{\mathrm{1PN}}(t,t) \ln\left(\frac{r^h_{12}(t)}{s}\right), \tag{2.10}$$

and

$$\Delta^{\mathrm{f-h}} H(t) \equiv +2\frac{G^2 H}{c^5} \mathcal{F}^{\mathrm{split}}_{\mathrm{1PN}}(t,t) \ln\left(f(t)\right). \tag{2.11}$$

Here, $\mathcal{F}^{\mathrm{split}}_{\mathrm{1PN}}(t,t) \equiv \mathcal{F}^{\mathrm{GW}}_{\mathrm{1PN}}(t)$ is the instantaneous gravitational wave energy flux so that the flexibility term $\Delta^{\mathrm{f-h}} H(t)$ is a purely local additional contribution to $\delta H_{\mathrm{nonloc,h}}$. In the following, we will keep indicating by a label f or h nonlocal (or local) contributions that depend on choosing as partie finie scale $2r^f_{12}/c$ or $2r^h_{12}/c$.

We now compute the time average of $\delta H^{4+5\mathrm{PN}}_{\mathrm{real,nonloc,h}}(t)$ along an ellipticlike bound-state motion, using its well-known (harmonic coordinates) 1PN-accurate quasi-Keplerian parametrization [56], i.e.,

$$\langle \delta H^{4+5\mathrm{PN}}_{\mathrm{nonloc,h}} \rangle \equiv \frac{1}{\oint dt} \oint \delta H^{4+5\mathrm{PN}}_{\mathrm{nonloc,h}}(t) dt. \tag{2.12}$$

The quasi-Keplerian parametrization of the orbit (needed at the 1PN level of accuracy for the purposes of the present paper) is summarized in Table I. The functional relations shown there are also valid at 1PN in ADM coordinates [which start differing only at 2PN level from harmonic coordinates] and in EOB coordinates (with different numerical values of the orbital elements a_r and e_t). See Ref. [57] for the 2PN generalization of the quasi-Keplerian parametrization.

TABLE I. Quasi-Keplerian representation of the 1PN motion (valid both in harmonic and EOB coordinates with $e_t = e_t^{\mathrm{coord}}$, etc). Here, $n = 2\pi/P$ (with P the radial period) and $K = \Phi/2\pi$ (periastron advance) are gauge invariant, while the various eccentricity parameters, e_t, e_r, and e_ϕ, and the semimajor axis, a_r, are coordinate dependent; u denotes here the eccentric anomaly (not to be confused with the inverse radial variable used below in various EOB potentials). We employ here mass-rescaled radial variables: $r \equiv r^{\mathrm{phys}}/(GM)$, $a_r \equiv a_r^{\mathrm{phys}}/(GM)$.

$n(t-t_0) = \ell = u - e_t \sin u$
$r = a_r(1 - e_r \cos u)$
$\dot{r} = \frac{n a_r e_r \sin u}{1 - e_t \cos u}$
$\phi - \phi_0 = 2K \arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2} \tan\frac{u}{2}\right]$
$\dot{\phi} = nK \frac{\sqrt{1-e_\phi^2}}{(1-e_t\cos u)(1-e_\phi \cos u)}$
$\tan\frac{u}{2} = \sqrt{\frac{1-e_\phi}{1+e_\phi}} \tan\frac{\bar{\phi}}{2}$, $\bar{\phi} \equiv \frac{\phi-\phi_0}{K}$
$\sin u = \frac{\sqrt{1-e_\phi^2}\sin\bar{\phi}}{1+e_\phi\cos\bar{\phi}}$, $\cos u = \frac{\cos\bar{\phi}+e_\phi}{1+e_\phi\cos\bar{\phi}}$

The temporal average is conveniently transformed as an integral over the azimuthal angle, namely,

$$\begin{aligned} \langle \delta H^h \rangle &= \frac{n}{2\pi} \int_0^{2\pi/n} \delta H^h dt \\ &= \frac{n}{2\pi} \int_0^{2\pi K} \frac{\delta H^h}{\dot{\phi}^h} d\phi^h \\ &= \frac{nK}{2\pi} \int_0^{2\pi} \left.\frac{\delta H^h}{\dot{\phi}^h}\right|_{(\phi^h - \phi^h_0) = K\bar{\phi}^h} d\bar{\phi}^h, \end{aligned} \tag{2.13}$$

where $\bar{\phi}^h \equiv (\phi^h - \phi^h_0)/K$ (see the caption of Table I for the definition of the various orbital parameters). The result of the average is a gauge-invariant function (say, F) of a set of independent orbital parameters. The latter are chosen here to be the (harmonic-coordinates) semimajor axis a^h_r and eccentricity e^h_t, so that

$$\langle \delta H^h \rangle \equiv \langle \delta H^{4+5\mathrm{PN}}_{\mathrm{nonloc}} \rangle = F^h(a^h_r, e^h_t), \tag{2.14}$$

with

$$F^h(a^h_r, e^h_t) = \frac{\nu^2}{(a^h_r)^5} \left[\mathcal{A}^{4\mathrm{PN}}(e^h_t) + \mathcal{B}^{4\mathrm{PN}}(e^h_t) \ln a^h_r\right] + \frac{\nu^2}{(a^h_r)^6} \left[\mathcal{A}^{5\mathrm{PN}}(e^h_t) + \mathcal{B}^{5\mathrm{PN}}(e^h_t) \ln a^h_r\right]. \tag{2.15}$$

The expansion in powers of e^h_t of the coefficients parametrizing $F^h(a^h_r, e^h_t)$ are listed in Table II up to the order $O((e^h_t)^{10})$ included. (We use here $G = 1 = c$.)

Let us also mention that the average of the f-related contribution [with the parametrization (2.7)], Eq. (2.11), to the nonlocal Hamiltonian starts at the 5PN order and reads

TABLE II. Coefficients of the averaged nonlocal Hamiltonian (with scale $2r_{12}^h/c$) in harmonic coordinates.

Coefficient	Expression
$\mathcal{A}^{4\mathrm{PN}}(e_t^h)$	$\frac{128}{5}\ln(2)+\frac{64}{5}\gamma+(\frac{1256}{15}\gamma-\frac{176}{5}+\frac{729}{5}\ln(3)+\frac{296}{15}\ln(2))(e_t^h)^2$ $+(\frac{29966}{15}\ln(2)-\frac{2681}{15}-\frac{13851}{20}\ln(3)+242\gamma)(e_t^h)^4$ $+(\frac{1953125}{576}\ln(5)+\frac{1526}{3}\gamma+\frac{419661}{320}\ln(3)-\frac{90017}{180}-\frac{116722}{15}\ln(2))(e_t^h)^6$ $+(\frac{3605}{4}\gamma-\frac{83984375}{4608}\ln(5)-\frac{306433}{288}+\frac{5381201}{180}\ln(2)+\frac{26915409}{2560}\ln(3))(e_t^h)^8$ $+(-\frac{4697998651}{54000}\ln(2)-\frac{138733913079}{2048000}\ln(3)+\frac{678223072849}{18432000}\ln(7)-\frac{18541327}{9600}$ $+\frac{18736328125}{442368}\ln(5)+\frac{114807}{80}\gamma)(e_t^h)^{10}$
$\mathcal{B}^{4\mathrm{PN}}(e_t^h)$	$-\frac{32}{5}-\frac{628}{15}(e_t^h)^2-121(e_t^h)^4-\frac{763}{3}(e_t^h)^6-\frac{3605}{8}(e_t^h)^8-\frac{114807}{160}(e_t^h)^{10}$
$\mathcal{A}^{5\mathrm{PN}}(e_t^h)$	$(-\frac{11708}{105}-\frac{112}{5}\nu)\gamma+(-\frac{25276}{105}+\frac{912}{35}\nu)\ln(2)+(-\frac{486}{7}\nu+\frac{243}{14})\ln(3)+\frac{32}{5}\nu-\frac{96}{5}$ $+[(\frac{19024}{35}-\frac{57284}{15}\nu)\ln(2)+(\frac{94041}{70}\nu-\frac{27702}{35})\ln(3)+(-\frac{10672}{35}-\frac{1364}{3}\nu)\gamma-\frac{5441}{35}+\frac{4672}{21}\nu](e_t^h)^2$ $+[(-\frac{599911}{105}+\frac{3476231}{105}\nu)\ln(2)+(-\frac{303993}{896}-\frac{6268671}{1120}\nu)\ln(3)+(-\frac{9765625}{672}\nu+\frac{9765625}{2688})\ln(5)$ $+(\frac{14003}{35}-2241\nu)\gamma+\frac{59756}{35}\nu-\frac{1160639}{840}](e_t^h)^4$ $+[(\frac{16502161}{945}-\frac{73289299}{315}\nu)\ln(2)+(-\frac{70956243}{896}\nu+\frac{23153769}{896})\ln(3)$ $+(-\frac{434140625}{24192}+\frac{162109375}{1152}\nu)\ln(5)+(-\frac{20027}{3}\nu+\frac{17797}{5})\gamma+\frac{474653}{72}\nu-\frac{15761437}{2520}](e_t^h)^6$ $+[(-\frac{604746629}{10080}+\frac{4166409179}{3780}\nu)\ln(2)+(-\frac{18879921207}{163840}+\frac{237686858217}{286720}\nu)\ln(3)$ $+(-\frac{843248046875}{1548288}\nu-\frac{2542507812 5}{6193152})\ln(5)+(\frac{96889010407}{884736}-\frac{96889010407}{221184}\nu)\ln(7)$ $+(\frac{356481}{32}-\frac{61285}{4}\nu)\gamma-\frac{168508293}{8960}+\frac{2591779}{144}\nu](e_t^h)^8$ $+[(\frac{189246461867}{126000}-\frac{990620463289}{108000}\nu)\ln(2)+(\frac{2103914638719}{14336000}-\frac{95555936957967}{28672000}\nu)\ln(3)$ $+(\frac{68322265625}{1032192}+\frac{2431021484375}{2064384}\nu)\ln(5)+(\frac{455281459902493}{110592000}\nu-\frac{11520188079967}{18432000})\ln(7)$ $+(\frac{2035869}{80}-\frac{4794867}{160}\nu)\gamma+\frac{255777929}{6400}\nu-\frac{1492974817}{33600}](e_t^h)^{10}$
$\mathcal{B}^{5\mathrm{PN}}(e_t^h)$	$\frac{5854}{105}+\frac{56}{5}\nu+(\frac{682}{3}\nu+\frac{5336}{35})(e_t^h)^2+(-\frac{14003}{70}+\frac{2241}{2}\nu)(e_t^h)^4+(\frac{20027}{6}\nu-\frac{17797}{10})(e_t^h)^6$ $+(\frac{61285}{8}\nu-\frac{356481}{64})(e_t^h)^8+(\frac{4794867}{320}\nu-\frac{2035869}{160})(e_t^h)^{10}$

$$\frac{1}{\oint dt}\oint \Delta^{\mathrm{f-h}}H(t)dt=\frac{\nu^2}{a_r^6}\left[\frac{64}{5}(c_2+c_3)+\frac{8}{15}(229c_3+12c_1+301c_2)e_t^2+\frac{2}{5}(1183c_3+1761c_2+97c_1)e_t^4\right.$$
$$+\frac{14}{15}(134c_1+1341c_3+2137c_2)e_t^6+\frac{7}{4}(1529c_3+171c_1+2543c_2)e_t^8$$
$$\left.+\frac{231}{80}(208c_1+2961c_2+1729c_3)e_t^{10}\right]+O(e_t^{12}). \tag{2.16}$$

The functional dependence on the orbital parameters a_r^h and e_t^h could be replaced by a dependence on the gauge-invariant energy and angular momentum (see Table III).

Using the known transformation between harmonic and EOB coordinates [32,33], the relations between the harmonic-coordinates orbital parameters (a_r^h, e_t^h) and the corresponding EOB-coordinate orbital parameters (a_r^e, e_t^e) read (with $\eta \equiv 1/c$)

$$a_r^h = a_r^e - \eta^2, \qquad e_t^h = e_t^e\left(1+\frac{\nu}{a_r^e}\eta^2\right). \tag{2.17}$$

The invariant function F^h can then be reexpressed in terms of the corresponding EOB parameters

$$F^h(a_r^h, e_t^h) = \tilde{F}^h(a_r^e, e_t^e). \tag{2.18}$$

We list the relations between the various harmonic-coordinate orbital parameters and the corresponding EOB-coordinates ones as functions of (rescaled) energy [$\bar{E}\equiv(H-Mc^2)/\mu$) and angular momentum ($j \equiv J/(Gm_1m_2)$] in Table III.[3]

Let us now use the (Delaunay-averaged) $4+5$PN information about the nonlocal action contained in Eq. (2.18) to compute a corresponding (squared, rescaled) effective nonlocal $4+5$PN-accurate EOB Hamiltonian, say,

$$\delta[\hat{H}^{4\mathrm{PN}+5\mathrm{PN}}_{\mathrm{eff,nonloc,h}}]^2 = \delta\hat{H}^2_{\mathrm{eff,h}}. \tag{2.19}$$

[3]Throughout the paper, we use several energy-related variables, $\bar{E}$, $\tilde{E}$, $\bar{E}_{\mathrm{eff}}$, etc., which are defined in the sections where they appear.

TABLE III. 1PN quasi-Keplerian orbital parameters (see Table I) as functions of $\bar{E} \equiv (H-M)/\mu$ and $j \equiv J/(GM\mu)$. We also include the (1PN-truncated) inverse relations, $(\bar{E}, j)$ vs (a_r, e_t), because they are often useful.

Orbital parameter, X	Harmonic, X^h	EOB, X^e
a_r	$\frac{1}{(-2\bar{E})}[1+\frac{7-\nu}{2}\bar{E}\eta^2]$	$\frac{1}{(-2\bar{E})}(1-\frac{\nu-3}{2}\bar{E}\eta^2)$
n	$(-2\bar{E})^{3/2}[1+\frac{15-\nu}{4}\bar{E}\eta^2]$	$(-2\bar{E})^{3/2}(1+\frac{15-\nu}{4}\bar{E}\eta^2)$
e_t	$\{1+2\bar{E}j^2[1+(\frac{17}{2}-\frac{7}{2}\nu)\bar{E}\eta^2+\frac{2-2\nu}{j^2}\eta^2]\}^{1/2}$	$\{1+2\bar{E}j^2[1+(\frac{\nu+17}{2}\bar{E}+\frac{2}{j^2})\eta^2]\}^{1/2}$
e_r	$\{1+2\bar{E}j^2[1+(-\frac{15}{2}+\frac{5}{2}\nu)\bar{E}\eta^2+\frac{-6+\nu}{j^2}\eta^2]\}^{1/2}$	$\{1+2\bar{E}j^2[1+(\frac{\nu-7}{2}\bar{E}-\frac{4}{j^2})\eta^2]\}^{1/2}$
e_ϕ	$\{1+2\bar{E}j^2[1+(-\frac{15}{2}+\frac{1}{2}\nu)\bar{E}\eta^2-\frac{6}{j^2}\eta^2]\}^{1/2}$	$\{1+2\bar{E}j^2[1+(\frac{\nu-15}{2}\bar{E}-\frac{6}{j^2})\eta^2]\}^{1/2}$
$\delta_r=\frac{e_r}{e_t}-1$	$(3\nu-8)\bar{E}\eta^2$	$-6\bar{E}\eta^2$
$\delta_\phi=\frac{e_\phi}{e_t}-1$	$(2\nu-8)\bar{E}\eta^2$	$-8\bar{E}\eta^2$
K	$1+\frac{3}{j^2}\eta^2$	$1+\frac{3}{j^2}\eta^2$
k	$\frac{3}{a_r(1-e_t^2)}$	$\frac{3}{a_r(1-e_t^2)}$
$\bar{E}$	$-\frac{1}{2a_r}-\frac{1}{2a_r^2}(-\frac{7}{4}+\frac{\nu}{4})\eta^2$	$-\frac{1}{2a_r}-\frac{1}{2a_r^2}(-\frac{3}{4}+\frac{\nu}{4})\eta^2$
j^2	$a_r(1-e_t^2)(1+\frac{4+2(\nu-3)e_t^2}{a_r(1-e_t^2)}\eta^2)$	$a_r(1-e_t^2)(1-\frac{5e_t^2-3}{a_r(1-e_t^2)}\eta^2)$

We recall the (universal) EOB link between the usual Hamiltonian and the effective one:

$$H_{\rm eob} = Mc^2\sqrt{1+2\nu(\hat{H}_{\rm eff}-1)}. \tag{2.20}$$

This is achieved (as at the 4PN level [23]) by parametrizing a general squared effective EOB Hamiltonian (in standard p_r gauge [33]) in terms of PN-expanded EOB potentials $A(u)$, $\bar{D}(u)$, and $Q(u,p_r)$ (where $u \equiv GM/(c^2 r^{\rm phys}) = \eta^2/r$, $p_r = \eta p_r^{\rm phys}/\mu$ and $p_\phi = j/\eta$):

$$\begin{aligned}\delta\hat{H}^2_{\rm eff}(u,p_r,p_\phi) &= (1+2p_r^2+p_\phi^2u^2)\delta A(u)\\ &\quad+(1-4u)p_r^2\delta\bar{D}(u)\\ &\quad+(1-2u)\delta Q(u,p_r).\end{aligned} \tag{2.21}$$

Here, the notation δ refers to the looked-for additional 4 + 5PN nonlocal contribution, and we have written the right-hand side at the needed 1PN fractional accuracy. The general parametrization of $\delta A(u)$, $\delta\bar{D}(u)$, and $\delta Q(u,p_r)$ reads

$$\begin{aligned}\delta A &= a_5^{\rm nonloc}u^5 + a_6^{\rm nonloc}u^6,\\ \delta\bar{D} &= \bar{d}_4^{\rm nonloc}u^4 + \bar{d}_5^{\rm nonloc}u^5,\\ \delta Q &= p_r^4(q_{43}^{\rm nonloc}u^3 + q_{44}^{\rm nonloc}u^4)\\ &\quad+p_r^6(q_{62}^{\rm nonloc}u^2 + q_{63}^{\rm nonloc}u^3)\\ &\quad+p_r^8(q_{81}^{\rm nonloc}u + q_{82}^{\rm nonloc}u^2)+\ldots.\end{aligned} \tag{2.22}$$

Here, $a_5^{\rm nonloc} = a_5^{\rm nl,c} + a_5^{\rm nl,ln}\ln(u)$, etc., are *a priori* unknown (logarithmically varying) coefficients parametrizing the 4 + 5PN nonlocal EOB Hamiltonian. Here, we have not indicated any additional label f or h, because the form of Eqs. (2.22) is valid for both cases. It is only when computing specific values of the greater than or equal to 5PN EOB parameters $a_6^{\rm nonloc}, \bar{d}_5^{\rm nonloc}, q_{44}^{\rm nonloc}, \ldots$, that we will need to specify whether they correspond to the unflexed h case or to some specified flexed case f. [$a_5^{\rm nonloc}$ belongs to the 4PN approximation and does not depend on the choice of $f = 1 + O(1/c^2)$.]

When doing explicit computations, one needs to truncate the p_r expansion of δQ to a finite order. The nth order in p_r^2 in δQ corresponds to the nth order in e^2 when correspondingly computing the redshift δz_1, as we shall do below.

Converting $\delta H^2_{\rm eff}$ into the usual Hamiltonian $\delta H_{\rm eob}$ is straightforward,

$$\delta H_{\rm eob}^{4+5\rm PN} = \frac{\mu M}{2H_{\rm eob}\hat{H}_{\rm eff}}\delta\hat{H}^2_{\rm eff}, \tag{2.23}$$

as is taking the Delaunay time average over the orbital motion. As before, the latter average is conveniently done in terms of an integral over the EOB azimuthal angle by using Hamilton's equations to express dt^e in terms of $d\phi^e$ along the orbit, i.e.,

$$dt^e = \frac{H_{\rm eob}\hat{H}_{\rm eff}}{Au^2 j}d\phi^e, \tag{2.24}$$

so that

$$\begin{aligned}\langle\delta H_{\rm eob}^{4+5\rm PN}\rangle &= \frac{\mu M}{2H_{\rm eob}\hat{H}_{\rm eff}}\frac{nK}{2\pi}\frac{H_{\rm eob}\hat{H}_{\rm eff}}{j}\int\frac{\delta\hat{H}^2_{\rm eff}}{Au^2}d\bar{\phi}^e\\ &= \frac{\nu M^2 nK}{4\pi j}\int\frac{\delta\hat{H}^2_{\rm eff}}{Au^2}d\bar{\phi}^e,\end{aligned} \tag{2.25}$$

where $\bar{\phi}^e = (\phi^e - \phi^e_0)/K$ and the 1PN-accurate expression for K is given in Table III.

Let us now focus on the r^h_{12} version of the nonlocal action (considered in both harmonic and EOB coordinates). Correspondingly to what we have done before in harmonic coordinates, the time average of the above parametrized EOB nonlocal Hamiltonian provides a function of the EOB orbital parameters, say,

$$F^{4+5\mathrm{PN}}_{\mathrm{eob\ nonloc,h}}(a^e_r, e^e_t; a^{\mathrm{nonloc}}_5, a^{\mathrm{nonloc,h}}_6, \ldots). \tag{2.26}$$

This invariant function must coincide with the above-computed function $\tilde{F}^h(a^e_r, e^e_t)$, Eq. (2.18), which came from the original (r^h_{12}-defined) nonlocal Hamiltonian, Eq. (2.10). Comparison order by order (in both $1/a^e_r$ and in e^e_t) fixes all the unknown coefficients at 5PN, besides checking all the (already known [23]) 4PN ones. The results are of the type

TABLE IV. Coefficients of the nonlocal (r^h_{12}-scaled) 4 + 5PN part of the EOB potentials, Eq. (2.22). We suppress the label h for brevity.

Coefficient	Expression
$a^{\mathrm{nl,c}}_5$	$(\frac{128}{5}\gamma + \frac{256}{5}\ln(2))\nu$
$a^{\mathrm{nl,ln}}_5$	$\frac{64}{5}\nu$
$\bar{d}^{\mathrm{nl,c}}_4$	$(-\frac{992}{5} + \frac{1184}{15}\gamma - \frac{6496}{15}\ln(2) + \frac{2916}{5}\ln(3))\nu$
$\bar{d}^{\mathrm{nl,ln}}_4$	$\frac{592}{15}\nu$
$q^{\mathrm{nl,c}}_{43}$	$(-\frac{5608}{15} + \frac{496256}{45}\ln(2) - \frac{33048}{5}\ln(3))\nu$
$q^{\mathrm{nl,ln}}_{43}$	0
$q^{\mathrm{nl,c}}_{62}$	$(-\frac{4108}{15} - \frac{2358912}{25}\ln(2) + \frac{1399437}{50}\ln(3) + \frac{390625}{18}\ln(5))\nu$
$q^{\mathrm{nl,ln}}_{62}$	0
$a^{\mathrm{nl,c}}_6$	$(-\frac{128}{5} - \frac{14008}{105}\gamma - \frac{31736}{105}\ln(2) + \frac{243}{7}\ln(3))\nu + (\frac{64}{5} - \frac{288}{5}\gamma + \frac{928}{35}\ln(2) - \frac{972}{7}\ln(3))\nu^2$
$a^{\mathrm{nl,ln}}_6$	$-\frac{7004}{105}\nu - \frac{144}{5}\nu^2$
$\bar{d}^{\mathrm{nl,c}}_5$	$(-\frac{7318}{35} - \frac{2840}{7}\gamma + \frac{120648}{35}\ln(2) - \frac{19683}{7}\ln(3))\nu + (\frac{67736}{105} - \frac{6784}{15}\gamma - \frac{326656}{21}\ln(2) + \frac{58320}{7}\ln(3))\nu^2$
$\bar{d}^{\mathrm{nl,ln}}_5$	$-\frac{1420}{7}\nu - \frac{3392}{15}\nu^2$
$q^{\mathrm{nl,c}}_{44}$	$(\frac{1007633}{315} + \frac{10856}{105}\gamma - \frac{40979464}{315}\ln(2) + \frac{14203593}{280}\ln(3) + \frac{9765625}{504}\ln(5))\nu + (\frac{74436}{35} - \frac{1184}{5}\gamma + \frac{33693536}{105}\ln(2) - \frac{6396489}{70}\ln(3) - \frac{9765625}{126}\ln(5))\nu^2$
$q^{\mathrm{nl,ln}}_{44}$	$\frac{5428}{105}\nu - \frac{592}{5}\nu^2$
$q^{\mathrm{nl,c}}_{63}$	$(\frac{1300084}{525} + \frac{6875745536}{4725}\ln(2) - \frac{23132628}{175}\ln(3) - \frac{101687500}{189}\ln(5))\nu + (\frac{160124}{75} - \frac{4998308864}{1575}\ln(2) - \frac{45409167}{350}\ln(3) + \frac{26171875}{18}\ln(5))\nu^2$
$q^{\mathrm{nl,ln}}_{63}$	0

$$a^{\mathrm{nl,c}}_5 = \left(\frac{128}{5}\gamma + \frac{256}{5}\ln(2)\right)\nu, \tag{2.27}$$

where we decomposed $a^{\mathrm{nonloc}}_5 = a^{\mathrm{nl,c}}_5 + a^{\mathrm{nl,ln}}_5 \ln u$. They are listed in Table IV. Note that the nonlocal 4PN coefficients listed here slightly differ from the corresponding 4PN coefficients, $a^{\mathrm{II,c}}_5, \ldots$, listed in Ref. [23], because the latter reference had used a fixed partie finie scale s/c and had thereby incorporated the effects linked to averaging

$$\mathcal{F}^{\mathrm{split}}_{\mathrm{1PN}}(t,t)\ln r^h_{12}(t)$$

in the "local" parts of the (real and EOB) Hamiltonians.

III. USING SELF-FORCE THEORY TO COMPUTE THE LOCAL-PLUS-NONLOCAL EOB HAMILTONIAN, IN STANDARD GAUGE, AT FIRST ORDER IN ν

In this section, we shall use SF theory to compute the full, local-plus-nonlocal (rescaled effective) EOB Hamiltonian, at first order in ν. [The *rescaled* effective EOB Hamiltonian $\hat{H}_{\mathrm{eff}} \equiv H_{\mathrm{eff}}/\mu$ being divided by $\mu = \nu M$, the $O(\nu)$ contributions to $\hat{H}_{\mathrm{eff}}$ correspond to $O(\nu^2)$ contributions to $H_{\mathrm{real}} = Mc^2 + \cdots$.] It is convenient to parametrize the full, local-plus-nonlocal dynamics in terms of the various potentials entering the general form of $\hat{H}^2_{\mathrm{eff}}$ in standard p_r gauge, namely,

$$\hat{H}^2_{\mathrm{eff}} = A(u)(1 + p^2_\phi u^2 + A(u)\bar{D}(u)p^2_r + Q(u,p_r)). \tag{3.1}$$

The full knowledge of $\hat{H}_{\mathrm{eff}}$ means the knowledge of the various potentials: $A(u)$, $\bar{D}(u)$, and $Q(u,p_r) = p^4_r q_4(u) + p^6_r q_6(u) + p^8_r q_8(u) + p^{10}_r q_{10}(u) + \cdots$. These potentials all have, at any given PN level, a polynomial structure in ν, and they can be written in the forms

$$\begin{aligned}
A(u) &= 1 - 2u + \nu a^{\nu^1}(u) + \nu^2 a^{\nu^2}(u) + \nu^3 a^{\nu^3}(u) + \ldots \\
\bar{D}(u) &= 1 + \nu \bar{d}^{\nu^1}(u) + \nu^2 \bar{d}^{\nu^2}(u) + \nu^3 \bar{d}^{\nu^3}(u) + \ldots \\
q_4(u) &= \nu q^{\nu^1}_4(u) + \nu^2 q^{\nu^2}_4(u) + \nu^3 q^{\nu^3}_4(u) + \ldots \\
q_6(u) &= \nu q^{\nu^1}_6(u) + \nu^2 q^{\nu^2}_6(u) + \nu^3 q^{\nu^3}_6(u) + \ldots \\
q_8(u) &= \nu q^{\nu^1}_8(u) + \nu^2 q^{\nu^2}_8(u) + \nu^3 q^{\nu^3}_8(u) + \ldots,
\end{aligned} \tag{3.2}$$

etc. SF theory is an efficient tool for analytically computing (in principle, at any given PN order) the linear-in-ν pieces of the above EOB potentials, i.e., $a^{\nu^1}(u) = 2u^3 + a_4 u^4 + \cdots$, $\bar{d}^{\nu^1}(u)$, $q^{\nu^1}_4(u)$, etc. Indeed, the self-force computation of the redshift invariant $\langle \delta z_1 \rangle$ [48,58] of a particle moving along eccentric equatorial orbits in a perturbed Schwarzschild background, combined with the first law [42–44], has already allowed one to compute the linear-in-ν pieces of most of the EOB potentials. More precisely, $a^{\nu^1}(u)$ is known from the redshift of a particle

TABLE V. Coefficients of the 4 + 5PN terms in the linear-in-ν (i.e., 1SF) parts of the EOB potentials, Eq. (3.2). Here, γ denotes Euler's constant.

Coefficient	Expression
$\nu a^{\nu^1}_{4\mathrm{PN}+5\mathrm{PN}}$	$\nu[(-\frac{4237}{60}+\frac{2275}{512}\pi^2+\frac{256}{5}\ln(2)+\frac{128}{5}\gamma+\frac{64}{5}\ln(u))u^5+(-\frac{1066621}{1575}+\frac{246367}{3072}\pi^2-\frac{31736}{105}\ln(2)-\frac{14008}{105}\gamma-\frac{7004}{105}\ln(u)+\frac{243}{7}\ln(3))u^6]$
$\nu\bar{d}^{\nu^1}_{4\mathrm{PN}+5\mathrm{PN}}$	$\nu[(\frac{1184}{15}\gamma+\frac{2916}{5}\ln(3)-\frac{6496}{15}\ln(2)-\frac{23761}{1536}\pi^2-\frac{533}{45}+\frac{592}{15}\ln(u))u^4+(-\frac{2840}{7}\gamma-\frac{19683}{7}\ln(3)+\frac{120648}{35}\ln(2)-\frac{63707}{512}\pi^2+\frac{294464}{175}-\frac{1420}{7}\ln(u))u^5]$
$\nu q^{\nu^1}_{4\,4\mathrm{PN}+5\mathrm{PN}}$	$\nu[(\frac{496256}{45}\ln(2)-\frac{33048}{5}\ln(3)-\frac{5308}{15})u^3+(\frac{10856}{105}\gamma-\frac{40979464}{315}\ln(2)+\frac{14203593}{280}\ln(3)+\frac{9765625}{504}\ln(5)-\frac{93031}{1536}\pi^2+\frac{1295219}{350}+\frac{5428}{105}\ln(u))u^4]$
$\nu q^{\nu^1}_{6\,4\mathrm{PN}+5\mathrm{PN}}$	$\nu[(-\frac{2358912}{25}\ln(2)+\frac{1399437}{50}\ln(3)+\frac{390625}{18}\ln(5)-\frac{827}{3})u^2+(-\frac{101687500}{189}\ln(5)+\frac{6875745536}{4725}\ln(2)+\frac{2613083}{1050}-\frac{23132628}{175}\ln(3))u^3]$

moving along a circular orbit, whereas $\bar{d}^{\nu^1}(u)$, $q^{\nu^1}_4(u)$, $q^{\nu^1}_6(u)$, etc., are known from the averaged redshift invariant $\langle\delta z_1\rangle$ of a particle moving along a (bound) eccentric orbit at successive orders in an expansion in powers of the eccentricity

$$\langle\delta z_1\rangle = \delta z_1^{e^0} + e^2\delta z_1^{e^2} + e^4\delta z_1^{e^4} + e^6\delta z_1^{e^6} + O(e^8). \tag{3.3}$$

$\bar{d}^{\nu^1}(u)$ follows from $\delta z_1^{e^2}$, $q^{\nu^1}_4(u)$ follows from $\delta z_1^{e^4}$, $q^{\nu^1}_6(u)$ follows from $\delta z_1^{e^6}$, etc. The current self-force analytical knowledge of $\langle\delta z_1\rangle$ is limited at order $O(e^4)$. For the purpose of the present work, it was necessary to extend this knowledge to $O(e^6)$. We have used SF theory to compute high-PN expressions for $\delta z_1^{e^6}$, and correspondingly $q^{\nu^1}_6(u)$. We present in Appendix A our newly derived complete expression for $\delta z_1^{e^6}$ up to 9.5PN as well as its transcription into the EOB potential $q^{\nu^1}_6(u)$. The known SF expressions for the other potentials can be found in the literature (see Refs. [45,49,50,59]). We list in Table V the 4 + 5PN contributions to $a^{\nu^1}(u)$, $\bar{d}^{\nu^1}(u)$, $q^{\nu^1}_4(u)$, and $q^{\nu^1}_6(u)$.

IV. OBTAINING THE 5PN-ACCURATE LOCAL EOB HAMILTONIAN AT LINEAR ORDER IN ν

In the previous section, we used SF theory to derive the linear-in-ν local-plus-nonlocal EOB potentials. Subtracting from the latter local-plus-nonlocal potentials, the nonlocal part of the EOB potentials obtained in Sec. II above allows us to write down the *local* part of the EOB potentials at the first order in ν (only). At this stage, the $O(\nu\geq 2)$ contributions to the local EOB potentials are known only at 4PN, but not beyond. To clarify our knowledge so far, let us parametrize the ν dependence of a generic quantity $X(\nu)$ by the notation

$$X(\nu) \equiv X^{\nu^1}\nu + X^{(\nu)}, \tag{4.1}$$

where

$$X^{(\nu)} = X^{\nu^2}\nu^2 + X^{\nu^3}\nu^3 + X^{\nu^4}\nu^4 + \ldots. \tag{4.2}$$

For example, the local part of the a potential at 5PN (i.e., $\propto u^6$) will be written in the form

$$a_{5\mathrm{PN,loc,f}} = \left(-\frac{1026301}{1575}\nu + \frac{246367}{3072}\nu\pi^2 + a^{(\nu)}_{6,f}\right)u^6. \tag{4.3}$$

As indicated here, as we shall always use a flexibility parameter $f(t)-1$ which is at least of order ν^1, and as the corresponding contribution to the (unrescaled) Hamiltonian (2.11) involves an extra factor $\mathcal{F}^{\mathrm{split}}(t,t) = O(\nu^2)$, the effect of the flexibility factor $f(t)$ on (both) the local and nonlocal EOB potentials will start at order ν^2 [corresponding to $O(\nu^3)$ in the unrescaled Hamiltonian]. Our SF computation thereby uniquely determines all the linear-in-ν contributions to the local EOB potentials.

Summarizing, the local EOB potentials at 4 + 5PN, obtained from our (MPM + SF) results so far, have the following form:

$$\begin{aligned}
a_{4\mathrm{PN}+5\mathrm{PN,loc,f}} &= \left[\left(\frac{2275}{512}\pi^2-\frac{4237}{60}\right)\nu+\left(\frac{41}{32}\pi^2-\frac{221}{6}\right)\nu^2\right]u^5+\left(-\frac{1026301}{1575}\nu+\frac{246367}{3072}\nu\pi^2+a^{(\nu)}_{6,f}\right)u^6,\\
\bar{d}_{4\mathrm{PN}+5\mathrm{PN,loc,f}} &= \left[\left(\frac{1679}{9}-\frac{23761}{1536}\pi^2\right)\nu+\left(-260+\frac{123}{16}\pi^2\right)\nu^2\right]u^4+\left(\frac{331054}{175}\nu-\frac{63707}{512}\nu\pi^2+\bar{d}^{(\nu)}_{5,f}\right)u^5,\\
q_{4\,4\mathrm{PN}+5\mathrm{PN,loc,f}} &= (20\nu+q^{(\nu)}_{43})u^3+\left(-\frac{93031}{1536}\nu\pi^2+\frac{1580641}{3150}\nu+q^{(\nu)}_{44,f}\right)u^4,\\
q_{6\,4\mathrm{PN}+5\mathrm{PN,loc,f}} &= \left(-\frac{9}{5}\nu+q^{(\nu)}_{62}\right)u^2+\left(\frac{123}{10}\nu+q^{(\nu)}_{63,f}\right)u^3.
\end{aligned} \tag{4.4}$$

Here, we completed our previous $O(\nu)$ SF-based results by including the 4PN-level $O(\nu^2)$ terms previously derived in Ref. [23].

Note the remarkable fact that the 5PN-accurate local EOB Hamiltonian is *logarithm free*. Not only have all the $\ln u$ terms disappeared (as expected because they have been known for a long time to be linked to the time nonlocality), but even the various numerical logarithms $\ln 2, \ln 3, \ldots$, as well as Euler's constant γ have all disappeared. Only rational numbers and $\pi^2 \sim \zeta(2)$ enter this local Hamiltonian. The expressions above include still undetermined nonlinear-in-ν terms in the parametrized form indicated above, $a^{(\nu)}_{6,f}$, $\bar{d}^{(\nu)}_{5,f}$, $q^{(\nu)}_{44,f}$, and $q^{(\nu)}_{63,f}$. (The 4PN-level terms $q^{(\nu)}_{43}$ and $q^{(\nu)}_{62}$ are already known and will be written below.)

Note that at the 5PN level there is the further term in the Q potential

$$q_{8\,5\mathrm{PN,loc,f}} = (q^{\nu^1}_{82}\nu + q^{(\nu)}_{82})u^2 \equiv q_{82}(\nu)u^2, \tag{4.5}$$

which is still undetermined from our SF computation [because of its limitation to the order $O(e^6)$]. On the other hand, there is no contribution in the local Hamiltonian of the type $q_{10\,5\mathrm{PN,loc}} = q_{10,1}(\nu)u$, because Q^{loc} starts at order G^2. Such a term can only enter the nonlocal part of the Hamiltonian, where it comes from the need to expand the nonlocal Hamiltonian as a formally infinite series of powers of p_r^2 [23].

We are going to determine below most of the so far unknown nonlinear-in-ν coefficients by using information concerning the scattering angle for hyperbolic encounters. However, to do so, it will be convenient to change the standard EOB p_r gauge used above, into the so-called energy gauge [28].

V. 5PN LOCAL EOB HAMILTONIAN: FROM THE STANDARD p_r GAUGE TO THE ENERGY GAUGE

In the previous section, we determined, at the linear-in-ν order, the 5PN local EOB Hamiltonian in the standard p_r gauge. We then incorporated the nonlinear-in-ν contributions to the various EOB potentials A, $\bar{D}$, and Q in a parametrized form (still in the standard p_r gauge). Let us start from such a (local, p_r-gauge) Hamiltonian, i.e.,

$$\hat{H}^2_{\mathrm{eff,loc}} = A^{\mathrm{loc}}(1 + j^2u^2 + A^{\mathrm{loc}}\bar{D}^{\mathrm{loc}}p_r^2 + Q^{\mathrm{loc}}), \tag{5.1}$$

with

$$A^{\mathrm{loc}} = 1 - 2u + \delta A^{\mathrm{loc}}, \qquad \bar{D}^{\mathrm{loc}} = 1 + \delta\bar{D}^{\mathrm{loc}}, \tag{5.2}$$

and

$$\begin{aligned}
\delta A^{\mathrm{loc}} &= 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41}{32}\pi^2\right)u^4 + a_{4\mathrm{PN}+5\mathrm{PN,loc}},\\
\delta\bar{D}^{\mathrm{loc}} &= 6\nu u^2 + (52\nu - 6\nu^2)u^3 + \bar{d}_{4\mathrm{PN}+5\mathrm{PN,loc}},\\
Q^{\mathrm{loc}} &= p_r^4[2(4-3\nu)\nu u^2 + q_{4\,4\mathrm{PN}+5\mathrm{PN,loc}}]\\
&\quad + p_r^6 q_{6\,4\mathrm{PN}+5\mathrm{PN,loc}} + p_r^8(q^{\nu^1}_{82}\nu + q^{(\nu)}_{82})u^2,
\end{aligned} \tag{5.3}$$

depending on the various unknown coefficients $a^{(\nu)}_6$, $\bar{d}^{(\nu)}_5$, $q^{(\nu)}_{44}$, $q^{(\nu)}_{62}$, $q^{(\nu)}_{63}$, and $q^{(\nu)}_{82}$. Here, we did not put any explicit label f or h because the discussion of the present section applies to both cases.

Let us now show how to transform the above p_r-gauge (local) EOB Hamiltonian, Eq. (5.1), into its energy-gauge form, i.e., into the post-Schwarzschild (squared) effective Hamiltonian

$$\hat{H}^2_{\mathrm{eff,EG}} = H_S^2 + (1-2u)Q_{\mathrm{EG}}(u, H_S), \tag{5.4}$$

where H_S denotes the (rescaled) Schwarzschild Hamiltonian, i.e., the square root of

$$H_S^2 = (1-2u)[1 + (1-2u)p_r^2 + j^2u^2], \tag{5.5}$$

and where the energy-gauge Q term reads

$$\begin{aligned}
Q_{\mathrm{EG}}(u, H_S) &= u^2 q_{2\mathrm{EG}}(H_S) + u^3 q_{3\mathrm{EG}}(H_S)\\
&\quad + u^4 q_{4\mathrm{EG}}(H_S) + u^5 q_{5\mathrm{EG}}(H_S)\\
&\quad + u^6 q_{6\mathrm{EG}}(H_S) + \ldots.
\end{aligned} \tag{5.6}$$

Here, a term $q_{n\mathrm{EG}}(\gamma)u^n$, being proportional to G^n, describes the nPM approximation. When working within some PN-approximation scheme, one can only determine a limited number of terms in the PN expansion (corresponding to an expansion in powers of $p_\infty^2 \equiv \gamma^2 - 1$) of each separate energy-gauge coefficient $q_{n\mathrm{EG}}(\gamma)$. We will discuss below which terms in the p_∞^2 expansion of the various $q_{n\mathrm{EG}}(\gamma)$'s correspond to the 5PN level.

The PN expansions of all the energy-gauge coefficients $q_{n\mathrm{EG}}(\gamma)$ are determined from the corresponding p_r-gauge coefficients entering the Hamiltonian (notably, the 5PN-level ones $a^{(\nu)}_6$, $\bar{d}^{(\nu)}_5$, $q^{(\nu)}_{44}$, $q^{(\nu)}_{62}$ and $q^{(\nu)}_{63}$) by computing the canonical transformation connecting the two gauges. The structure of this canonical transformation is

$$g(r,p_r) = (rp_r)\frac{1}{r^2}\left[\frac{3}{2}\eta^4\nu + \eta^6\left(\frac{g_1}{r}+\frac{g_2 j^2}{r^2}+g_3 p_r^2\right) + \eta^8\left(\frac{h_1}{r^2}+\frac{h_2 j^4}{r^4}+h_3 p_r^4+\frac{h_4 j^2}{r^3}+h_5\frac{p_r^2}{r} + \frac{h_6 j^2 p_r^2}{r^2}\right) + \eta^{10}\left(\frac{n_1}{r^3}+\frac{n_2 j^2}{r^4} + \frac{n_3 p_r^2}{r^2}+\frac{n_4 j^4}{r^5}+\frac{n_5 j^2 p_r^2}{r^3}+\frac{n_6 p_r^4}{r} + \frac{n_7 j^6}{r^6}+\frac{n_8 j^4 p_r^2}{r^4}+\frac{n_9 j^2 p_r^4}{r^2}+n_{10}p_r^6\right)\right]. \quad (5.7)$$

Here, we have factored out the term (rp_r) that corresponds to the identity transformation, and we have ordered g by means of the PN-counting paramer $\eta \equiv \frac{1}{c}$. In addition, we are using here rescaled coordinates, namely, $r = r^{\rm phys}/(GM)$, $p_r = p_r^{\rm phys}/\mu$, and $j \equiv p_\phi = p_\phi^{\rm phys}/(GM\mu)$. As a consequence, each factor $\frac{p^m}{r^n}$ beyond the first factor (rp_r) in the canonical transformation (5.7) has to be seen as containing a factor G^n, and therefore to correspond to the nPM approximation. (When doing this counting, one must count each factor $j = |\mathbf{r}\times\mathbf{p}|$ as being $\sim rp \propto G^{-1}$, i.e., use the equivalence $\frac{j}{r} \sim p \sim p_r$.) In particular, we see that, in view of the overall prefactor $\frac{1}{r^2} = (\frac{GM}{r^{\rm phys}})^2$, the whole canonical transformation g, Eq. (5.7), starts at the 2PM (G^2) level.

Previous work has determined the canonical transformation g, Eq. (5.7), up to the 4PN level, i.e., $O(\eta^8)$. The 2PN ($\frac{3}{2}\eta^4\nu$) and 3PN ($\eta^6 g_i$) gauge parameters were derived in Ref. [28], while the 4PN ones ($\eta^8 h_i$) were derived in Appendix A of Ref. [30]. We have extended this determination to the 5PN level, by imposing that the two (effective, squared) Hamiltonians (5.1) and (5.4) be equivalent (at 5PN accuracy), through this canonical transformation. The explicit expressions of the 5PN coefficients n_i will be displayed later, in their final form, after we determine, using our strategy, all possible unknowns. However, as we discuss in the next section, the linear-in-ν results of the previous section suffice, at this stage, to uniquely determine the 3PM (G^3) energy-gauge coefficient $q_{3\rm EG}(\gamma;\nu)$, and thereby to test the all-PN-orders 3PM result of Refs. [29,31].

VI. DETERMINATION OF THE 3PM DYNAMICS UP TO THE 5PN (AND 6PN) LEVELS

Let us show here how the linear-in-ν results of Sec. IV suffice to determine the 3PM (G^3) energy-gauge coefficient $q_{3\rm EG}(\gamma;\nu)$, and thereby the 3PM scattering angle. This fact follows from three other facts. First, as explicitly shown in Ref. [51], the nonlocal part of the Hamiltonian starts contributing to the scattering angle only at the 4PM (G^4) level. Second, as emphasized in Ref. [41], thanks to the special ν dependence of the scattering angle, the knowledge of the $O(\nu^1)$ contribution to the scattering angle suffices to know its exact-in-ν value. Third, the local PN-expanded canonical transformation g, Eq. (5.7), being, at each PN order, a polynomial in G, cannot decrease the PM order of any contribution to the Hamiltonian. Putting these facts together, we conclude that it suffices to determine the linear-in-ν and less than or equal to G^3 value of g to compute the exact-in-ν value of the 3PM-level energy-gauge coefficient $q_{3\rm EG}(\gamma;\nu)$, at the same PN accuracy at which we know the local linear-in-ν, p_r-gauge Hamiltonian.

Before discussing the determination of the 3PM coefficient $q_{3\rm EG}(\gamma;\nu)$, let us recall that the value of the 2PM coefficient $q_{2\rm EG}(\gamma)$ has been determined to all PN orders, i.e., as an exact function of $\gamma = H_S$, in Ref. [28]. (It was then checked by other calculations [29,38].) It reads

$$q_{2\rm EG}(\gamma,\nu) = \frac{3}{2}(5\gamma^2-1)\left(1-\frac{1}{h(\gamma,\nu)}\right), \quad (6.1)$$

where

$$h(\gamma;\nu) \equiv [1+2\nu(\gamma-1)]^{1/2}. \quad (6.2)$$

We show in Appendix B how one can compute in *closed form* the result of transforming the energy-gauge 2PM Hamiltonian contribution $Q^{\rm 2PM}_{\rm EG}(u,H_S) = u^2 q_{2\rm EG}(H_S)$ into its standard p_r-gauge version $Q^{\rm 2PM}(u,p_r) = u^2 q_2(p_r^2)$. This is achieved by using an Abel transform. Note that the knowledge of the exact $Q^{\rm 2PM}$ will be crucial for the computation of the 5PN-level term $q_{82}p_r^8 u^2$.

Let us now come to the value of the 3PM coefficient $q_{3\rm EG}(\gamma,\nu)$. The structure of the ν-dependence of $q_{3\rm EG}(\gamma)$ has been shown to depend on the knowledge of two functions of γ, $A_{q3}(\gamma)$, and $B_{q3}(\gamma)$, namely [29,31,41],

$$q_{3\rm EG}(\gamma;\nu) = A_{q3}(\gamma) + \frac{B_{q3}(\gamma)}{h(\gamma;\nu)} - \frac{A_{q3}(\gamma)+B_{q3}(\gamma)}{h^2(\gamma;\nu)}. \quad (6.3)$$

Among the two functions of γ entering $q_{3\rm EG}(\gamma;\nu)$, the B_{q3} function is exactly known to be

$$B_{q3}(\gamma) = \frac{3}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{(\gamma^2-1)}. \quad (6.4)$$

Concerning the value of the other 3PM-level function $A_{q3}(\gamma)$, PN-based work previous to Ref. [26] had determined its 4PN-accurate value, namely,

$$A^{\rm PN}_{q3}(\gamma) = \frac{1}{(\gamma^2-1)}\left[2+\frac{37}{2}(\gamma^2-1)+\frac{117}{10}(\gamma^2-1)^2 + A_{5\rm PN}(\gamma^2-1)^3 + A_{6\rm PN}(\gamma^2-1)^4+\ldots\right], \quad (6.5)$$

where the coefficient $+\frac{117}{10}$ corresponds to the 4PN level and where the further coefficients $A_{5\rm PN}$ and $A_{6\rm PN}$,

respectively, parametrize the 5PN and 6PN contributions that we shall discuss next.

Only one line of work has so far been able to compute the exact value of the function $A_{q3}(\gamma)$, and thereby of the 3PM coefficient $q_{3\mathrm{EG}}(\gamma;\nu)$. Namely, the quantum-amplitude-based computation of Bern *et al.* Refs. [29,31] led to the (partly conjectural) exact expression for the function $A_{q3}(\gamma)$

$$A_{q3}^{\mathrm{Bern}}(\gamma) = -B_{q3}(\gamma) + \frac{\bar{C}^B(\gamma)}{\gamma - 1}, \tag{6.6}$$

with B_{q3} given in Eq. (6.4) and $\bar{C}^B$ given by (see, e.g., Eq. (3.69) of Ref. [41])

$$\bar{C}^B(\gamma) = \frac{2}{3}\gamma(14\gamma^2 + 25) + 4\frac{4\gamma^4 - 12\gamma^2 - 3}{\sqrt{\gamma^2 - 1}}\operatorname{arcsinh}\left(\sqrt{\frac{\gamma - 1}{2}}\right). \tag{6.7}$$

The expansion in powers of $\gamma^2 - 1$ of A_{q3}^{Bern} (describing its PN expansion) reads

$$A_{q3}^{\mathrm{Bern}}(\gamma) = \frac{1}{(\gamma^2 - 1)}\left[2 + \frac{37}{2}(\gamma^2 - 1) + \frac{117}{10}(\gamma^2 - 1)^2 + \frac{219}{140}(\gamma^2 - 1)^3 - \frac{7079}{10080}(\gamma^2 - 1)^4 + \frac{989}{2240}(\gamma^2 - 1)^5 + \ldots\right]. \tag{6.8}$$

Let us explain how the results presented in the previous sections allows us to compute the value of the 5PN coefficient $A_{5\mathrm{PN}}$ and how the recent extension of our method [39] has also allowed us to compute the 6PN coefficient $A_{6\mathrm{PN}}$.

Comparing the effect of the canonical transformation g, Eq. (5.7), on the linear-in-ν local (p_r-gauge) Hamiltonian given in Sec. IV to the corresponding $O(G^3)$-truncated energy-gauge Hamiltonian (5.4), with

$$Q_{\mathrm{EG}}^{\leq 3\mathrm{PM}}(u, H_S) = u^2 q_{2\mathrm{EG}}(H_S, \nu) + u^3 q_{3\mathrm{EG}}(H_S, \nu) + O(G^4) \tag{6.9}$$

yields enough equations to determine the linear-in-ν values of the 5PN-level coefficients parametrizing the $\leq G^3$ terms in Eq. (5.7). In view of the overall factor $\frac{1}{r^2} \propto G^2$ in g, the only less than or equal to G^3, 5PN coefficients are the n_i's with $i = 4, 5, 6, 7, 8, 9, 10$. For instance, the value of n_4 is determined to be

$$n_4 = \frac{603}{1120}\nu + O(\nu^2). \tag{6.10}$$

See below for the linear-in-ν values of the remaining 3PM-level (and 5PN level) gauge parameters n_i; $i = 4, 5, 6, 7, 8, 9, 10$. In addition, this less than or equal to G^3 determination of the gauge transformation g also determines the linear-in-ν, 5PN contribution to the 3PM energy-gauge coefficient $q_{3\mathrm{EG}}(H_S, \nu)$. In turn, as explained above [see Eq. (6.3)], the linear-in-ν value of the 3PM coefficient $q_{3\mathrm{EG}}(H_S, \nu)$ uniquely determines a corresponding knowledge of the (ν-independent) function $A_{q3}(\gamma)$. We thereby deduced (as already announced in Ref. [26]) from the results of Sec. IV the value of the 5PN-level coefficient in $A_{q3}(\gamma)$

$$A_{5\mathrm{PN}} = \frac{219}{140}, \tag{6.11}$$

in agreement with the result of Bern *et al.*, Eq. (6.8).

Recently, we have been able to extend our computation to the 6PN level by (i) pushing the computation of the nonlocal action to the 6PN order and (ii) pushing the SF redshift computation explained in Sec. III to the eighth order in eccentricity. This has allowed us to extend the knowledge of the local Hamiltonian to the 6PN order (see below for the 5.5PN, purely nonlocal contribution). We report our complete 6PN results somewhere else [39]. Let us here only cite the crucial new term allowing us to compute the 3PM-level, 6PN-accurate coefficient $A_{6\mathrm{PN}}$. It is the following $O(p_r^8 u^3)$ contribution to the EOB Q potential (in p_r gauge):

$$Q_{p_r^8 u^3}^{6\mathrm{PN}} = \left(-\frac{7447}{560}\nu + O(\nu^2)\right)p_r^8 u^3. \tag{6.12}$$

Transforming the p_r-gauge (3PM-6PN) knowledge of Eq. (6.12) into its energy-gauge correspondent [by extending the canonical transformation (5.7) to the 6PN level] then determines the 6PN-level, linear-in-ν contribution to the 3PM energy-gauge coefficient $q_{3\mathrm{EG}}(H_S, \nu)$. Expressing the latter result in terms of the parametrization, Eq. (6.3) finally leads to the value

$$A_{6\mathrm{PN}} = -\frac{7079}{10080}, \tag{6.13}$$

for the 6PN term in the function $A_{q3}(\gamma)$. This value agrees with the result of Bern *et al.*, Eq. (6.8).

While we were preparing our work for publication, an effective-field-theory computation of Blümlein *et al.* [40] reported an independent (purely PN-based[4]) derivation of the two coefficients (6.11), (6.13). Reference [41] had tried to reconcile an apparent tension between the high-energy limit of the result of Refs. [29,31] and the high-energy

[4]By contrast with our method which combines several different approximation schemes.

behavior of an older SF computation [60] by conjecturing another value of the function $A_{q3}(\gamma)$, which has a softer high-energy behavior and which starts to differ from Eq. (6.8) at the 6PN level. The latter conjecture is now disproved by the result (6.13). (See, however, Ref. [41], for other conjectural possibilities for reconciling the results of Refs. [29,31] and Ref. [60].)

VII. NONLOCAL PART OF THE SCATTERING ANGLE

Reference [41] has recently pointed out the existence of a restricted functional dependence of the scattering angle $\chi(\gamma, j; \nu)$ on the symmetric mass ratio ν.[5] This generic constraint applies to the total (local-plus-nonlocal) scattering angle. In the present section, we compute the nonlocal contribution, $\chi_{\rm nonloc,f}(\gamma, j; \nu)$, to the scattering angle with sufficient accuracy to be able to fully exploit the structure pointed out in Ref. [41]. Let us consider the large-j expansion of $\chi_{\rm nonloc,f}(\gamma, j; \nu)$, namely (see Ref. [51]),

$$\chi_{\rm nonloc,f}(\gamma, j; \nu) = \frac{\nu p_\infty^4}{j^4}\left[A_0(p_\infty;\nu) + \frac{A_1(p_\infty;\nu)}{p_\infty j} + \frac{A_2(p_\infty;\nu)}{(p_\infty j)^2} + O\left(\frac{1}{(p_\infty j)^3}\right)\right] \tag{7.1}$$

(where we recall that $p_\infty^2 \equiv \gamma^2 - 1$). As we shall see below, it is enough, for our present 5PN accuracy, to compute the coefficient $A_0(p_\infty;\nu)$ entering the leading order in $1/j$. The difficulty, however, is to compute it at the 1PN fractional accuracy: $A_0(p_\infty;\nu) = A_{00} + \eta^2 A_{02} p_\infty^2 + O(\eta^4 p_\infty^4)$. As will become clear, the small expansion parameter $\frac{1}{p_\infty j}$ (which happens to be of *Newtonian* order, i.e., η^0) used in Eq. (7.1) is equivalent to the inverse of the Newtonian eccentricity.

To compute the nonlocal contribution, $\chi_{\rm nonloc,f}$, to the scattering angle, we extend the strategy used at the leading PN order in Ref. [51]. It was shown there that

$$\frac{1}{2}\chi_{\rm nonloc,f}(\gamma, j; \nu) = \frac{1}{2\nu}\frac{\partial}{\partial j} W_{\rm nonloc,f}(\gamma, j; \nu), \tag{7.2}$$

where

$$W_{\rm nonloc,f}(\gamma, j; \nu) = \int dt \delta H_{\rm nonloc,f}. \tag{7.3}$$

Inserting the expression, Eq. (2.9), of $\delta H_{\rm nonloc,f}$ then leads to expressing $W_{\rm nonloc,f}$ as a sum of three terms, say,

$$W_{\rm nonloc,f} = W^{\rm split\ flux} + W^{\rm flux} + W^{\rm f-h}, \tag{7.4}$$

where

$$W^{\rm split\ flux} = -\frac{G^2 H}{c^5}\int dt {\rm Pf}_{2s/c} \int \frac{dt'}{|t'-t|}\mathcal{F}^{\rm split}_{\rm 1PN}(t,t'), \tag{7.5}$$

$$W^{\rm flux} = +\frac{2G^2 H}{c^5}\int dt \mathcal{F}^{\rm split}_{\rm 1PN}(t,t)\ln\left(\frac{r^h(t)}{s}\right), \tag{7.6}$$

and

$$W^{\rm f-h} = +2\frac{G^2 H}{c^5}\int dt \mathcal{F}^{\rm split}_{\rm 1PN}(t,t)\ln\left(f(t)\right). \tag{7.7}$$

To this end, we need to evaluate the split flux, as given in Eq. (2.3), along hyperbolic orbits at the fractional 1PN order. We then need (as will be made clear below) to compute the leading-order term of the expansion of $W_{\rm nonloc,f}(\gamma, j; \nu)$ in the large-j limit, corresponding to a large-eccentricity limit for the considered hyperbolic orbit.

To evaluate such a large-eccentricity limit, we start from the 1PN-accurate harmonic-coordinate quasi-Keplerian parametrization [56] of the hyperbolic motion, namely,

$$\begin{aligned} r &= \bar{a}_r(e_r \cosh v - 1),\\ \ell &= \bar{n}(t - t_0) = e_t \sinh v - v,\\ \bar{\phi} &= \frac{\phi - \phi_0}{K} = V = 2\arctan\left[\sqrt{\frac{e_\phi + 1}{e_\phi - 1}}\tanh\frac{v}{2}\right], \end{aligned} \tag{7.8}$$

where $K = 1 + \frac{3}{j^2}\eta^2$ and where the orbital parameters $\bar{n}$, $\bar{a}_r$, e_r, e_t, and e_ϕ are the functions of $\bar{E} = (E_{\rm tot} - Mc^2)/\mu$ and j listed in Table VI.

As shown in Ref. [56], the hyperbolic representation (7.8) is an analytic continuation of the corresponding ellipticlike quasi-Keplerian parametrization.

It is then useful to change the integration variables t and t' entering the definition of $W_{\rm nonloc,f}$ into the variables $T = \tanh v/2$ and $T' = \tanh v'/2$, where v and v' are the variables entering the quasi-Keplerian representation (7.8). This operation maps the original integration domain $(t, t') \in \mathbb{R}\times\mathbb{R}$ onto the compact domain $(T, T') \in [-1, 1]\times[-1, 1]$. It also transforms the singular line $t = t'$ into $T = T'$, together with a transformation of the constant cutoff $|t' - t| = 2s/c$ implied by the Pf operation into a corresponding T-dependent cutoff (see below).

TABLE VI. Quasi-Keplerian representation [Eq. (7.8)] of the hyperbolic 1PN motion, in terms of $\bar{E} = (E_{\rm tot} - Mc^2)/\mu$ and j.

$\bar{n}$	$(2\bar{E})^{3/2}[1 + \frac{1}{4}(15 - \nu)\bar{E}\eta^2]$
$\bar{a}_r$	$\frac{1}{2\bar{E}}[1 + \frac{1}{2}(7 - \nu)\bar{E}\eta^2]$
e_t^2	$1 + 2\bar{E}j^2 + \bar{E}[-\bar{E}j^2(-17 + 7\nu) + 4(1 - \nu)]\eta^2$
e_r^2	$1 + 2\bar{E}j^2 + \bar{E}[-5\bar{E}j^2(3 - \nu) + 2(-6 + \nu)]\eta^2$
e_ϕ^2	$1 + 2\bar{E}j^2 + \bar{E}[-\bar{E}j^2(15 - \nu) - 12]\eta^2$

[5]Note that it is important here to use the effective EOB energy $\gamma = \hat{\mathcal{E}}_{\rm eff}$ as an energy argument, besides $j = J/(GM\mu)$ and $\nu = \mu/M$.

We succeeded in computing the large-eccentricity limit of $W^{\rm split\ flux}$, at the leading order in eccentricity but including the fractional 1PN contribution. Both integrals in T' (with Pf) and in T can be performed exactly, within this limit. Note that during the various computational steps we take e_r as fundamental eccentricity and expand in powers of $\frac{1}{a_r} \sim p_\infty^2$. Some details follow.

The structure of $\mathcal{F}^{\rm split}_{\rm 1PN}(T,T')$ is

$$\mathcal{F}^{\rm split}_{\rm 1PN}(T,T') = \frac{\nu^2}{\bar{a}_r^5 e_r^4}[\mathcal{F}_{00} + \eta^2(\mathcal{F}_{20} + \nu\mathcal{F}_{21})], \tag{7.9}$$

where, for example,

$$\mathcal{F}_{00}(T,T') = \frac{32}{15}\frac{(1-T^2)^2(1-T'^2)^2}{(1+T^2)^5(1+T'^2)^5}\mathcal{P}_{00}(T,T'), \tag{7.10}$$

with

$$\begin{aligned}\mathcal{P}_{00}(T,T') = {}& 3(1-T^2)(1-T'^2)\\ &\times (T'^4 - 4T'^2 + 1)(T^4 - 4T^2 + 1)\\ &+ TT'\{37(T'^4+1)(T^4+1)\\ &- 52[T^2(T'^4+1) + T'^2(T^4+1)]\\ &+ 76T'^2T^2\}.\end{aligned} \tag{7.11}$$

Similarly, the expression for the integration measure $d\mathcal{M}_{(t,t')} = dtdt'/|t-t'|$, at 1PN, transformed in the variables T, T' is

$$\begin{aligned}d\mathcal{M}_{(T,T')} = {}& 2e_r\bar{a}_r^{3/2}\left[1 - \frac{1+2\nu}{2\bar{a}_r}\eta^2\right]\\ &\times \frac{(1+T'^2)(1+T^2)dTdT'}{(1-T'^2)(1-T^2)(1+TT')|T-T'|},\end{aligned} \tag{7.12}$$

at the leading order in a large-eccentricity expansion.

The (PN-expanded) transformed integrand $d\mathcal{M}_{(T,T')} \times \mathcal{F}(T,T')$ is then written as

$$d\mathcal{M}_{(T,T')}\mathcal{F}(T,T') = \mathcal{G}(T,T')\frac{dTdT'}{|T-T'|}. \tag{7.13}$$

The original integral was singular at $t = t'$, i.e., along the bisecting line of the t–t' plane. This singularity line becomes the bisecting line in the plane $T - T'$, but endowed with a T-dependent slit [equivalent to a Pf scale $2f(T)s/c$, where $f(T)$ is identified from the relation $dT = f(T)dt$]. In the large-eccentricity limit, one finds

$$f(T) = \frac{\bar{n}}{2e_t}\frac{(1-T^2)^2}{1+T^2}. \tag{7.14}$$

In other words, the integration interval of the split-flux integral over T' is divided into the two parts

$$[-1, T - \epsilon f(T)] \cup [T + \epsilon f(T), 1], \tag{7.15}$$

where ϵ is initially considered as being infinitesimal, before replacing it by the finite value $2s/c$ at the end. This leads to the partie finie integral

$$\begin{aligned}\mathcal{I}(T) = {\rm Pf}_{2sf(T)/c}\int_{-1}^{1} dT'\frac{\mathcal{G}(T,T')}{|-T'+T|} &= \int_{-1}^{1} dT'\frac{\mathcal{G}(T,T') - \mathcal{G}(T,T)}{|-T'+T|}\\ &+ \mathcal{G}(T,T)\left[\int_{-1}^{T-\epsilon f(T)}\frac{dT'}{T-T'} + \int_{T+\epsilon f(T)}^{1}\frac{dT'}{T'-T}\right]_{\epsilon\mapsto 2s/c}\\ &= \int_{-1}^{1} dT'\frac{q(T,T')}{|-T'+T|} + \mathcal{G}(T,T)\ln\left(\frac{1-T^2}{\epsilon^2 f^2(T)}\right)_{\epsilon\mapsto 2s/c},\end{aligned} \tag{7.16}$$

where we denoted the subtracted integrand as

$$q(T,T') \equiv \mathcal{G}(T,T') - \mathcal{G}(T,T). \tag{7.17}$$

Further integration in T then gives

$$\mathcal{J} \equiv \int_{-1}^{1}\mathcal{I}(T)dT = \int_{-1}^{1}dT\int_{-1}^{1}dT'\frac{q(T,T')}{|-T'+T|} - 2\ln\left(\frac{2s}{c}\right)\int_{-1}^{1}dT\mathcal{G}(T,T) + \int_{-1}^{1}dT\mathcal{G}(T,T)\ln\left(\frac{1-T^2}{f^2(T)}\right) \tag{7.18}$$

so that the first term in $W^{\rm nonloc,f}$ is given by

$$W^{\rm split\ flux} = -H_{\rm real}\mathcal{J}. \tag{7.19}$$

We find

$$W^{\text{split flux}} = \frac{2}{15}\frac{\pi\nu^2}{e_r^3\bar{a}_r^{7/2}} H_{\text{real}}\left\{100 + 37\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right) + \left[\frac{685}{4} - \frac{1017}{14}\nu + \left(\frac{3429}{56} - \frac{37}{2}\nu\right)\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\right]\frac{\eta^2}{\bar{a}_r}\right\}. \tag{7.20}$$

Note the presence of logarithmic terms.

The second contribution to $W^{\text{nonloc,f}}$, namely, Eq. (7.6), can be similarly computed, leading to

$$W^{\text{flux}} = \frac{2}{15}\frac{\pi\nu^2}{e_r^3\bar{a}_r^{7/2}} H_{\text{real}}\left\{-\frac{85}{4} - 37\ln\left(\frac{s}{2e_r\bar{a}_r}\right) + \left[-\frac{9679}{224} + \frac{981}{56}\nu + \left(-\frac{3429}{56} + \frac{37}{2}\nu\right)\ln\left(\frac{s}{2e_r\bar{a}_r}\right)\right]\frac{\eta^2}{\bar{a}_r}\right\}. \tag{7.21}$$

Summarizing, and reexpressing e_r and $\bar{a}_r$ in terms of p_∞ and j, we find for the h contribution to $W^{\text{nonloc,f}}$:

$$\begin{aligned} W^{\text{nonloc,h}} &\equiv W^{\text{split flux}} + W^{\text{flux}} \\ &= \frac{2}{15}\frac{\nu^2 p_\infty^4\pi}{j^3}\left[\frac{315}{4} + 37\ln\left(\frac{p_\infty}{2}\right) + \left[\frac{2753}{224} - \frac{1071}{8}\nu + \left(\frac{1357}{56} - \frac{111}{2}\nu\right)\ln\left(\frac{p_\infty}{2}\right)\right]p_\infty^2\eta^2\right]. \end{aligned} \tag{7.22}$$

This result is accurate modulo two types of corrections: $O(\eta^4)$ and $O(\frac{1}{p_\infty j})$. This suffices to compute the $1/j^4$ contribution to $\chi^{\text{nonloc,h}}$ at the fractional 1PN accuracy, namely,

$$\frac{1}{2}\chi^{\text{nonloc,h}} = \frac{1}{2\nu}\frac{\partial}{\partial j}W^{\text{nonloc,h}} = \frac{\chi_4^{\text{nonloc}}}{j^4} + O\left(\frac{1}{j^5}\right), \tag{7.23}$$

where (indicating for clarity the greater than or equal to 4PN nature of $\chi_4^{\text{nonloc,h}}$)

$$\chi_4^{\text{nonloc,h}} = \nu\eta^8 p_\infty^4\pi\left[a_{4\text{PN}} + \eta^2 a_{5\text{PN}} + \ln\left(\frac{p_\infty}{2}\right)(b_{4\text{PN}} + \eta^2 b_{5\text{PN}})\right], \tag{7.24}$$

with

$$\begin{aligned} a_{4\text{PN}} &= -\frac{63}{4}, \\ a_{5\text{PN}} &= -\frac{2753}{1120} + \frac{1071}{40}\nu, \\ b_{4\text{PN}} &= -\frac{37}{5}, \\ b_{5\text{PN}} &= -\frac{1357}{280} + \frac{111}{10}\nu. \end{aligned} \tag{7.25}$$

The meaning of this result will be further discussed in the next section.

VIII. USE AND DETERMINATION OF THE FLEXIBILITY FACTOR $f(t)$

The general rule uncovered in Ref. [41] restricts, at each PM order G^n, the ν dependence of the energy-rescaled scattering angle $\tilde{\chi}_n \equiv h^{n-1}\chi_n$, where we recall the definition,

$$h(\gamma,\nu) \equiv \sqrt{1 + 2\nu(\gamma - 1)}, \qquad \gamma = \sqrt{1 + p_\infty^2\eta^2}. \tag{8.1}$$

In the present case, we are interested in the $O(G^4)$ contribution χ_4 to the scattering angle. Let us then reexpress the result (7.24) in terms of the quantity $\tilde{\chi}_4 = h^3\chi_4$. We find

$$\begin{aligned} h^3\chi_4^{\text{nonloc,h}}(\gamma,\nu) &= \nu\eta^8 p_\infty^4\pi\left\{-\frac{63}{4} - \frac{37}{5}\ln\left(\frac{p_\infty}{2}\right)\right. \\ &\quad \left. + \eta^2 p_\infty^2\left[-\frac{2753}{1120} - \frac{1357}{280}\ln\left(\frac{p_\infty}{2}\right) + \frac{63}{20}\nu\right]\right\}. \end{aligned} \tag{8.2}$$

The general rule of Ref. [41] states, in this case, that the product $(h^3\chi_4)(\gamma,\nu)$ should be (at most) *linear* in ν. Taking into account the overall factor ν in $\chi_4^{\text{nonloc,h}}$, we see that this is true, at the fractional 1PN accuracy (i.e., at 5PN) for the logarithmic contributions to $\chi_4^{\text{nonloc,h}}$. On the other hand, the last term, $\propto \frac{63}{20}\nu$, in the expression of $h^3\chi_4^{\text{nonloc,h}}$ corresponds to a 5PN-level contribution equal to

$$\delta^{\nu^2}\left[\frac{h^3\chi_4^{\text{nonloc,h}}}{j^4}\right] = \nu^2\pi\frac{63}{20}\eta^{10}\frac{p_\infty^6}{j^4}. \tag{8.3}$$

The latter contribution is quadratic in ν. However, the general rule of Ref. [41] applies to the *total* scattering angle and therefore says that

$$h^3\chi_4^{\rm tot} = h^3\chi_4^{\rm loc,f} + h^3\chi_4^{\rm nonloc,f} = h^3\chi_4^{\rm loc,f} + h^3\chi_4^{\rm nonloc,h} + h^3\chi_4^{\rm f-h} \tag{8.4}$$

should be linear in ν. The presence of the $O(\nu^2)$ contribution (8.3) in $h^3\chi_4^{\rm nonloc,h}$ is then telling us that there should be compensating $O(\nu^2)$ contributions in the other terms $h^3\chi_4^{\rm loc,f} + h^3\chi_4^{\rm f-h}$. This can be arranged in many ways. On the one hand, if we were to insist on defining the nonlocal action by the h route, i.e., by systematically using $2r_{12}^h/c$ as Pf scale, we just need to adapt the $O(\nu^2)$ structure of the local (5PN-level) EOB potentials [so far only determined at the $O(\nu^1)$ level] so as to absorb the term (8.3). On the other hand, it seems advantageous to use the natural flexibility in the definition of the Pf scale (i.e., in using a flexed Pf scale $2r_{12}^f/c$) to make $h^3\chi_4^{\rm nonloc,f} = h^3\chi_4^{\rm nonloc,h} + h^3\chi_4^{\rm f-h}$ separately linear in ν. This allows us to better separate the determinations of the local and nonlocal parts of the dynamics. We shall see below that this second route has several nice properties.

If we choose the f route (as we shall do here), we need to determine the coefficients c_1, c_2, c_3 entering the 5PN-relevant flexibility factor $f(t) = 1 + \eta^2 f_1(t) + O(\eta^4)$, namely,

$$f_1 = c_1 p_r^2 + c_2 p^2 + c_3 \frac{1}{r}, \tag{8.5}$$

so that

$$h^3\chi_4^{\rm f-h} = \chi_4^{\rm f-h} + O(\eta^{12}) = -\nu^2\pi\frac{63}{20}\eta^{10}p_\infty^6 + O(\eta^{12}). \tag{8.6}$$

Here, we used the facts that $h = 1 + O(\eta^2)$ and that we require $\chi_4^{\rm f-h} = O(\eta^{10})$. It is not difficult to write the constraint on the coefficients c_1, c_2, c_3 implied by the equation (8.6). Indeed, we can write

$$\frac{1}{2}\chi^{\rm f-h} = \frac{1}{2\nu}\frac{\partial}{\partial j}W^{\rm f-h}, \tag{8.7}$$

where [see Eq. (7.7)]

$$W^{\rm f-h} = +2\frac{G^2H}{c^5}\int dt\mathcal{F}_{\rm 1PN}^{\rm split}(t,t)\ln(1+f_1(t)) = +2\frac{G^2H}{c^5}\int dt\mathcal{F}_{\rm 1PN}^{\rm GW}(t)f_1(t). \tag{8.8}$$

Recalling that the leading-order GW flux reads (in terms of scaled variables, and henceforth using $G = c = 1$)

$$\mathcal{F}_{\rm 1PN}^{\rm GW}(t) = \nu^2\frac{8}{15}\frac{1}{r^4}(12p^2 - 11p_r^2), \tag{8.9}$$

we have

$$\Delta W^{\rm f-h} = \nu^2\frac{16}{15}\int\frac{dt}{r^4}(12p^2 - 11p_r^2)\left(c_1p_r^2 + c_2p^2 + \frac{c_3}{r}\right). \tag{8.10}$$

This integral should be evaluated at Newtonian order. Though, for our present purpose of compensating the term 8.6)), we only need to compute the integral (8.10) to leading order in inverse eccentricity, let us give its exact value, as computed along a Newtonian orbit of squared effective energy $\gamma^2 = 1 + p_\infty^2$ and angular momentum j, with associated eccentricity $e^2 \equiv 1 + p_\infty^2 j^2$ and associated Newtonian-like energy $\bar{E} \equiv \frac{1}{2}p_\infty^2$,

$$\Delta W^{\rm f-h} = \frac{16\nu^2\bar{E}^{9/2}}{15}\left[\frac{A_1}{(e^2-1)^{9/2}}\arctan\sqrt{\frac{e+1}{e-1}} + \frac{A_2}{(e^2-1)^4}\right], \tag{8.11}$$

where

$$\begin{aligned} A_1 &= 4\sqrt{2}[(13c_1+74c_2)e^6 + (150c_1+1242c_2+366c_3)e^4 \\ &\quad + (96c_1+1544c_2+968c_3)e^2 + 192(c_2+c_3)], \\ A_2 &= \frac{2\sqrt{2}}{15}[(1437c_1+9866c_2+1568c_3)e^4 \\ &\quad + (2356c_1+28758c_2+14734c_3)e^2 \\ &\quad + 92c_1+7156c_2+6588c_3]. \end{aligned} \tag{8.12}$$

The beginning of the more immediately relevant large-j expansion of $\Delta W^{\rm f-h}$ reads

$$\Delta W^{\rm f-h} = \frac{\nu^2}{15}\left(\pi(13c_1+74c_2)\frac{p_\infty^6}{j^3} + \frac{64}{15}(51c_1+343c_2+49c_3)\frac{p_\infty^5}{j^4} + 3\pi(63c_1+488c_2+122c_3)\frac{p_\infty^4}{j^5} + \ldots\right). \tag{8.13}$$

Let us now compare the result (8.13) to the additional contribution to $W = \int dtH$, namely,

$$W^{\rm f-h,needed} = \nu^3\pi\frac{21}{10}\eta^{10}\frac{p_{\rm inf}^6}{j^3}, \tag{8.14}$$

which, according to Eqs. (8.6) and (8.7), is needed to compensate for the undesired ν^2/j^4 contribution to $\chi^{\rm nonloc,h}$.

By comparing Eq. (8.13) to Eq. (8.14), we see that it is enough that the coefficients c_1, c_2, c_3 parametrizing f_1 satisfy the single constraint

$$13c_1 + 74c_2 = \frac{63}{2}\nu. \tag{8.15}$$

The third flexibility parameter c_3 does not enter this constraint because it starts contributing to W at the $\propto \frac{\nu^2 c_3}{j^4}$ level, corresponding to a term $\propto \frac{\nu^2 c_3}{j^5}$ in χ. Such a flexed contribution is not needed at 5PN. The choice of the value of c_3 is free. We could simply take $c_3 = 0$. See below for the effect of choosing a nonzero value of c_3.

Equation (8.15) yields only one constraint on the two flexibility parameters c_1, c_2. The numerically simplest solution of Eq. (8.15) (having the smallest denominators) would be $c_1 = \frac{39}{2}\nu$, $c_2 = -3\nu$. On the other hand, similarly to the choice of p_r gauge, or energy gauge, for the EOB Hamiltonian, we could choose here, respectively, a flexibility factor f_1 containing either only p_r^2, namely,

$$f_1 = \frac{63}{26}\nu p_r^2, \tag{8.16}$$

or only p^2, namely,

$$f_1 = \frac{63}{148}\nu p^2. \tag{8.17}$$

By straightforward computations, we showed that the additional contribution $\Delta^{\mathrm{f-h}}H$ to the (nonlocal) Hamiltonian is equivalent, modulo a canonical transformation, to the following (p_r-gauge-type) Hamiltonian

$$\Delta^{\mathrm{f-h}}H = \frac{16}{15}\nu^2\left[(13c_1 + 74c_2)\frac{p_r^4}{r^4} + (12c_1 + 121c_2 + 49c_3)\frac{p_r^2}{r^5} + 12(c_2 + c_3)\frac{1}{r^6}\right]. \tag{8.18}$$

Let us note in passing that an efficient way of showing that $\Delta^{\mathrm{f-h}}H$ is canonically equivalent to Eq. (8.18) is to compute its integral along an ellipticlike orbit [instead of an hyperboliclike one, as in Eq. (8.11)]. (The time integral of the change of a Hamiltonian under an infinitesimal canonical transformation vanishes both along hyperbolic orbits and along elliptic motions.) The latter integral is much simpler than Eq. (8.11) and reads

$$\left[\oint dt \Delta H^{\mathrm{f-h}}\right]_{\mathrm{elliptic}} = \frac{\nu^2}{15j^9}[(74c_2 + 13c_1)e_r^6 + (1242c_2 + 366c_3 + 150c_1)e_r^4 + (96c_1 + 968c_3 + 1544c_2)e_r^2 + 192(c_2 + c_3)]. \tag{8.19}$$

In view of Eq. (8.18), it is easy to see that the Hamiltonian variation $\Delta^{f-h}H$ associated with a general f_1 (with arbitrary parameters c_1, c_2, c_3) is equivalent to varying the potentials $A, \bar{D}, Q$ parametrizing a p_r-gauge EOB Hamiltonian by the amounts

$$\Delta^{\mathrm{f}}A = \Delta^{\mathrm{f}}a_6 u^6,$$
$$\Delta^{\mathrm{f}}\bar{D} = \Delta^{\mathrm{f}}\bar{d}_5 u^5,$$
$$\Delta^{\mathrm{f}}Q = \Delta^{\mathrm{f}}q_{44} p_r^4 u^4, \tag{8.20}$$

where

$$\Delta^{\mathrm{f}}a_6 = \frac{128}{5}\nu(c_2 + c_3),$$
$$\Delta^{\mathrm{f}}\bar{d}_5 = \frac{32}{15}\nu(12c_1 + 121c_2 + 49c_3),$$
$$\Delta^{\mathrm{f}}q_{44} = \frac{32}{15}\nu(13c_1 + 74c_2). \tag{8.21}$$

The latter changes parametrize the contribution $\Delta^{f-h}H$ which is a part of the *nonlocal* Hamiltonian, $H^{\mathrm{nonloc,f}}$; see Eq. (2.9). They are absent in the h part of the nonlocal Hamiltonian $H^{\mathrm{nonloc,h}}$. More generally, both parts of the h-type Hamiltonian, the local one, $H^{\mathrm{loc,h}}$, and the nonlocal one, $H^{\mathrm{nonloc,h}}$, are totally independent of the choice of the flexibility factor f. Therefore, when one decides (as is our preferred choice) to use the f route[6] for computing the local Hamiltonian, one ends up with f-type EOB potentials (say in p_r gauge) parametrizing the complementary local Hamiltonian that are related to the corresponding h-type ones in the following way:

$$a_6^{\mathrm{loc,f}} = a_6^{\mathrm{loc,h}} - \Delta^{\mathrm{f}}a_6,$$
$$\bar{d}_5^{\mathrm{loc,f}} = \bar{d}_5^{\mathrm{loc,h}} - \Delta^{\mathrm{f}}\bar{d}_5,$$
$$q_{44}^{\mathrm{loc,f}} = q_{44}^{\mathrm{loc,h}} - \Delta^{\mathrm{f}}q_{44}. \tag{8.22}$$

The minus signs on the right-hand sides are needed because $\Delta^{\mathrm{f}}a_6$, etc., parametrize the additional contribution $+\Delta^{f-h}H \in H^{\mathrm{nonloc,f}}$; see Eq. (2.9).

The changes (8.22) have been written for a general flexibility factor of the form (8.5). Let us now apply these general results to the relevant case in which the parameters c_1, c_2, c_3 satisfy the constraint (8.15). We are going to see below that, when using the f route, the 5PN-level value of the EOB coefficient $q_{44}^{\mathrm{loc,f}}$ is fully determined and takes the value indicated in Table VII. We therefore conclude from the last Eq. (8.22) that the 5PN value of the EOB coefficient $q_{44}^{\mathrm{loc,h}}$ that would be derived by using the h route is also fully determined and differs from the f one by

[6]We mean by f route the use of a tuned flexibility factor f such that $\chi^{\mathrm{nonloc,f}}$ separately satisfies the rule of Ref. [41].

TABLE VII. Coefficients of the f-route local 4 + 5PN part of the EOB potentials, Eq. (11.7).

Coefficient	Expression
$a_5^{\rm loc}$	$(\frac{2275}{512}\pi^2-\frac{4237}{60})\nu+(\frac{41}{32}\pi^2-\frac{221}{6})\nu^2$
$a_6^{\rm loc}$	$(-\frac{1026301}{1575}+\frac{246367}{3072}\pi^2)\nu+a_6^{\nu^2}\nu^2+4\nu^3$
$\bar{d}_4^{\rm loc}$	$(\frac{1679}{9}-\frac{23761}{1536}\pi^2)\nu+(-260+\frac{123}{16}\pi^2)\nu^2$
$\bar{d}_5^{\rm loc}$	$(\frac{331054}{175}-\frac{63707}{512}\pi^2)\nu+\bar{d}_5^{\nu^2}\nu^2$ $+(\frac{1069}{3}-\frac{205}{16}\pi^2)\nu^3$
$q_{43}^{\rm loc}$	$20\nu-83\nu^2+10\nu^3$
$q_{44}^{\rm loc}$	$(\frac{1580641}{3150}-\frac{93031}{1536}\pi^2)\nu+(-\frac{2075}{3}+\frac{31633}{512}\pi^2)\nu^2$ $+(640-\frac{615}{32}\pi^2)\nu^3$
$q_{62}^{\rm loc}$	$-\frac{9}{5}\nu-\frac{27}{5}\nu^2+6\nu^3$
$q_{63}^{\rm loc}$	$\frac{123}{10}\nu-\frac{69}{5}\nu^2+116\nu^3-14\nu^4$
$q_{82}^{\rm loc}$	$\frac{6}{7}\nu+\frac{18}{7}\nu^2+\frac{24}{7}\nu^3-6\nu^4$

$$q_{44}^{\rm loc,h}=q_{44}^{\rm loc,f}+\frac{336}{5}\nu^2. \tag{8.23}$$

This has the effect of changing the rational $O(\nu^2)$ contribution $-\frac{2075}{3}\nu^2$ in $q_{44}^{\rm loc,f}$ into $-\frac{9367}{15}\nu^2$.

The corresponding changes in the values of $a_6^{\rm loc}$ and $\bar{d}_5^{\rm loc}$ are (currently) irrelevant because they only shift the two $O(\nu^2)$ parameters $a_6^{\nu^2}$ and $\bar{d}_5^{\nu^2}$ that are left undetermined by our method. If wished, the three flexibility parameters c_1, c_2, c_3 can be chosen so as to satisfy, besides the constraint Eq. (8.15), the two other equations $\Delta^{\rm f}a_6=0$ and $\Delta^{\rm f}\bar{d}_5=0$, ensuring that the two undetermined $O(\nu^2)$ parameters of the f route coincide with their corresponding h-route values. This yields the following specific values:

$$\begin{aligned} c_1&=\frac{189}{4}\nu,\\ c_2&=-\frac{63}{8}\nu,\\ c_3&=\frac{63}{8}\nu. \end{aligned} \tag{8.24}$$

These values define a sort of *minimal* choice for the flexibility factor, ensuring that the corresponding nonlocal scattering angle $h^3\chi_4^{\rm nonloc,f}$ is linear in ν, while leaving fixed the two $O(\nu^2)$ parameters $a_6^{\nu^2}$ and $\bar{d}_5^{\nu^2}$ entering the local dynamics.

Let us mention at this point that the formulation used in the published version of Ref. [26] contains an inconsistency related to the present discussion. Indeed, the value of the local Hamiltonian defined (in p_r gauge) by Eqs. (17) there is the f-route value, while Eq. (5) states that one was using the h route. The simplest way to correct this inconsistency is to multiply the Pf scale r_{12}^h entering Eq. (5) by a factor $f=1+f_1$, solution of Eq. (8.15). Alternatively, if one insists on using the h route (i.e., r_{12}^h as Pf length scale in the nonlocal action), one should replace the value of $q_{44}^{\rm loc}=q_{44}^{\rm loc,f}$ given in Eqs. (17) there, by $q_{44}^{\rm loc,h}$, as given in Eq. (8.23) above. [Correlatively, the f-route value of $\chi_{4{\rm loc}}^{5{\rm PN}}=\chi_{4{\rm loc,f}}^{5{\rm PN}}$ given in Eq. (19) there should then be changed into its h-route value $\chi_{4{\rm loc,h}}^{5{\rm PN}}=\chi_{4{\rm loc,f}}^{5{\rm PN}}+\chi^{\rm f-h}$, where $\chi^{\rm f-h}=-\frac{63}{20}\pi p_\infty^6\nu^2$; see Eqs. (8.4) and (8.6)].

IX. USING THE MASS-RATIO DEPENDENCE OF THE SCATTERING ANGLE TO DETERMINE MOST OF THE $\nu^{n\geq 2}$ STRUCTURE OF THE f-ROUTE LOCAL HAMILTONIAN

In the following, we assume that we define the nonlocal Hamiltonian by using a flexed Pf scale $r_{12}^f=f(t)r_{12}^h$, with a flexibility factor $f(t)=1+\eta^2 f_1(t)$ satisfying the constraint discussed in the previous section. This allows us to separately apply the constraints found in Ref. [41] to the scattering angle deriving from the corresponding local Hamiltonian, $H^{\rm loc,f}$. We are going to see that these constraints determine most of the nonlinear-in-ν contributions to $H^{\rm loc,f}$.

Let us start by recalling that, given any (local) Hamiltonian, the scattering angle of hyperboliclike motions is given by the integral ($u=1/r$) [61]

$$\frac{1}{2}(\chi(E,j)+\pi)=-\int_0^{u_{\rm max}}\frac{\partial}{\partial j}p_r(u;E,j)\frac{du}{u^2}, \tag{9.1}$$

where $u_{\rm max}=u_{\rm max}(E,j)=1/r_{\rm min}$ corresponds to the distance of closest approach of the two bodies and where the radial momentum $p_r=p_r(u;E,j)$ is obtained from writing the energy conservation at a given angular momentum. As $j=J/(GM\mu)$, the PM expansion of the scattering angle is an expansion in powers of $1/j\propto G$,

$$\frac{\chi^{\rm loc}(\hat{\mathcal{E}}_{\rm eff},j,\nu)}{2}=\chi_1^{\rm loc}(p_\infty,\nu)\frac{1}{j}+\chi_2^{\rm loc}(p_\infty,\nu)\frac{1}{j^2}+\chi_3^{\rm loc}(p_\infty,\nu)\frac{1}{j^3}+\ldots \tag{9.2}$$

where we replaced $\gamma=\hat{\mathcal{E}}_{\rm eff}$ by $p_\infty\equiv\sqrt{\hat{\mathcal{E}}_{\rm eff}^2-1}$.

The test-mass (Schwarzschild) limit $\chi_n^{\rm Sch}$ corresponds to setting $\nu=0$.

With this notation, let us consider the function (which vanishes for $\nu=0$)

$$T_n(p_\infty,\nu)=h^{n-1}(p_\infty,\nu)\chi_n^{\rm loc}-\chi_n^{\rm Sch}(p_\infty),\quad n\geq 2, \tag{9.3}$$

where $h\equiv\sqrt{1+2\nu(\hat{\mathcal{E}}_{\rm eff}-1)}$. Reference [41] has shown that T_n must be a polynomial in ν of order (at most)

$d_n = [\frac{n-1}{2}]$, where $[x]$ denotes the integer part of x: $T_n \sim \nu + \nu^2 + \cdots + \nu^{d_n}$. Therefore, we have the following conditions C_n:

(i) $[C_2:]$ $T_2 = 0$ does not depend on ν;
(ii) $[C_3:]$ $T_3 \sim \nu$;
(iii) $[C_4:]$ $T_4 \sim \nu$;
(iv) $[C_5:]$ $T_5 \sim \nu + \nu^2$;
(v) $[C_6:]$ $T_6 \sim \nu + \nu^2$.

Here, we shall apply these results at the 5PN level, using the 5PN expansion of h, namely,

$$\begin{aligned} h &= \{1 + 2\nu[(1 + p_\infty^2)^{1/2} - 1]\}^{1/2} \\ &= 1 + \frac{1}{2}\nu p_\infty^2 + \left(-\frac{1}{8}\nu - \frac{1}{8}\nu^2\right) p_\infty^4 + \left(\frac{1}{16}\nu^2 + \frac{1}{16}\nu + \frac{1}{16}\nu^3\right) p_\infty^6 \\ &\quad + \left(-\frac{5}{128}\nu^2 - \frac{5}{128}\nu - \frac{3}{64}\nu^3 - \frac{5}{128}\nu^4\right) p_\infty^8 \\ &\quad + \left(\frac{7}{256}\nu^2 + \frac{7}{256}\nu + \frac{9}{256}\nu^3 + \frac{5}{128}\nu^4 + \frac{7}{256}\nu^5\right) p_\infty^{10} + O(p_\infty^{12}), \end{aligned} \tag{9.4}$$

and, correspondingly, of T_n,

$$T_n^{\mathrm{5PN}}(p_\infty, \nu) = [h^{n-1}(p_\infty, \nu)\chi_n^{\mathrm{loc}} - \chi_n^{\mathrm{Sch}}]^{\mathrm{5PN}}, \quad n \geq 2. \tag{9.5}$$

Our SF-based computation above has heretofore determined only the coefficients of the $O(\nu^1)$ terms in the (μ-rescaled) local Hamiltonian. The determination of most of the $O(\nu^{\geq 2})$ coefficients in the local EOB Hamiltonian will now be obtained by first computing the PN expansion of the (local part of the conservative) scattering angle, χ^{loc} [using in Eq. (9.1) a PN-expanded expression for p_r] and then computing the various T_n's at the 5PN accuracy.

From the condition C_2, we find

$$\begin{aligned} q_{62}^{\mathrm{loc}} &= -\frac{9}{5}\nu - \frac{27}{5}\nu^2 + 6\nu^3, \\ q_{82}^{\mathrm{loc}} &= \frac{18}{7}\nu^2 + \frac{6}{7}\nu + \frac{24}{7}\nu^3 - 6\nu^4. \end{aligned} \tag{9.6}$$

From the condition C_3, we find

$$\begin{aligned} q_{43}^{\mathrm{loc}} &= 20\nu - 83\nu^2 + 10\nu^3, \\ q_{63}^{\mathrm{loc}} &= \frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4. \end{aligned} \tag{9.7}$$

From the condition C_4, we find

$$\begin{aligned} q_{44}^{\mathrm{loc}} &= \left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu \\ &\quad + \left(-\frac{2075}{3} + \frac{31633}{512}\pi^2\right)\nu^2 \\ &\quad + \left(640 - \frac{615}{32}\pi^2\right)\nu^3. \end{aligned} \tag{9.8}$$

From the condition C_5, we fix $\bar{d}_5^{\nu^3}$ (and $\bar{d}_5^{\nu^4} = 0$) so that

$$\begin{aligned} \bar{d}_5^{\mathrm{loc}} &= \left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + \bar{d}_5^{\nu^2}\nu^2 \\ &\quad + \left(\frac{1069}{3} - \frac{205}{16}\pi^2\right)\nu^3, \end{aligned} \tag{9.9}$$

where the $O(\nu^2)$ coefficient $\bar{d}_5^{\nu^2}$ remains undetermined.

Finally, from the condition C_6, we fix $a_6^{\nu^3}$ (and $a_6^{\nu^4} = 0$) so that

$$a_6^{\mathrm{loc}} = \left(-\frac{1026301}{1575} - \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2}\nu^2 + 4\nu^3, \tag{9.10}$$

where the $O(\nu^2)$ coefficient $a_6^{\nu^2}$ remains undetermined.

The additional condition C_7 (meaning that $T_7 \sim \nu + \nu^2 + \nu^3$) does not carry any new information.

Summarizing, the conditions C_n have allowed us to determine all the terms in the 5PN-accurate (gauge-fixed) f-flexed local effective EOB Hamiltonian apart from the two $O(\nu^2)$ terms parametrized by $a_6^{\nu^2}$ and $\bar{d}_5^{\nu^2}$.

X. VALUES OF THE 5PN-ACCURATE f-ROUTE LOCAL SCATTERING ANGLE AT PM ORDERS G^3, G^4, G^5, AND G^6

Having determined most of the coefficients parametrizing the local Hamiltonian, we can write down the following (PN-expanded) values for the f-flexed local parts of the successive n-PM contributions, χ_n, to the scattering angle (subtracted by their Schwarzschild values):

$$
\begin{aligned}
\pi^{-1}(\chi_2^{\rm loc}-\chi_2^{\rm Sch}) &= -\frac{3}{4}p_\infty^2\nu+\left(\frac{9}{16}\nu^2-\frac{3}{4}\nu\right)p_\infty^4\\
&\quad+\left(\frac{9}{64}\nu+\frac{27}{64}\nu^2-\frac{15}{32}\nu^3\right)p_\infty^6+\left(-\frac{45}{256}\nu^2-\frac{15}{256}\nu-\frac{15}{64}\nu^3+\frac{105}{256}\nu^4\right)p_\infty^8,\\
\chi_3^{\rm loc}-\chi_3^{\rm Sch} &= -8p_\infty\nu+(8\nu^2-36\nu)p_\infty^3+\left(-\frac{91}{5}\nu+34\nu^2-8\nu^3\right)p_\infty^5+\left(\frac{69}{70}\nu+\frac{51}{5}\nu^2-32\nu^3+8\nu^4\right)p_\infty^7,\\
\pi^{-1}(\chi_4^{\rm loc}-\chi_4^{\rm Sch}) &= -\frac{15}{4}\nu+\left(\frac{45}{8}\nu^2-\frac{109}{2}\nu+\frac{123}{256}\pi^2\nu\right)p_\infty^2\\
&\quad+\left(-\frac{225}{32}\nu^3+\frac{33601}{16384}\pi^2\nu-\frac{19597}{192}\nu+\frac{4827}{64}\nu^2-\frac{369}{512}\pi^2\nu^2\right)p_\infty^4\\
&\quad+\left(-\frac{94899}{32768}\pi^2\nu^2+\frac{93031}{32768}\pi^2\nu-\frac{1945583}{33600}\nu+\frac{1937}{16}\nu^2-\frac{2895}{32}\nu^3+\frac{525}{64}\nu^4+\frac{1845}{2048}\pi^2\nu^3\right)p_\infty^6,\\
\chi_5^{\rm loc}-\chi_5^{\rm Sch} &= -\frac{8\nu}{p_\infty}+\left(\frac{41}{8}\pi^2\nu-\frac{1168}{3}\nu+24\nu^2\right)p_\infty\\
&\quad+\left(-\frac{227059}{135}\nu+\frac{5069}{144}\pi^2\nu-\frac{287}{24}\nu^2\pi^2+\frac{7342}{9}\nu^2-40\nu^3\right)p_\infty^3\\
&\quad+\left(-\frac{11108}{9}\nu^3-\frac{1460479}{525}\nu+\frac{41026}{15}\nu^2+56\nu^4-\frac{4}{15}\bar{d}_5^{\nu^2}\nu^2+\frac{451}{24}\nu^3\pi^2-\frac{40817}{640}\nu^2\pi^2\right.\\
&\quad\left.+\frac{111049}{960}\pi^2\nu\right)p_\infty^5,\\
\pi^{-1}(\chi_6^{\rm loc}-\chi_6^{\rm Sch}) &= -\frac{625}{4}\nu+\frac{615}{256}\pi^2\nu+\frac{105}{16}\nu^2+\left(\frac{35065}{64}\nu^2+\frac{257195}{8192}\pi^2\nu-\frac{293413}{192}\nu-\frac{615}{64}\pi^2\nu^2-\frac{525}{32}\nu^3\right)p_\infty^2\\
&\quad+\left(\frac{3675}{128}\nu^4-\frac{15}{32}\bar{d}_5^{\nu^2}\nu^2+\frac{2321185}{16384}\pi^2\nu-\frac{15}{32}a_6^{\nu^2}\nu^2+\frac{39975}{2048}\pi^2\nu^3-\frac{63277573}{13440}\nu-\frac{34325}{32}\nu^3\right.\\
&\quad\left.+\frac{444955}{128}\nu^2-\frac{2584605}{32768}\pi^2\nu^2\right)p_\infty^4.
\end{aligned}
\tag{10.1}
$$

The corresponding Schwarzschild terms ($\nu=0$) are given by

$$
\begin{aligned}
&\chi_1^{\rm Schw}(p_\infty)=\frac{1}{p_\infty}+2p_\infty,\qquad \chi_2^{\rm Schw}(p_\infty)=\pi\left(\frac{3}{2}+\frac{15}{8}p_\infty^2\right),\qquad \chi_3^{\rm Schw}(p_\infty)=-\frac{1}{3p_\infty^3}+\frac{4}{p_\infty}+24p_\infty+\frac{64}{3}p_\infty^3,\\
&\chi_4^{\rm Schw}(p_\infty)=\pi\left(\frac{105}{8}+\frac{315}{8}p_\infty^2+\frac{3465}{128}p_\infty^4\right),\qquad \chi_5^{\rm Schw}(p_\infty)=\frac{1}{5p_\infty^5}-\frac{2}{p_\infty^3}+\frac{32}{p_\infty}+320p_\infty+640p_\infty^3+\frac{1792}{5}p_\infty^5,\\
&\chi_6^{\rm Schw}(p_\infty)=\pi\left(\frac{1155}{8}+\frac{45045}{64}p_\infty^2+\frac{135135}{128}p_\infty^4+O(p_\infty^6)\right),\\
&\chi_7^{\rm Schw}(p_\infty)=-\frac{1}{7p_\infty^7}+\frac{8}{5p_\infty^5}-\frac{16}{p_\infty^3}+\frac{320}{p_\infty}+4480p_\infty+14336p_\infty^3+O(p_\infty^5),\\
&\chi_8^{\rm Schw}(p_\infty)=\pi\left(\frac{225225}{128}+\frac{765765}{64}p_\infty^2+O(p_\infty^4)\right),\\
&\chi_9^{\rm Schw}(p_\infty)=\frac{1}{9p_\infty^9}-\frac{10}{7p_\infty^7}+\frac{96}{7}\frac{1}{p_\infty^5}-\frac{448}{3}\frac{1}{p_\infty^3}+\frac{3584}{p_\infty}+64512p_\infty+O(p_\infty^3),\\
&\chi_{10}^{\rm Schw}(p_\infty)=\pi\frac{2909907}{128}+O(p_\infty^2).
\end{aligned}
\tag{10.2}
$$

Here, the results for $\chi_1^{\rm Schw}(p_\infty)$ up to $\chi_5^{\rm Schw}(p_\infty)$ are exact, while the error terms in $\chi_6^{\rm Schw}(p_\infty)$ up to $\chi_{10}^{\rm Schw}(p_\infty)$ correspond to the 5PN accuracy.

These results for the scattering angle provide a lot of new information that offers gauge-invariant checks for future independent computations of the dynamics of binary systems.

In particular, using the fact (explicitly proven in Ref. [51]) that the nonlocal dynamics starts contributing to the scattering angle only at $O(G^4)$, so that $\chi_3 = \chi_3^{\rm loc} + \chi_3^{\rm nonloc} = \chi_3^{\rm loc}$, our result above for $\chi_3^{\rm loc}$ actually describes the total 3PM-level scattering angle. Its explicit expression [when combining the test mass and $\nu^{\geq 1}$ piece and adding our recent 6PN extension, embodied in Eq. (6.13)] reads

$$\begin{aligned}\chi_3 = &-\frac{1}{3p_\infty^3} + \frac{4}{p_\infty} + (24-8\nu)p_\infty + \left(\frac{64}{3} - 36\nu + 8\nu^2\right)p_\infty^3 \\ &+ \left(-\frac{91}{5}\nu + 34\nu^2 - 8\nu^3\right)p_\infty^5 \\ &+ \left(\frac{69}{70}\nu + \frac{51}{5}\nu^2 - 32\nu^3 + 8\nu^4\right)p_\infty^7 \\ &+ \left(\frac{1447}{5040}\nu - \frac{93}{56}\nu^2 - \frac{27}{10}\nu^3 + 30\nu^4 - 8\nu^5\right)p_\infty^9 \\ &+ O(p_\infty^{11}),\end{aligned} \tag{10.3}$$

In this expression, the last term $\propto p_\infty^9$ is the 6PN contribution to χ_3. As already mentioned, this result is in agreement with the corresponding 6PN-level term in the PN expansion of the 3PM-level recent result of Refs. [29,31]. It has also been recently obtained in Ref. [40]. Let us note in passing that all the rather complicated ν structure of χ_3 is actually described by the simple rule C_3 mentioned above [i.e., the linearity of T_3, Eq. (9.3), in ν]. Indeed, we have

$$h^2\chi_3 = (1 + 2\nu(\gamma-1))\chi_3^{\rm Schw}(p_\infty) - 2\nu p_\infty \bar{C}(p_\infty), \tag{10.4}$$

where

$$\chi_3^{\rm Schw}(p_\infty) = \frac{1}{3p_\infty^3}(-1 + 12p_\infty^2 + 72p_\infty^4 + 64p_\infty^6), \tag{10.5}$$

and

$$\bar{C}(\gamma) \equiv (\gamma-1)(A_{q3}(\gamma) + B_{q3}(\gamma)), \tag{10.6}$$

whose 6PN-accurate expansion reads

$$\begin{aligned}\bar{C}^{\rm 6PN}(p_\infty) = 4 + 18p_\infty^2 + \frac{91}{10}p_\infty^4 - \frac{69}{140}p_\infty^6 \\ - \frac{1447}{10080}p_\infty^8 + O(p_\infty^{10}).\end{aligned} \tag{10.7}$$

In addition, our results also provide a complete, 5PN-accurate value for the 4PM-level scattering angle $\chi_4 = \chi_4^{\rm loc,f} + \chi_4^{\rm nonloc,f}$. It is convenient to reexpress the result for χ_4 in terms of its rescaled version, $\tilde{\chi}_4 = h^3\chi_4$. We have evaluated in Sec. VIII the nonlocal contribution to $h^3\chi_4$, namely (when using a flexibility factor f_1 of the type discussed there),

$$h^3\chi_4^{\rm nonloc,f}(\gamma,\nu) = \nu\eta^8 p_\infty^4 \pi\left\{-\frac{63}{4} - \frac{37}{5}\ln\left(\frac{p_\infty}{2}\right) + \eta^2 p_\infty^2\left[-\frac{2753}{1120} - \frac{1357}{280}\ln\left(\frac{p_\infty}{2}\right)\right]\right\}. \tag{10.8}$$

Concerning the corresponding complementary f-type local contribution $\chi_4^{\rm loc,f}$, we have already given its explicit value in Eqs. (10.1) above. Let us also cite the much simpler expression of its rescaled version, which is linear in ν. Similarly to the rescaled version of χ_3 written above, it can be written as

$$h^3\chi_4^{\rm loc,f}(\gamma,\nu) = (1 + 2\nu(\gamma-1))\chi_4^{\rm Schw}(p_\infty) + \nu\hat{\chi}_4^{\rm loc,f}(p_\infty), \tag{10.9}$$

where

$$\hat{\chi}_4^{\rm loc,f}(p_\infty) = \pi\left[-\frac{15}{4} + \left(\frac{123}{256}\pi^2 - \frac{767}{16}\right)p_\infty^2 + \left(-\frac{4033}{48} + \frac{33601}{16384}\pi^2\right)p_\infty^4 + \left(-\frac{6514457}{134400} + \frac{93031}{32768}\pi^2\right)p_\infty^6\right] + O(p_\infty^8). \tag{10.10}$$

Finally, concerning the 5PM and 6PM local scattering angles, $\chi_5^{\rm loc,f}$ and $\chi_6^{\rm loc,f}$, most of the information displayed in Eqs. (10.1) above comes from the h^{n-1}-rescaling rule. (It is, however, important to confirm this rule by explicit computations.) Apart from the latter rule, the new 5PM and 6PM information derived here concerns the linear-in-ν contributions. We can write the expressions

$$\nu^{-1}h^4[\chi_5^{\mathrm{loc,f}}(\gamma,\nu)-\chi_5^{\mathrm{Schw}}(p_\infty)] = -\frac{8}{p_\infty}+\left(8\nu+\frac{41}{8}\pi^2-\frac{1168}{3}\right)p_\infty+\left(\frac{370}{9}\nu+\frac{5069}{144}\pi^2-\frac{41}{24}\nu\pi^2-\frac{227059}{135}\right)p_\infty^3 + \left(-\frac{4}{15}\nu\bar{d}_5^{\nu^2}+\frac{23407}{5760}\nu\pi^2+\frac{111049}{960}\pi^2-\frac{1460479}{525}-\frac{58874}{135}\nu\right)p_\infty^5+O(p_\infty^7),$$

$$\nu^{-1}\pi^{-1}h^5[\chi_6^{\mathrm{loc,f}}(\gamma,\nu)-\chi_6^{\mathrm{Schw}}(p_\infty)] = \frac{615}{256}\pi^2+\frac{105}{16}\nu-\frac{625}{4}+\left(\frac{10065}{64}\nu-\frac{293413}{192}+\frac{257195}{8192}\pi^2-\frac{1845}{512}\nu\pi^2\right)p_\infty^2 + \left(-\frac{15}{32}\nu(\bar{d}_5^{\nu^2}+a_6^{\nu^2})-\frac{23675}{96}\nu-\frac{61855}{32768}\nu\pi^2+\frac{2321185}{16384}\pi^2-\frac{63277573}{13440}\right)p_\infty^4 + O(p_\infty^6), \tag{10.11}$$

which emphasize that the coefficients of the ν^2 contributions are currently not fully determined, since they involve the $O(\nu^2)$ terms $\bar{d}_5^{\nu^2}$ and $a_6^{\nu^2}$.

Let us also rewrite this information in a form more directly connected with exhibiting the simple $\sim 1+\nu+\nu^2$ structure of $h^{n-1}\chi_n^{\mathrm{loc,f}}-\chi_n^{\mathrm{Schw}}$:

$$\nu^{-1}(h^4\chi_5^{\mathrm{loc,f}}-\chi_5^{\mathrm{Schw}}) = \frac{2}{5p_\infty^3}+\left(-\frac{121}{10}+\frac{1}{5}\nu\right)\frac{1}{p_\infty}+\left(-\frac{19457}{60}+\frac{59}{10}\nu+\frac{41}{8}\pi^2\right)p_\infty + \left(-\frac{41}{24}\nu\pi^2+\frac{10681}{144}\nu+\frac{5069}{144}\pi^2-\frac{4572503}{4320}\right)p_\infty^3 + \left(\frac{111049}{960}\pi^2+\frac{23407}{5760}\nu\pi^2-\frac{4}{15}\nu\bar{d}_5^{\nu^2}-\frac{55558621}{33600}-\frac{573577}{4320}\nu\right)p_\infty^5 + O(p_\infty^7),$$

$$\nu^{-1}\pi^{-1}(h^5\chi_6^{\mathrm{loc,f}}-\chi_6^{\mathrm{Schw}}) = \left(-\frac{625}{4}+\frac{105}{16}\nu+\frac{615}{256}\pi^2\right)+\left(-\frac{1845}{512}\nu\pi^2+\frac{257195}{8192}\pi^2+\frac{10065}{64}\nu-\frac{224113}{192}\right)p_\infty^2 + \left(-\frac{15}{32}\nu a_6^{\nu^2}-\frac{15}{32}\nu\bar{d}_5^{\nu^2}-\frac{61855}{32768}\nu\pi^2+\frac{2321185}{16384}\pi^2+\frac{4625}{192}\nu-\frac{20420849}{6720}\right)p_\infty^4+O(p_\infty^6). \tag{10.12}$$

XI. FINAL RESULTS FOR THE 5PN-ACCURATE f-ROUTE LOCAL EOB HAMILTONIAN

Let us gather the 5PN-accurate results (for the f-type local dynamics) obtained so far in the previous sections. They concern various forms of the local Hamiltonian: (i) the energy-gauge version of the local effective EOB Hamiltonian, (ii) the p_r-gauge version of the local effective EOB Hamiltonian, (iii) the local *real* Hamiltonian, and (iv) the canonical transformation connecting the p_r gauge to the energy gauge. Before listing our results, let us recall again the link between the usual "real" Hamiltonian, H, and the dimensionless (μ-rescaled) effective EOB Hamiltonian $\hat{H}_{\mathrm{eff}}$:

$$H^{\mathrm{loc}} = H^{\mathrm{loc}}_{\mathrm{eob}} = M\sqrt{1+2\nu(\hat{H}^{\mathrm{loc}}_{\mathrm{eff}}-1)}. \tag{11.1}$$

Note that we sometimes (as indicated here) add a subscript eob to the real, local Hamiltonian H^{loc} when we wish to emphasize that it is expressed in terms of EOB canonical coordinates. But, numerically, $H^{\mathrm{loc}}_{\mathrm{eob}}$ is equal to the usual (local) Hamiltonian, whose conserved value is equal to the total, c.m. conserved energy of the binary system [minus the nonlocal $4+5$PN contribution linked to Eq. (2.2)].

A. 5PN-accurate f-flexed local effective EOB Hamiltonian in energy gauge

We recall that the energy-gauge, squared effective EOB Hamiltonian is written as

$$\hat{H}^2_{\mathrm{eff,EG}} = H_S^2+(1-2u)Q_{\mathrm{EG}}(u,H_S), \tag{11.2}$$

where the rescaled Schwarzschild Hamiltonian $H_S=\sqrt{(1-2u)[1+(1-2u)p_r^2+j^2u^2]}$ and where the energy-gauge Q potential is written as

$$Q_{\rm EG}(u,H_S) = u^2 q_{\rm 2EG}(H_S;\nu) + u^3 q_{\rm 3EG}(H_S;\nu) + u^4 q_{\rm 4EG}(H_S;\nu) + u^5 q_{\rm 5EG}(H_S;\nu) + u^6 q_{\rm 6EG}(H_S;\nu) + \ldots. \tag{11.3}$$

The exact value of the 2PM coefficient $q_{\rm 2EG}(\gamma;\nu)$ is given by Eq. (6.1) (where we recall that $h(\gamma;\nu) \equiv [1+2\nu(\gamma-1)]^{1/2}$). The exact ν dependence of the 3PM coefficient $q_{\rm 3EG}(\gamma;\nu)$ is described by

$$q_{\rm 3EG}(\gamma;\nu) = A_{q3}(\gamma)\left(1-\frac{1}{h^2(\gamma;\nu)}\right) + \frac{B_{q3}(\gamma)}{h(\gamma;\nu)}\left(1-\frac{1}{h(\gamma;\nu)}\right), \tag{11.4}$$

where the exact value of $B_{q3}(\gamma)$ is given in Eq. (6.4) and where our new method has allowed us to compute the 6PN-accurate value of the function $A_{q3}(\gamma)$, as given by Eqs. (6.5), (6.11), and (6.13). According to Refs. [29,31], the exact value of $A_{q3}(\gamma)$ is given by Eqs. (6.6) and (6.7).

Less PN information is known about the higher PM coefficients $q_{n\rm EG}(\gamma;\nu)$, though the analog of the exact ν structure displayed for $n=3$ in Eq. (11.4) has been given in Ref. [41]. Here, we shall parametrize their PN expansions as follows:

$$\begin{aligned} q_{\rm 4EG}(\gamma;\nu) &= \nu\left(\frac{175}{3}-\frac{41}{32}\pi^2\right)-\frac{7}{2}\nu^2 + q^1_{\rm 4EG}(\nu)(\gamma^2-1) + q^2_{\rm 4EG}(\nu)(\gamma^2-1)^2,\\ q_{\rm 5EG}(\gamma;\nu) &= q^0_{\rm 5EG}(\nu) + q^1_{\rm 5EG}(\nu)(\gamma^2-1),\\ q_{\rm 6EG}(\gamma;\nu) &= q^0_{\rm 6EG}(\nu). \end{aligned} \tag{11.5}$$

Here, the first term in $q_{\rm 4EG}$ is at the 3PN level; the first term in $q_{\rm 5EG}$ is at the 4PN level; and the first, and only, term in $q_{\rm 6EG}$ is at the 5PN level.

The final form of the ν-dependent PN-expansion parameters $q^p_{n\rm EG}(\nu)$ entering the f-flexed energy-gauge (squared) effective Hamiltonian, Eqs. (11.3) and (11.5), is the following:

$$\begin{aligned} q^1_{\rm 4EG}(\nu) &= \left(\frac{5632}{45}-\frac{33601}{6144}\pi^2\right)\nu+\left(-\frac{405}{4}+\frac{123}{64}\pi^2\right)\nu^2+\frac{13}{2}\nu^3,\\ q^2_{\rm 4EG}(\nu) &= \left(\frac{699761}{7200}-\frac{93031}{12288}\pi^2\right)\nu+\left(-\frac{77443}{480}+\frac{31633}{4096}\pi^2\right)\nu^2+\left(130-\frac{615}{256}\pi^2\right)\nu^3-\frac{293}{32}\nu^4,\\ q^0_{\rm 5EG}(\nu) &= \left(\frac{44357}{360}-\frac{29917}{6144}\pi^2\right)\nu+\left(\frac{205}{64}\pi^2-\frac{2387}{24}\right)\nu^2+\frac{9}{4}\nu^3,\\ q^1_{\rm 5EG}(\nu) &= \left(\frac{15540691}{25200}-\frac{2590847}{61440}\pi^2\right)\nu+\left(-\frac{15581}{80}+\frac{1}{5}\bar{d}^{\nu^2}_5+\frac{347673}{20480}\pi^2\right)\nu^2+\left(\frac{5131}{24}-\frac{1763}{256}\pi^2\right)\nu^3-\frac{93}{16}\nu^4,\\ q^0_{\rm 6EG}(\nu) &= \left(-\frac{69733}{350}+\frac{541363}{10240}\pi^2\right)\nu+\left(\frac{11717}{60}+\frac{1}{5}\bar{d}^{\nu^2}_5+\frac{17857}{5120}\pi^2+a^{\nu^2}_6\right)\nu^2+\left(\frac{326}{3}-\frac{287}{64}\pi^2\right)\nu^3-\frac{11}{8}\nu^4. \end{aligned} \tag{11.6}$$

B. 5PN-accurate f-type local effective EOB Hamiltonian in p_r gauge

In the standard p_r gauge, the final form of the 5PN-accurate building blocks $A(u;\nu)$, $\bar{D}(u;\nu)$, and $Q(u,p_r;\nu)$ of the f-type local effective EOB Hamiltonian $\hat{H}^{\rm loc}_{\rm eff}$ are

$$\begin{aligned} A_{\rm loc} &= 1-2u+2\nu u^3+\nu\left(\frac{94}{3}-\frac{41}{32}\pi^2\right)u^4 + a^{\rm loc}_5 u^5 + a^{\rm loc}_6 u^6,\\ \bar{D}_{\rm loc} &= 1+6\nu u^2+(52\nu-6\nu^2)u^3+\bar{d}^{\rm loc}_4 u^4+\bar{d}^{\rm loc}_5 u^5,\\ Q_{\rm loc} &= p_r^4[2(4-3\nu)\nu u^2+q^{\rm loc}_{43}u^3+q^{\rm loc}_{44}u^4] + p_r^6(q^{\rm loc}_{62}u^2+q^{\rm loc}_{63}u^3)+q^{\rm loc}_{82}p_r^8u^2. \end{aligned} \tag{11.7}$$

The values of the coefficients $a^{\rm loc}_5$, $a^{\rm loc}_6$, $\bar{d}^{\rm loc}_4$, $\bar{d}^{\rm loc}_5$, $q^{\rm loc}_{43}$, $q^{\rm loc}_{44}$, $q^{\rm loc}_{62}$, $q^{\rm loc}_{63}$, and $q^{\rm loc}_{82}$ parametrizing the 4 + 5PN structure of $\hat{H}^{\rm loc}_{\rm eff}$ are summarized in Table VII.

C. Standard (f-type local) EOB Hamiltonian at 5PN

For completeness, let also display the 5PN (f-type local) real Hamiltonian as function of u, p_r, and p^2, where

$$p^2 \equiv p_r^2 + j^2u^2.$$

Inserting the results of the previous subsection in the EOB energy map (11.1), one gets the following explicit (real) EOB Hamiltonian:

TABLE VIII. Coefficients entering the real EOB Hamiltonian. All coefficients $C^{(2l)}_{2k}(\nu)$ start linearly in ν, except $C^{(6)}_{2}(\nu)$ and $C^{(8)}_{0}(\nu)$, which begin instead at $O(\nu^2)$. Furthermore, the highest power of ν in the coefficients $C^{(0)}_{2k}(\nu)$ is ν^6, whereas the remaining coefficients stop at ν^5. The number of nonzero coefficients is then 95.

Coefficient	Powers	Value
$C^{(0)}_{12}(\nu)$	$p_r^0 p^{12} u^0$	$-\frac{21}{1024}\nu-\frac{21}{1024}\nu^2-\frac{7}{256}\nu^3-\frac{35}{1024}\nu^4$ $-\frac{35}{1024}\nu^5-\frac{21}{1024}\nu^6$
$C^{(0)}_{10}(\nu)$	$p_r^0 p^{10} u^1$	$-\frac{7}{256}\nu-\frac{7}{256}\nu^2-\frac{3}{128}\nu^3+\frac{5}{256}\nu^4$ $+\frac{35}{256}\nu^5+\frac{63}{256}\nu^6$
$C^{(0)}_{8}(\nu)$	$p_r^0 p^8 u^2$	$\frac{5}{256}\nu+\frac{5}{256}\nu^2+\frac{3}{64}\nu^3+\frac{35}{256}\nu^4$ $+\frac{35}{256}\nu^5-\frac{315}{256}\nu^6$
$C^{(0)}_{6}(\nu)$	$p_r^0 p^6 u^3$	$-\frac{1}{32}\nu+\frac{1}{32}\nu^2-\frac{5}{32}\nu^4-\frac{45}{32}\nu^5+\frac{105}{32}\nu^6$
$C^{(0)}_{4}(\nu)$	$p_r^0 p^4 u^4$	$\frac{5}{64}\nu+(\frac{41}{512}\pi^2-\frac{385}{192})\nu^2$ $+(-\frac{91}{48}+\frac{41}{512}\pi^2)\nu^3$ $+(-\frac{123}{512}\pi^2+\frac{363}{64})\nu^4+\frac{155}{64}\nu^5-\frac{315}{64}\nu^6$
$C^{(0)}_{2}(\nu)$	$p_r^0 p^2 u^5$	$-\frac{7}{16}\nu+(\frac{1619}{2048}\pi^2-\frac{1141}{120})\nu^2$ $+(-\frac{2275}{2048}\pi^2+\frac{3997}{240})\nu^3$ $+(-\frac{671}{48}+\frac{41}{64}\pi^2)\nu^4-\frac{25}{16}\nu^5+\frac{63}{16}\nu^6$
$C^{(0)}_{0}(\nu)$	$p_r^0 p^0 u^6$	$-\frac{21}{16}\nu+(\frac{254113}{6144}\pi^2-\frac{8478053}{25200})\nu^2$ $+(\frac{1}{2}a_6^{\nu^2}-\frac{1199}{40}+\frac{1947}{1024}\pi^2)\nu^3$ $+(\frac{355}{48}-\frac{41}{128}\pi^2)\nu^4+\frac{5}{16}\nu^5-\frac{21}{16}\nu^6$
$C^{(2)}_{8}(\nu)$	$p_r^2 p^8 u^1$	$-\frac{35}{128}\nu-\frac{35}{128}\nu^2-\frac{45}{128}\nu^3-\frac{25}{64}\nu^4-\frac{35}{128}\nu^5$
$C^{(2)}_{6}(\nu)$	$p_r^2 p^6 u^2$	$-\frac{5}{16}\nu-\frac{5}{4}\nu^2-\frac{9}{8}\nu^3-\frac{1}{2}\nu^4+\frac{5}{4}\nu^5$
$C^{(2)}_{4}(\nu)$	$p_r^2 p^4 u^3$	$+\frac{3}{16}\nu+\frac{111}{16}\nu^2+\frac{99}{16}\nu^3+\frac{69}{8}\nu^4-\frac{33}{16}\nu^5$
$C^{(2)}_{2}(\nu)$	$p_r^2 p^2 u^4$	$-\frac{1}{4}\nu+(-\frac{611}{36}+\frac{25729}{6144}\pi^2)\nu^2$ $+(\frac{1549}{36}+\frac{13921}{6144}\pi^2)\nu^3$ $+(\frac{59}{2}-\frac{123}{64}\pi^2)\nu^4+2\nu^5$
$C^{(2)}_{0}(\nu)$	$p_r^2 p^0 u^5$	$\frac{5}{8}\nu+(\frac{447313}{700}-\frac{36359}{1024}\pi^2)\nu^2$ $+(-\frac{61153}{3072}\pi^2+\frac{31397}{72}+\frac{1}{2}\bar{d}_5^{\nu^2})\nu^3$ $+(-\frac{41}{16}\pi^2+\frac{1031}{12})\nu^4-\frac{11}{8}\nu^5$
$C^{(4)}_{4}(\nu)$	$p_r^4 p^4 u^2$	$-\frac{15}{16}\nu+\frac{9}{16}\nu^2-\frac{3}{4}\nu^3-\frac{9}{16}\nu^4-\frac{9}{8}\nu^5$
$C^{(4)}_{2}(\nu)$	$p_r^4 p^2 u^3$	$-\frac{3}{4}\nu-\frac{33}{4}\nu^2+\frac{47}{4}\nu^3+10\nu^4+2\nu^5$
$C^{(4)}_{0}(\nu)$	$p_r^4 p^0 u^4$	$+\frac{1}{4}\nu+(-\frac{93031}{3072}\pi^2+\frac{405004}{1575})\nu^2$ $+(\frac{31633}{1024}\pi^2-\frac{1697}{6})\nu^3$ $+(-\frac{615}{64}\pi^2+\frac{1115}{4})\nu^4+\frac{1}{2}\nu^5$
$C^{(6)}_{2}(\nu)$	$p_r^6 p^2 u^2$	$\frac{9}{20}\nu^2+\frac{9}{5}\nu^3-\frac{3}{20}\nu^4-\frac{3}{2}\nu^5$
$C^{(6)}_{0}(\nu)$	$p_r^6 p^0 u^3$	$-\frac{1}{2}\nu+\frac{211}{20}\nu^2-\frac{23}{5}\nu^3+\frac{493}{10}\nu^4-4\nu^5$
$C^{(8)}_{0}(\nu)$	$p_r^8 p^0 u^2$	$\frac{3}{7}\nu^2+\frac{9}{7}\nu^3+\frac{12}{7}\nu^4-3\nu^5$

$$H^{\rm loc,5PN}_{\rm eob}=\sum_{0\le k\le 6,0\le l\le 4,k+l\le 6} C^{(2l)}_{2k}(\nu)\,p^{2k}p_r^{2l}u^{6-k-l}. \tag{11.8}$$

The ν-dependent coefficients $C^{(2l)}_{2k}(\nu)=\sum_n C^{(2l)}_{2k,n}\nu^n$ are listed in Table VIII.

Let us recall once more that, modulo the only two undetermined coefficients $a_6^{\nu^2}$ and $\bar{d}_5^{\nu^2}$, the full 5PN-accurate dynamics has been determined here. It is given by adding to the local action defined by $H^{\le 5\rm PN}_{\rm loc}$ the f-flexed $4+5$PN nonlocal one written down in Eqs. (2.10) and (2.11). We find it remarkable that, though the real local Hamiltonian finally involves 95 different numerical coefficients keying the various powers of u, p^2, p_r^2, and ν (as listed in Table VIII), our combination of tools has allowed us to determine all these coefficients, except for two of them. To help visualize the structure of the 5PN Hamiltonian [encoded in the ν-dependent coefficients $C^{(2l)}_{2k}(\nu)$], we present the matrix of the nonzero numerical coefficients $C^{(2l)}_{2k,n}$ entering $C^{(2l)}_{2k}(\nu)=\sum_n C^{(2l)}_{2k,n}\nu^n$ in Fig. 1.

We summarize in Fig. 2 the source of information having allowed us to determine each one of these 95 coefficients. Figure 2 is a schematic version of Fig. 1, in which we do not distinguish p^2 from p_r^2, so that there seems to appear only 36 coefficients: the test-particle limit determines the ν^1 row; the 1SF computations determine the ν^2 row; the first two columns are, respectively, determined by the 1PM and 2PM exact EOB Hamiltonians; and the $\nu^{\ge 3}$ dependence of the next third and fourth columns (respectively, corresponding to 3PM and 4PM) is completely determined by the EOB-PM result concerning the ν-polynomial structure of T_n, Eq. (9.3). The latter result also determines the coefficients in the last two columns (5PM and 6PM) except for the two coefficients $h^{\nu^3}_{25}$ and $h^{\nu^3}_{06}$.

D. Canonical transformation between the p_r gauge and the energy gauge

Let us finally give the values of the parameters g_i, h_i, and n_i entering the generating function $g(q,p)$, Eq. (5.7), of the canonical transformation connecting the p_r-gauge and the energy-gauge f-flexed local Hamiltonians. If we denote the p_r-gauge phase-space variables as (r,p_r) [with Hamiltonian $H(r,p_r)$] and the energy-gauge ones as (r',p'_r) [with Hamiltonian $H'(r',p'_r)$], we have $H(r,p_r)=H'(r',p'_r)$ with the following link between the phase-space variables (besides $p_\phi=j=p'_\phi$):

$$r'=r+\partial_{p'_r}g(r,p'_r);\qquad p_r=p'_r+\partial_r g(r,p'_r). \tag{11.9}$$

In Table IX, we list the final form of the gauge parameters, necessary to pass from the standard EOB gauge to the energy gauge.

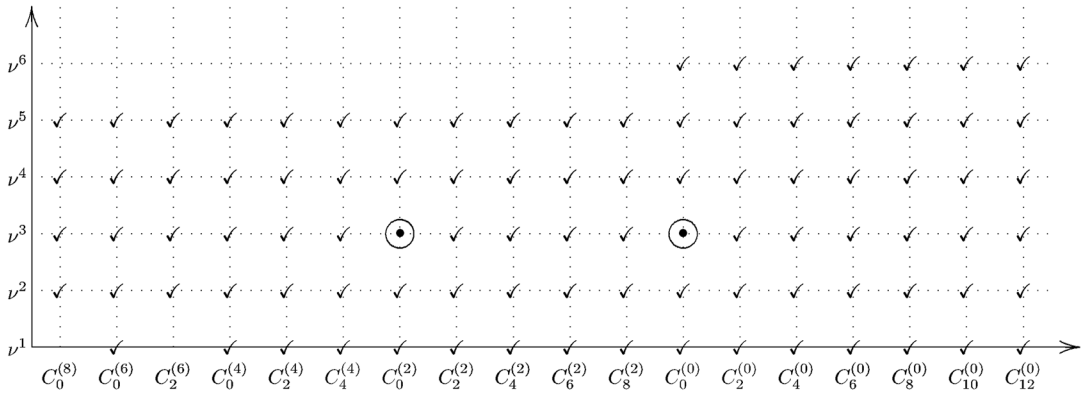

FIG. 1. Matrix of the 95 nonzero numerical coefficients $C^{(2l)}_{2k,n}$ encoding the various powers of ν in the Hamiltonian (11.8). The checks indicate the coefficients determined in the present work. The only two missing coefficients are indicated by circled dots.

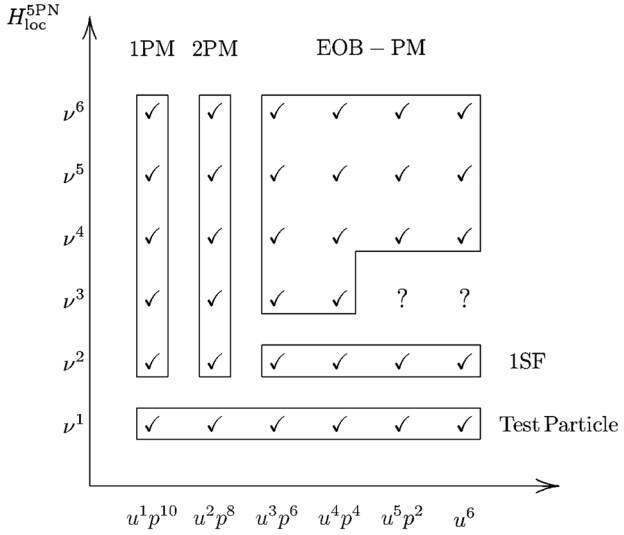

FIG. 2. Schematic representation of the theoretical tools used to obtain the various contributions to the 5PN-accurate local Hamiltonian, adapted from Ref. [26]. These contributions are keyed, on the horizontal axis, by powers of $u = GM/r$ and squared momentum $p^2 \sim p_r^2$ and, on the vertical axis, by powers of $\nu \equiv m_1 m_2/(m_1+m_2)^2$. The checks indicate the coefficients determined in the present work. The question marks denote the only two missing coefficients. Note that, even if certain coefficients in Table VIII only include terms up to $O(\nu^5)$, the identification $p^2 \sim p_r^2$ done in this schematic figure lumps terms together so that $O(\nu^6)$ terms arise in each column.

XII. 5.5PN-LEVEL ACTION AND ITS TRANSCRIPTION INTO THE EOB STANDARD GAUGE HAMILTONIAN

A somewhat surprising result of SF computations was the discovery [46] of half-integer-power PN contributions (starting at the 5.5PN level) to the near-zone metric and to the Hamiltonian. This was quickly understood [45–47] as coming from second-order tail (or tail-of-tail, or simply tail^2) effects. The conservative action term associated with such tail^2 effects was obtained in Ref. [23] [see Sec. IXB there, Eq. (9.19)]. It reads

$$S_{5.5\mathrm{PN}} = -\int dt H_{5.5\mathrm{PN}}, \tag{12.1}$$

where the 5.5 PN Hamiltonian is given by the nonlocal tail^2 expression

$$H_{5.5\mathrm{PN}} = H_{\mathrm{tail}^2} = \frac{B}{2}\left(\frac{G\mathcal{M}}{c^3}\right)^2 \int_{-\infty}^{\infty} \frac{d\tau}{\tau} [\mathcal{G}^{\mathrm{split}}(t, t+\tau) - \mathcal{G}^{\mathrm{split}}(t, t-\tau)], \tag{12.2}$$

with $B = -\frac{107}{105}$. Similarly to the tail^1 effect discussed above, this action involves a time-split bilinear function of the multipole moments that is closely linked to the gravitational wave flux, namely,

$$\mathcal{G}^{\mathrm{split}}(t, t') = \frac{G}{5c^5} I^{(3)}_{ij}(t) I^{(4)}_{ij}(t') + \ldots. \tag{12.3}$$

At the present 5.5PN accuracy, it is enough to use the leading-order version of the time-split function $\mathcal{G}^{\mathrm{split}}(t, t')$, obtained by keeping only the lowest-order quadrupolar contribution (neglecting higher multipole terms) with $I_{ij} \approx \mu r_{\langle ij \rangle}$ evaluated at the Newtonian level. In addition, we can also neglect the difference between $\mathcal{M}$ and M.

An important conceptual point is that, though Eqs. (12.1) and (12.2), seem to define only the nonlocal part of the 5.5PN action, actually they give the *complete* 5.5PN action. Indeed, the usual PN-expanded way of computing the local part of the action (e.g., by integrating the near-zone Hamiltonian density, as in Ref. [10]) cannot generate any half-integral PN contribution. In addition, the nonlocal action, Eqs. (12.1) and (12.2), has (contrary to the 4 + 5 PN one) no ultraviolet divergence at small $\tau = t' - t$. This indicates the completeness of the 5.5PN action written above. Actually, the correctness of this action has been directly checked by satisfactorily comparing its predictions

TABLE IX. Gauge parameters entering the canonical transformation (5.7).

PN order	Parameter	Value
	g_1	$\frac{13}{2}\nu+\frac{1}{2}\nu^2$
3PN	g_2	$\frac{3}{2}\nu-\frac{9}{8}\nu^2$
(see Ref. [28]))	g_3	$\frac{5}{2}\nu-\frac{15}{8}\nu^2$
	h_1	$(\frac{2419}{144}-\frac{25729}{12288}\pi^2)\nu+(-\frac{353}{16}+\frac{123}{128}\pi^2)\nu^2+\frac{5}{8}\nu^3$
	h_2	$-\frac{9}{32}\nu-\frac{27}{32}\nu^2+\frac{15}{16}\nu^3$
	h_3	$-\frac{99}{160}\nu-\frac{297}{160}\nu^2+\frac{33}{16}\nu^3$
4PN	h_4	$\frac{263}{80}\nu-\frac{101}{16}\nu^2-\frac{13}{8}\nu^3$
(see Ref. [30])	h_5	$\frac{281}{80}\nu-\frac{199}{16}\nu^2-\frac{11}{8}\nu^3$
	h_6	$-\frac{3}{4}\nu-\frac{9}{4}\nu^2+\frac{5}{2}\nu^3$
	n_1	$(\frac{3082519}{25200}-\frac{224053}{30720}\pi^2)\nu+(\frac{15499}{240}+\frac{1}{10}\bar{d}_5^{\nu^2}-\frac{96211}{20480}\pi^2)\nu^2+(-\frac{7}{24}+\frac{41}{256}\pi^2)\nu^3+\frac{7}{8}\nu^4$
	n_2	$(\frac{798353}{25200}-\frac{93031}{24576}\pi^2)\nu+(\frac{31633}{8192}\pi^2-\frac{11227}{240})\nu^2+(\frac{357}{16}-\frac{615}{512}\pi^2)\nu^3-\frac{21}{8}\nu^4$
	n_3	$(\frac{758123}{10800}-\frac{651217}{73728}\pi^2)\nu+(-\frac{71857}{720}+\frac{221431}{24576}\pi^2)\nu^2+(\frac{3331}{48}-\frac{1435}{512}\pi^2)\nu^3-\frac{7}{3}\nu^4$
	n_4	$\frac{603}{1120}\nu-\frac{321}{160}\nu^2+\frac{91}{16}\nu^3+\frac{41}{16}\nu^4$
	n_5	$\frac{97}{35}\nu-\frac{51}{20}\nu^2+\frac{31}{2}\nu^3+\frac{11}{2}\nu^4$
5PN	n_6	$\frac{3657}{1120}\nu-\frac{171}{160}\nu^2+\frac{273}{16}\nu^3+\frac{39}{16}\nu^4$
	n_7	$\frac{15}{128}\nu+\frac{45}{128}\nu^2+\frac{15}{32}\nu^3-\frac{105}{128}\nu^4$
	n_8	$\frac{55}{128}\nu+\frac{165}{128}\nu^2+\frac{55}{32}\nu^3-\frac{385}{128}\nu^4$
	n_9	$\frac{73}{128}\nu+\frac{219}{128}\nu^2+\frac{73}{32}\nu^3-\frac{511}{128}\nu^4$
	n_{10}	$\frac{279}{896}\nu+\frac{837}{896}\nu^2+\frac{279}{224}\nu^3-\frac{279}{128}\nu^4$

with SF computations (that automatically include all local and nonlocal effects); see Ref. [23].

As before, we can use the Delaunay averaging technique to relate the 5.5PN Hamiltonian (12.2) to its EOB counterpart. The time average of $H_{5.5\mathrm{PN}}$ was already considered in Ref. [23] and shown there to be expressible as

$$\langle H_{5.5\mathrm{PN}}\rangle=-\pi\frac{2B}{5}\frac{G}{c^5}\left(\frac{G\mathcal{M}}{c^3}\right)^2 n_{\mathrm{phys}}^7 S_7^{\mathrm{quad}},\tag{12.4}$$

where $n_{\mathrm{phys}}=2\pi/P_{\mathrm{phys}}$ is the (physical) orbital frequency and where

$$S_7^{\mathrm{quad}}=\sum_{p=1}^{\infty}p^7|I_{ij}(p)|^2,\tag{12.5}$$

with $I_{ij}(p)$ denoting the Fourier coefficients of the quadrupole moment $I_{ij}(t)$.

Extending the results of Ref. [23], we have computed [starting directly from the integral expression (12.2)] the orbital average of $H_{5.5\mathrm{PN}}$ to the 16th order in eccentricity, with the result

$$\langle H_{5.5\mathrm{PN}}\rangle=-\frac{\mu^2}{M}c^2\frac{6848}{525}\frac{\pi}{a_r^{13/2}}\varphi(e),\tag{12.6}$$

where a_r is dimensionless and

$$\begin{aligned}\varphi(e)=1&+\frac{2335}{192}e^2+\frac{42955}{768}e^4+\frac{6204647}{36864}e^6\\&+\frac{352891481}{884736}e^8+\frac{286907786543}{353894400}e^{10}\\&+\frac{6287456255443}{4246732800}e^{12}+\frac{5545903772613817}{2219625676800}e^{14}\\&+\frac{422825073954708079}{106542032486400}e^{16}+O(e^{18}).\end{aligned}\tag{12.7}$$

The last two terms will not be used below. [The first two terms in $\varphi(e)$ were previously computed in Refs. [62,63].] Note that the rescaled function $\tilde{\varphi}(e)=\varphi(e)(1-e^2)^{13/2}$, once reexpanded in e, becomes

$$\begin{aligned}\tilde{\varphi}(e)=1&+\frac{1087}{192}e^2-\frac{4027}{768}e^4-\frac{172009}{36864}e^6\\&+\frac{1758725}{884736}e^8+\frac{211269943}{353894400}e^{10}+\frac{976098889}{4246732800}e^{12}\\&+\frac{796425035243}{6658877030400}e^{14}+\frac{2583007392829}{35514010828800}e^{16}\\&+O(e^{18}),\end{aligned}\tag{12.8}$$

with coefficients which remain of order 1.

Let us now transcribe the 5.5PN-level tail^2 time-averaged nonlocal original $H_{5.5\mathrm{PN}}$ into its corresponding EOB version, parametrized (in standard p_r gauge) by an effective EOB Hamiltonian expanded as a series in powers of p_r^2:

$$\delta\hat{H}^2{}_{\text{eff 5.5PN}} = A_{65}u^{13/2} + \bar{D}_{55}u^{11/2}p_r^2 + q_{4,4.5}p_r^4u^{9/2} + q_{6,3.5}p_r^6u^{7/2} + q_{8,2.5}p_r^8u^{5/2} + O(p_r^{10}). \tag{12.9}$$

We can compute the orbital average of $\delta\hat{H}^2_{\text{eff}}$ (henceforth omitting the additional 5.5 PN subscript), by writing

$$\langle\delta\hat{H}^2_{\text{eff}}\rangle = \frac{n}{2\pi}\int\frac{\delta\hat{H}^2_{\text{eff}}}{\dot{\phi}}d\phi, \tag{12.10}$$

where, at this leading order, we can use the Newtonian relations for $r = r(\phi)$ and $p_r = \dot{r}$,

$$r = \frac{a_r(1-e^2)}{1+e\cos(\phi)}, \qquad p_r = \frac{e}{\sqrt{a_r(1-e^2)}}\sin\phi, \tag{12.11}$$

with the (rescaled) orbital frequency of the radial motion given by $GMn_{\text{phys}} = n = a_r^{-3/2}$. The result reads

$$\langle\delta\hat{H}^2_{\text{eff}}\rangle = \frac{1}{a_r^{13/2}}\left[A_{6.5} + \left(\frac{143}{16}A_{6.5} + \frac{1}{2}\bar{D}_{5.5}\right)e^2 + \left(\frac{36465}{1024}A_{6.5} + \frac{3}{8}q_{4,4.5} + \frac{195}{64}\bar{D}_{5.5}\right)e^4 + \left(\frac{20995}{2048}\bar{D}_{5.5} + \frac{1616615}{16384}A_{6.5} + \frac{5}{16}q_{6,3.5} + \frac{255}{128}q_{4,4.5}\right)e^6 + \left(\frac{929553625}{4194304}A_{6.5} + \frac{3380195}{131072}\bar{D}_{5.5} + \frac{101745}{16384}q_{4,4.5} + \frac{1615}{1024}q_{6,3.5} + \frac{35}{128}q_{8,2.5}\right)e^8\right] + O(e^{10}). \tag{12.12}$$

Comparison (at the Newtonian level) among these two gauge-invariant quantities,

$$\langle\delta\hat{H}^2_{\text{eff}}\rangle = \frac{2}{\mu c^2}\langle H_{\text{5.5PN}}\rangle, \tag{12.13}$$

allows us to determine all tail^2 coefficients:

$$\begin{aligned} A_{6.5} &= \nu\frac{13696}{525}\pi, \\ \bar{D}_{5.5} &= \nu\frac{264932}{1575}\pi, \\ q_{4,4.5} &= \nu\frac{88703}{1890}\pi, \\ q_{6,3.5} &= -\nu\frac{2723471}{756000}\pi, \\ q_{8,2.5} &= \nu\frac{5994461}{12700800}\pi. \end{aligned} \tag{12.14}$$

The coefficients $A_{6.5}$, $\bar{D}_{5.5}$, $q_{4,4.5}$, and $q_{6,3.5}$ agree with previous results (both from Ref. [23] and from self-force computations). The last coefficient, $q_{8,2.5}$, is instead new and constitutes a prediction for future self-force computations of the averaged redshift invariant at order $O(e^8)$. Note that the entire 5.5 PN action is linear in ν (and proportional to ν). Therefore, self-force computations at the 5.5 PN level allow one to compute exact, ν-dependent 5.5 PN observables.

In the present section, we have considered 5.5PN-level gauge-invariant quantities linked to ellipticlike motions. We shall leave to future work the 5.5PN contribution to the scattering angle implied by the action (12.1).

XIII. ACTION VARIABLES AND DELAUNAY HAMILTONIAN FOR THE (f-ROUTE) LOCAL EFFECTIVE 5PN DYNAMICS

We have derived above the 5PN-accurate local Hamiltonian (in its f version), notably by making use of the special ν-dependent structure of the scattering angle [41]. The so-obtained local 5PN dynamics has been so far expressed within the EOB formalism, using two special gauges (p_r gauge and energy gauge). As these gauges are uniquely fixed by their definitions, all our results above can be considered gauge invariant. Our discussion above of the gauge-invariant scattering angle has, in particular, confirmed the fully gauge-fixed nature of the p_r gauge. The same holds for the energy gauge (as shown in Refs. [28,41]). It is, however, interesting to complete our study of the 5PN local dynamics by discussing another gauge-invariant description of the dynamics, applicable to bound-state motions (rather than scattering motions), whose usefulness for relativistic gravity was first emphasized in Ref. [57], namely, the Delaunay Hamiltonian,

$$H^{\text{loc,f}} = H(I_r, I_\phi), \tag{13.1}$$

i.e., the Hamiltonian expressed in terms of the action variables

$$\begin{aligned} I_r &= \frac{1}{2\pi}\oint p_r dr, \\ I_\phi &= \frac{1}{2\pi}\oint p_\phi d\phi = p_\phi = j. \end{aligned} \tag{13.2}$$

Note that we work here with dimensionless scaled variables $I_r = I_r^{\text{phys}}/(GM\mu)$, $I_\phi = j = J/GM\mu$.

Equivalently [modulo solving Eq. (13.1) with respect to I_r], one can consider the gauge-invariant functional link between the radial action I_r and the energy and the angular momentum, say,

$$I_r = I_r(\gamma, j). \tag{13.3}$$

As indicated here, we are going to see that great simplifications are reached if we use as energy variable the (scaled) effective EOB energy

$$\gamma \equiv \hat{\mathcal{E}}_{\rm eff} = \hat{H}_{\rm eff}, \tag{13.4}$$

which is related to the total local c.m. energy by the usual EOB energy map

$$H^{\rm loc,f} = M\sqrt{1 + 2\nu(\gamma - 1)}. \tag{13.5}$$

We use the same notation γ as in our previous discussion of scattering states, but one must note that we are now going to consider bound states for which $\gamma < 1$. This implies that the above-defined squared asymptotic EOB momentum p_∞^2 is now a negative quantity:

$$\gamma^2 - 1 \equiv p_\infty^2 \equiv -|p|^2. \tag{13.6}$$

Before studying the precise structure of the gauge-invariant function $I_r = I_r(\gamma, j)$, let us recall how this function acts as a potential for deriving both the periastron advance (Φ) and the radial period (P):

$$\frac{\Phi}{2\pi} = K = -\frac{\partial I_r(\gamma, j)}{\partial j},$$
$$\frac{P}{2\pi GM} = +h(\gamma, \nu)\frac{\partial I_r(\gamma, j)}{\partial \gamma}. \tag{13.7}$$

The factor $h(\gamma;\nu) \equiv [1 + 2\nu(\gamma - 1)]^{1/2}$ in the last equation comes from the "redshift" factor $dH/dH_{\rm eff}$ connecting the real-time period to the effective-time period [32].

We have computed the function $I_r(\gamma, j)$ associated with the f-route local 5PN-accurate Hamiltonian by using the technique explained in Ref. [57] (and used there at the 2PN level). We start from the local effective EOB (p_r-gauge) Hamiltonian at 5PN

$$\hat{H}_{\rm eff}^2 = A[1 + u^2 j^2 + p_r^2 A\bar{D} + q_4 p_r^4 + q_6 p_r^6 + q_8 p_r^8], \tag{13.8}$$

where

$$A(u,\nu) = 1 - 2u + 2\nu u^3 + a_4(\nu)u^4 + a_5(\nu)u^5 + a_6(\nu)u^6,$$
$$\bar{D}(u,\nu) = 1 + 6\nu u^2 + \bar{d}_3(\nu)u^3 + \bar{d}_4(\nu)u^4 + \bar{d}_5(\nu)u^5,$$
$$q_4(u,\nu) = q_{42}(\nu)u^2 + q_{43}(\nu)u^3 + q_{44}(\nu)u^4,$$
$$q_6(u,\nu) = q_{62}(\nu)u^2 + q_{63}(\nu)u^3,$$
$$q_8(u,\nu) = q_{82}(\nu)u^2. \tag{13.9}$$

We then use the energy conservation law

$$\gamma^2 = \hat{H}_{\rm eff}^2(p_r^2, j^2, u) \tag{13.10}$$

to iteratively solve for the radial momentum p_r as a function of γ, j, and $u = 1/r$. This is done in a PN-expanded way, after restoring a placeholder $\eta = 1/c$ for PN orders, with the following PN orders:

$$p_r \mapsto \eta p_r, \qquad j \mapsto \frac{j}{\eta}, \qquad u \mapsto \eta^2 u, \qquad p_\infty^2 \mapsto \eta^2 p_\infty^2. \tag{13.11}$$

Under this scaling, the quantity

$$e^2 \equiv 1 + p_\infty^2 j^2 \tag{13.12}$$

is fixed as $\eta \to 0$ and describes the eccentricity of the limiting Newtonian-like dynamics. (We are considering the case in which $0 < -p_\infty^2 j^2 < 1$, so that $0 < e^2 < 1$.) The PN-expanded value of the radial momentum has the structure

$$p_r(u; p_\infty^2, j) = \sum_{k=0}^{5} p_r^{(2k)}(u; p_\infty^2, j)\eta^{2k} + O(\eta^{12}), \tag{13.13}$$

with leading-order contribution

$$p_r^{(0)}(u; p_\infty^2, j) = (p_\infty^2 + 2u - u^2 j^2)^{1/2} \tag{13.14}$$

and 1PN correction given by

$$p_r^{(2)}(u; p_\infty^2, j) = \frac{u^3 j^2}{p_r^{(0)}} + 2u p_r^{(0)}. \tag{13.15}$$

The roots of the second-order polynomial $p_\infty^2 + 2u - u^2 j^2$,

$$u_\pm = \frac{1 \pm e}{j^2}, \tag{13.16}$$

are the Newtonian-like values associated with the periastron and apoastron passages.

Following Ref. [57], one can compute the PN-expansion of the radial integral

$$I_r = \frac{1}{2\pi}\oint dr\left(\sum_{k=0}^{5} p_r^{(2k)}(u; p_\infty^2, j)\eta^{2k}\right), \tag{13.17}$$

by taking the Hadamard partie finie of the resulting integrals. This leads to an explicit PN-expanded expression for the radial integral,

$$I_r(p_\infty^2, j; \nu) = \sum_{k=0}^{5} \eta^{2k} I_r^{(2k)}(p_\infty^2, j; \nu) + O(\eta^{12}), \tag{13.18}$$

starting with the Newtonian-like value ($k = 0$):

$$I_r^{(0)}(p_\infty^2, j; \nu) = -j + \frac{1}{\sqrt{-p_\infty^2}} = -j + \frac{1}{\sqrt{1-\gamma^2}}. \tag{13.19}$$

We recall that we are here considering ellipticlike motions with $\gamma^2 < 1$.

The function $I_r(\gamma, j; \nu)$ exhibits a remarkably simple structure, which is the reflection of the simple ν dependence of the PM-expanded scattering angle [41]. (The latter structure separately applies to the presently considered f-route local dynamics.) We can write the 5PN-accurate local $I_r(\gamma, j; \nu)$ in the form

$$\begin{aligned} I_r^{\mathrm{5PN,loc}}&(\gamma, j, \nu) \\ &= -j + I_0(\gamma) + \frac{I_1^S(\gamma)}{hj} + \frac{I_3^S(\gamma) + \nu I_3^{\nu^1}(\gamma)}{h^3 j^3} \\ &\quad + \frac{I_5^S(\gamma) + \nu I_5^{\nu^1}(\gamma) + \nu^2 I_5^{\nu^2}(\gamma)}{h^5 j^5} \\ &\quad + \frac{I_7^S(\gamma) + \nu I_7^{\nu^1}(\gamma) + \nu^2 I_7^{\nu^2}(\gamma) + \nu^3 I_7^{\nu^3}(\gamma)}{h^7 j^7} \\ &\quad + \frac{I_9^S(\gamma) + \nu I_9^{\nu^1}(\gamma) + \nu^2 I_9^{\nu^2}(\gamma) + \nu^3 I_9^{\nu^3}(\gamma) + \nu^4 I_9^{\nu^4}(\gamma)}{h^9 j^9}. \end{aligned} \tag{13.20}$$

Here, each line does not correspond to a well-defined PN order, though the successive lines start at some minimum PN order which increases linearly with the power of j present in the denominators. On the first line, the term $-j$ can be considered to be of Newtonian order, while the second term is a function of γ (given below) which, when it is expanded in powers of $\gamma - 1 = O(\eta^2)$, starts at the Newtonian order but then contains higher PN corrections of arbitrarily high PN orders. Similarly, the next line (proportional to $1/j$) starts at 1PN order but includes higher PN orders when expanded in powers of $\gamma - 1 = O(\eta^2)$. Each extra power of $1/j^2$ represents an extra PN order. The last term, $\propto 1/j^9$, is $1/j^{10}$ smaller than the first, Newtonian term $-j$, which corresponds (in view of the scaling $j \mapsto \frac{j}{\eta}$) to a relative factor η^{10}, corresponding indeed to a 5PN accuracy.

There are several remarkable features in the structure (13.20). First, the only j-independent term in this PN expansion (on the first line) starts at the Newtonian order and is exactly given by the simple formula

$$I_0(\gamma) = \frac{2\gamma^2 - 1}{\sqrt{1-\gamma^2}}. \tag{13.21}$$

This result was already obtained in Ref. [64] and connected there (see also Ref. [65]) with the analytic continuation (from $\gamma > 1$ to $\gamma < 1$) of the 1PM scattering coefficient χ_1. Second, and most remarkably, after pairing all the powers of j in the denominators with the same power of $h(\gamma; \nu) \equiv [1 + 2\nu(\gamma - 1)]^{1/2}$, the complicated ν dependence of the original PN-expanded I_r is reduced to a simple polynomial ν dependence of the numerators. Indeed, these exhibit the simple rule[7] that the numerator I_{2n+1} corresponding to the denominator $(hj)^{2n+1}$ is a polynomial in ν of order n. [The latter rule follows from the rule about $h^{n-1}\chi_n$ via the analytic continuation in γ allowing one to identify $\Phi(\gamma, j)$ with a suitably defined analytic continuation of $\chi(\gamma, j) + \chi(\gamma, -j)$ [64].]

The last remarkably simple feature of the expansion (13.20) is that the $\nu \to 0$ limits of each numerator, i.e., the coefficients $I_{2n+1}^S(\gamma)$ are polynomial functions of γ, which are given by the following expressions:

$$\begin{aligned} I_1^S(\gamma) &= -\frac{3}{4} + \frac{15}{4}\gamma^2, \\ I_3^S(\gamma) &= \frac{35}{64} - \frac{315}{32}\gamma^2 + \frac{1155}{64}\gamma^4, \\ I_5^S(\gamma) &= -\frac{231}{256} + \frac{9009}{256}\gamma^2 - \frac{45045}{256}\gamma^4 + \frac{51051}{256}\gamma^6 \\ I_7^S(\gamma) &= \frac{32175}{16384} - \frac{546975}{4096}\gamma^2 + \frac{10392525}{8192}\gamma^4 \\ &\quad - \frac{14549535}{4096}\gamma^6 + \frac{47805615}{16384}\gamma^8 \\ I_9^S(\gamma) &= -\frac{323323}{65536} + \frac{33948915}{65536}\gamma^2 - \frac{260275015}{32768}\gamma^4 \\ &\quad + \frac{1301375075}{32768}\gamma^6 - \frac{5019589575}{65536}\gamma^8 \\ &\quad + \frac{3234846615}{65536}\gamma^{10}. \end{aligned} \tag{13.22}$$

Note the $\gamma \to 1$ values of the latter test-mass polynomials

$$\begin{aligned} &I_1^S(1) = 3, \qquad I_3^S(1) = \frac{35}{4}, \qquad I_5^S(1) = \frac{231}{4}, \\ &I_7^S(1) = \frac{32175}{64}, \qquad I_9^S(1) = \frac{323323}{64}. \end{aligned} \tag{13.23}$$

The simple "Schwarzschild" polynomials $I_{2n+1}(\gamma)$ can be exactly computed by considering the $\nu \to 0$ limit of the radial action (Schwarzschild limit). Indeed, the test-particle limit of the radial action,

[7] This result was not uncovered in previous discussions [57,64] of the radial action.

TABLE X. Coefficients entering the j expansion, Eq. (13.20), of $I_r(\gamma, j; \nu)$.

Coefficient	Value
$I_3^{\nu^1}(\gamma)$	$-\frac{5}{2}+(-\frac{557}{12}+\frac{41}{64}\pi^2)(\gamma-1)+(-\frac{10873}{72}+\frac{35569}{6144}\pi^2)(\gamma-1)^2+(-\frac{7199407}{25200}+\frac{15829}{768}\pi^2)(\gamma-1)^3$
$I_5^{\nu^1}(\gamma)$	$(-\frac{125}{2}+\frac{123}{128}\pi^2)+(-\frac{224113}{240}+\frac{51439}{2048}\pi^2)(\gamma-1)+(-\frac{89527351}{16800}+\frac{979913}{4096}\pi^2)(\gamma-1)^2$
$I_5^{\nu^2}(\gamma)$	$\frac{21}{8}+(\frac{2013}{16}-\frac{369}{128}\pi^2)(\gamma-1)+(\frac{9739}{96}-\frac{3}{4}\bar{d}_5^{\nu^2}-\frac{18275}{4096}\pi^2-\frac{3}{4}a_6^{\nu^2})(\gamma-1)^2$
$I_7^{\nu^1}(\gamma)$	$(-\frac{248057}{288}+\frac{425105}{24576}\pi^2)+(-\frac{99111883}{6720}+\frac{2310485}{8192}\pi^2)(\gamma-1)$
$I_7^{\nu^2}(\gamma)$	$(-\frac{1025}{256}\pi^2+\frac{18925}{96})+(-\frac{5}{4}\bar{d}_5^{\nu^2}-\frac{15}{4}a_6^{\nu^2}-\frac{1290275}{12288}\pi^2+\frac{1089349}{288})(\gamma-1)$
$I_7^{\nu^3}(\gamma)$	$-\frac{45}{16}+(\frac{3075}{512}\pi^2-\frac{7595}{32})(\gamma-1)$
$I_9^{\nu^1}(\gamma)$	$-\frac{6817563}{640}+\frac{121807}{1024}\pi^2$
$I_9^{\nu^2}(\gamma)$	$-\frac{7}{16}\bar{d}_5^{\nu^2}+\frac{572999}{128}-\frac{1755159}{16384}\pi^2-\frac{35}{16}a_6^{\nu^2}$
$I_9^{\nu^3}(\gamma)$	$-\frac{42665}{96}+\frac{10045}{1024}\pi^2$
$I_9^{\nu^4}(\gamma)$	$\frac{385}{128}$

$$I_r^{\rm Sch}(\hat{\mathcal{E}}_{\rm eff}, j)=\frac{1}{2\pi}\oint dr p_r^{\rm Sch}(\hat{\mathcal{E}}_{\rm eff}, j), \tag{13.24}$$

is easily written down by solving $\hat{\mathcal{E}}_{\rm eff}^2=H_S^2=(1-2u)\times[1+(1-2u)p_r^2+j^2u^2]$ and reads (remembering $\hat{\mathcal{E}}_{\rm eff}=\gamma$)

$$I_r^{\rm Sch}(\gamma, j)=\frac{1}{2\pi}\oint du\frac{\sqrt{\gamma^2-(1-2u)(1+j^2u^2)}}{u^2(1-2u)}, \tag{13.25}$$

where the integral is taken around the two roots of the cubic polynomial $P_3(u)=\gamma^2-(1-2u)(1+j^2u^2)$ that are close to the Newtonian roots $u_\pm$ used in our PN-expanded computation above. We see that $I_r^{\rm Sch}(\gamma, j)$ is a complete elliptic integral (i.e., a period of an elliptic curve), so that it can be written down explicitly, e.g., in terms of a combination of usual Legendre complete elliptic integrals (however, the third type of Legendre elliptic integral appears).

TABLE XI. Coefficients entering the PN expansion, Eq. (13.27), of the Delaunay effective Hamiltonian $\hat{H}_{\rm eff}(I_2, I_3; \nu)-1$.

Coefficient	Value
$\bar{E}_{\rm eff}^0$	$-\frac{1}{2I_3^2}$
$\bar{E}_{\rm eff}^2$	$\frac{15}{8I_3^4}-\frac{3}{I_2I_3^3}$
$\bar{E}_{\rm eff}^4$	$(\frac{5}{2}\nu-\frac{35}{4})\frac{1}{I_2^3I_3^3}-\frac{27}{2}\frac{1}{I_2^2I_3^4}+(\frac{105}{4}-\frac{3}{2}\nu)\frac{1}{I_2I_3^5}-\frac{145}{16}\frac{1}{I_3^6}$
$\bar{E}_{\rm eff}^6$	$[-\frac{231}{4}-\frac{21}{8}\nu^2+(\frac{125}{2}-\frac{123}{128}\pi^2)\nu]\frac{1}{I_2^5I_3^3}+(-\frac{315}{4}+\frac{45}{2}\nu)\frac{1}{I_2^4I_3^4}+[\frac{303}{8}+\frac{15}{4}\nu^2+(\frac{41}{128}\pi^2-\frac{661}{12})\nu]\frac{1}{I_2^3I_3^5}$
	$+(-\frac{45}{2}\nu+225)\frac{1}{I_2^2I_3^6}+(-\frac{9}{8}\nu^2-\frac{825}{4}+\frac{75}{4}\nu)\frac{1}{I_2I_3^7}+\frac{6363}{128}\frac{1}{I_3^8}$
$\bar{E}_{\rm eff}^8$	$[-\frac{32175}{64}+\frac{45}{16}\nu^3+(-\frac{18925}{96}+\frac{1025}{256}\pi^2)\nu^2+(\frac{248057}{288}-\frac{425105}{24576}\pi^2)\nu]\frac{1}{I_2^7I_3^3}$
	$+[-\frac{20307}{32}-33\nu^2+(-\frac{1107}{128}\pi^2+\frac{5025}{8})\nu]\frac{1}{I_2^6I_3^4}+[\frac{7749}{32}-\frac{105}{16}\nu^3+(\frac{7643}{32}-\frac{123}{32}\pi^2)\nu^2+(-\frac{453613}{480}+\frac{80959}{4096}\pi^2)\nu]\frac{1}{I_2^5I_3^5}$
	$+[\frac{21435}{16}+75\nu^2+(-\frac{3755}{4}+\frac{615}{128}\pi^2)\nu]\frac{1}{I_2^4I_3^6}+[\frac{46275}{64}+\frac{75}{16}\nu^3+(-\frac{2989}{32}+\frac{123}{256}\pi^2)\nu^2+(-\frac{124129}{24576}\pi^2+\frac{120763}{288})\nu]\frac{1}{I_2^3I_3^7}$
	$+(-\frac{63}{2}\nu^2+\frac{3465}{8}\nu-\frac{85365}{32})\frac{1}{I_2^2I_3^8}+(\frac{585}{32}\nu^2-\frac{5745}{32}\nu+\frac{50703}{32}-\frac{15}{16}\nu^3)\frac{1}{I_2I_3^9}-\frac{75303}{256}\frac{1}{I_3^{10}}$
$\bar{E}_{\rm eff}^{10}$	$[-\frac{323323}{64}-\frac{385}{128}\nu^4+(-\frac{10045}{1024}\pi^2+\frac{42665}{96})\nu^3+(\frac{7}{16}\bar{d}_5^{\nu^2}-\frac{572999}{128}+\frac{35}{16}a_6^{\nu^2}+\frac{1755159}{16384}\pi^2)\nu^2+(-\frac{121807}{1024}\pi^2$
	$+\frac{6817563}{640})\nu]\frac{1}{I_2^9I_3^3}$
	$+[-\frac{386595}{64}+45\nu^3+(-\frac{18495}{8}+\frac{5535}{128}\pi^2)\nu^2+(\frac{314417}{32}-\frac{1481955}{8192}\pi^2)\nu]\frac{1}{I_2^8I_3^4}$
	$+[\frac{344637}{128}+\frac{315}{32}\nu^4+(-\frac{79655}{96}+\frac{17425}{1024}\pi^2)\nu^3+(-\frac{2344095}{16384}\pi^2-\frac{5}{8}\bar{d}_5^{\nu^2}+\frac{392325}{64}-\frac{15}{8}a_6^{\nu^2})\nu^2$
	$+(\frac{1792917}{8192}\pi^2-\frac{76818229}{6720})\nu]\frac{1}{I_2^7I_3^5}$
	$+[\frac{769545}{64}-165\nu^3+(\frac{116065}{24}-\frac{8815}{128}\pi^2)\nu^2+(-\frac{1907369}{96}+\frac{1345585}{4096}\pi^2)\nu]\frac{1}{I_2^6I_3^6}$
	$+[\frac{125235}{16}-\frac{735}{64}\nu^4+(\frac{16145}{32}-\frac{7995}{1024}\pi^2)\nu^3+(\frac{1241145}{16384}\pi^2+\frac{3}{16}\bar{d}_5^{\nu^2}+\frac{3}{16}a_6^{\nu^2}-\frac{205425}{64})\nu^2$
	$+(-\frac{860567}{4096}\pi^2+\frac{47882269}{16800})\nu]\frac{1}{I_2^5I_3^7}$
	$+[-\frac{715575}{64}+\frac{315}{2}\nu^3+(-\frac{5677}{2}+\frac{861}{64}\pi^2)\nu^2+(\frac{1418221}{96}-\frac{937783}{8192}\pi^2)\nu]\frac{1}{I_2^4I_3^8}$
	$+[-\frac{2464245}{128}+\frac{175}{32}\nu^4+(\frac{615}{1024}\pi^2-\frac{4255}{32})\nu^3+(\frac{149563}{192}-\frac{153649}{16384}\pi^2)\nu^2+(\frac{1388971}{24576}\pi^2-\frac{21149141}{100800})\nu]\frac{1}{I_2^3I_3^9}$
	$+(-\frac{181845}{32}\nu-\frac{81}{2}\nu^3+\frac{1769931}{64}+\frac{2835}{4}\nu^2)\frac{1}{I_2^2I_3^{10}}$
	$+(-\frac{105}{128}\nu^4-\frac{26595}{128}\nu^2+\frac{200445}{128}\nu+\frac{75}{4}\nu^3-\frac{1550595}{128})\frac{1}{I_2I_3^{11}}+\frac{1874587}{1024}\frac{1}{I_3^{12}}$

The latter exact, elliptic-integral representation is rather complex, but it is relatively easy to compute both its PN expansion [i.e., its expansion in powers of η; see Eq. (13.11)] and its expansion in inverse powers of j. This is discussed in detail in Appendix C, which also includes a discussion of the simpler complete elliptic integral giving the test-mass periastron advance.

Finally, the *primitive information* (beyond the test-mass limit) contained in the 5PN radial action $I_r(\gamma, j; \nu)$ is fully described by the small number of γ-dependent coefficients of the various powers of ν in the numerators of Eq. (13.20). These coefficients [contrary to their corresponding $\nu \to 0$ limits $I^S_{2n+1}(\gamma)$] are not known as exact functions of γ but only as limited expansions in powers of $\gamma - 1 = O(\eta^2)$. For instance, $I^{\nu^1}_3(\gamma)$ is known to fractional 3PN accuracy, i.e., up to the third order in $\gamma - 1$. The PN knowledge of the higher terms $I^{\nu^p}_{2n+1}(\gamma)$ linearly decreases as n increases, until the last terms $I^{\nu^p}_9(\gamma)$, with $p = 1, 2, 3, 4$, which are only known at the lowest (Newtonian) accuracy, i.e., only for $\gamma = 1$. The known information carried by all these $I^{\nu^p}_{2n+1}(\gamma)$ is gathered in Table X.

By inverting the functional relation $I_r = I_r(\hat{\mathcal{E}}_{\rm eff}, j)$, one can finally obtain the explicit value of the corresponding (effective) Delaunay Hamiltonian, $\hat{H}_{\rm eff}(I_r, j)$. This is conveniently done by defining the variables

$$I_3 \equiv I_r + j, \qquad j = I_2, \tag{13.26}$$

in terms of which one can get the PN expansion of $H_{\rm eff}(I_2, I_3)/\mu = \gamma c^2$ in the form

$$\frac{H^{\rm 5PN,loc,f}_{\rm eff}(I_2, I_3; \nu)}{\mu} = \eta^{-2} + \sum_{k=0}^{5} \eta^{2k} \bar{E}^{2k}_{\rm eff}(I_2, I_3; \nu) + O(\eta^{12}). \tag{13.27}$$

The values of the coefficients $\bar{E}^{2k}_{\rm eff}(I_2, I_3; \nu)$ are displayed in Table XI. Note, however, that the structure of this (effective) gauge-invariant Delaunay Hamiltonian is not particularly illuminating. The simple ν structure exhibited by the radial action function (13.20) is lost in the Delaunay Hamiltonian (13.1). Indeed, the hidden simplicity of the 5PN local dynamics is more transparent when encoding it either in the EOB potentials displayed above or in the radial action (13.20). Let us emphasize again that, given a specific gauge choice (say, p_r gauge or energy gauge), the corresponding EOB potentials are completely gauge fixed and can therefore be considered as being as gauge-invariantly defined as the more traditional gauge-invariant functions $I_r = I_r(E, j; \nu)$ or $H(I_r, j; \nu)$.

XIV. CONCLUSIONS

We have shown how to successfully combine several different theoretical tools to develop a new methodology [26] for extending the analytical computation of the conservative two-body dynamics beyond the current post-Newtonian knowledge (4PN). Our approach has allowed us to derive an almost complete expression for the 5PN-level action, given by the sum of a 4PN + 5PN nonlocal action, Eq. (2.2), and of a local one $\int p dq - H^{\leq 5{\rm PN}}_{\rm loc,f} dt$. We succeeded in determining the full functional structure of $H^{\leq 5{\rm PN}}_{\rm loc,f}$ (which contains 95 nonzero numerical coefficients), except for two (ν^3-level) unknown coefficients (ν^2 level in the EOB potentials A and $\bar{D}$). The two main derivations underlying our new results are i) the computation of the Delaunay average of the nonlocal action around eccentric orbits to the tenth order in eccentricity included and ii) the self-force computation of the redshift along eccentric orbits (around a Schwarzschild black hole) to sixth order in eccentricity.

We completed our results beyond the 5PN level in two different directions. On the one hand, we added the 5.5PN contribution to the action (which is purely nonlocal) and transcribed it into its EOB (p_r-gauge) form up to the eight order in p_r. On the other hand, we used a recent extension of our self-force computation to the eighth order in eccentricity to improve the determination of the third post-Minkowskian [$O(G^3)$] part of the dynamics to the 6PN level. This allowed us to compute the $O(G^3)$ contribution to the scattering angle up to the 6PN level included. Our 6PN-accurate $O(G^3)$ scattering angle agrees with the recent third post-Minkowskian [$O(G^3)$] result of Bern *et al.* [29,31].

We computed both the nonlocal and the local contributions to the 5PN-accurate, $O(G^4)$ scattering angle. As our 5PN (and 5.5PN) results are complete at the $O(G^4)$ order, the latter result offers checks for future fourth post-Minkowskian calculations. We could conveniently separate the study of the nonlocal versus local contributions to the scattering angle by flexing (at the 5PN level) the scale $2r^f_{12}/c$ entering the definition of the nonlocal action.

We point out a remarkable hidden simplicity of the local 5PN dynamics. This hidden simplicity only manifests itself when using a gauge-invariant description of the dynamics. There are several (complementary) ways of viewing the (local) 5PN dynamics in a gauge-invariant fashion. One can use the EOB description, in one of its gauge-fixed versions (p_r gauge or energy gauge). When comparing the EOB encoding of 5PN-level information (and ν structure) to the (simplified) $h^{n-1}\chi_n$ scattering encoding, one can see not only that they are one-to-one but also that the EOB encoding is as minimal as the $h^{n-1}\chi_n$ one. (See, Sec. X.) An alternative gauge-invariant approach is to focus on gauge-invariant observables. Two of them have a particularly interesting structure: the scattering function $\chi(\hat{\mathcal{E}}_{\rm eff}, j)$ and the radial action $I_r(\hat{\mathcal{E}}_{\rm eff}, j)$. We have emphasized that the (f-flexed local) radial action (when expressed in terms of the EOB effective energy $\hat{\mathcal{E}}_{\rm eff}$ and of the product hj, where $h = E^{\rm tot}/M$) has a remarkably simple structure, see

Eq. (13.20), which parallels the simple structure of $\chi(\hat{\mathcal{E}}_{\rm eff}, j)$. This simplicity is, essentially, already automatically incorporated in the structure of the EOB Hamiltonian [see Table VII and Eq. (11.6)]. Let us also note that the local 5PN dynamics is completely logarithm free and that all its numerical coefficients are rational at PM orders $G^{\leq 3}$ and include π^2 at PM orders $G^{\geq 4}$. We have relegated most of the technical details of our computation to various Appendixes. More precisely,

(1) Appendix A displays our new self-force result on the time-averaged redshift $\langle z_1 \rangle$ at the sixth order in eccentricity, $O(e^6)$, and its conversion into the corresponding EOB potential $q_6(u)$.
(2) Appendix B shows how to obtain a closed-form expression for the 2PM Hamiltonian in the standard (p_r) EOB gauge by computing the (inverse) Abel transform of its corresponding (closed-form) energy-gauge expression.
(3) Appendix C discusses the radial action and the Delaunay Hamiltonian for the test-mass limit.

Most of the coefficients entering long expressions, like the redshift invariant at the sixth order in eccentricity, have been given in the form of tables. (They are available in electronic format upon request.)

Standard PN approaches to binary dynamics (in their various flavors: Hamiltonian, Lagrangian, or effective-field-theory) have reached their limits, in view of the complexity of the required computations and of the subtle infrared issues linked to time nonlocality. Our work, which tackles nonlocality from the beginning, offers an alternative approach to standard computations, combining information from different contexts and using it in a synergetic way. It is therefore expected that it may lead to further progress in analytically controlling the dynamics of binary systems. It would be interesting to explore combining our new approach with the recently pioneered new approach to binary dynamics based on focussing on (classical or quantum) scattering motions [28,29,31,38].

The techniques we have been defining here can be extended to higher PN orders. We will separately present our complete, recent 6PN-level results [39].

Two coefficients are still missing in the 5PN Hamiltonian of a two-body system. Several routes for determining the two missing coefficients are conceivable, notably, second-order self-force computations or partial standard PN computations of the 5PN dynamics targeted toward a selected mass dependence. (The recent progress in computer-aided evaluation of the PN-expanded interaction potential of binary systems [16,18,40] gives hope that the two missing coefficients might be soon derived.) Also, high-accuracy numerical simulations might enter the game.

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APPENDIX A: THE TIME-AVERAGED REDSHIFT $\langle z_1 \rangle$ AT $O(e^6)$ AND ITS EOB TRANSCRIPTION $q_6(u)$

A redshift invariant for slightly eccentric orbits was introduced in the spacetime of a nonrotating black hole by Barack and Sago [48] as the orbital averaged value of the linear-in-mass-ratio correction δU to the coordinate time component of the particle's 4-velocity. The latter has been computed through the 9.5PN level in Ref. [50] up to the fourth order in the eccentricity, improving the previous

TABLE XII. Coefficients entering the PN expansion of $\delta z_1^{e^6}$, Eq. (A2).

Coefficient	Value
$c_3^{\rm c}$	$\frac{1}{4}$
$c_4^{\rm c}$	$-\frac{53}{12}-\frac{41}{128}\pi^2$
$c_5^{\rm c}$	$-\frac{178288}{5}\ln(2)+\frac{1994301}{160}\ln(3)-\frac{38471}{360}+\frac{6455}{4096}\pi^2+16\gamma+\frac{1953125}{288}\ln(5)$
$c_5^{\ln}$	8
$c_6^{\rm c}$	$-\frac{1694}{5}\gamma+\frac{66668054}{135}\ln(2)-\frac{29268135}{448}\ln(3)+\frac{782899}{4096}\pi^2-\frac{2027890625}{12096}\ln(5)-\frac{17344111}{5040}$
$c_6^{\ln}$	$-\frac{847}{5}$
$c_{6.5}^{\rm c}$	$-\frac{18404963}{151200}\pi$
$c_7^{\rm c}$	$-\frac{10083929027}{2835}\ln(2)+\frac{11019270343}{340200}+\frac{10727453}{2835}\gamma+\frac{4609218071875}{2612736}\ln(5)$ $+\frac{4663687}{524288}\pi^4-\frac{130309059379}{28311552}\pi^2-\frac{14238373347}{17920}\ln(3)+\frac{96889010407}{373248}\ln(7)$
$c_7^{\ln}$	$\frac{10727453}{5670}$
$c_{7.5}^{\rm c}$	$\frac{629926159}{470400}\pi$
$c_8^{\rm c}$	$-\frac{138663992506361}{20528640}\ln(7)+\frac{1097743020107}{4026531840}\pi^4-\frac{1044921875}{3024}\ln(5)^2-\frac{159152}{15}\zeta(3)$ $-\frac{4891192867}{70875}\gamma-\frac{160452171}{400}\ln(3)^2$ $-\frac{133972817261}{2764800}\pi^2+\frac{9033082952}{1575}\ln(2)^2+\frac{1216376}{225}\gamma^2+\frac{105459653332171}{16372125}\ln(2)$

(Table continued)

TABLE XII. *(Continued)*

Coefficient	Value
	$-\frac{160452171}{200}\gamma\ln(3)$
	$+\frac{84510345271221}{4928000}\ln(3)-\frac{116526439405625}{18289152}\ln(5)-\frac{1044921875}{1512}\gamma\ln(5)$
	$+\frac{1912624751720539}{3929310000}-\frac{160452171}{200}\ln(2)\ln(3)$
	$+\frac{4574838928}{1575}\gamma\ln(2)-\frac{1044921875}{1512}\ln(2)\ln(5)$
$c_8^{\ln}$	$-\frac{4891192867}{141750}+\frac{1216376}{225}\gamma-\frac{160452171}{400}\ln(3)-\frac{1044921875}{3024}\ln(5)+\frac{2287419464}{1575}\ln(2)$
$c_8^{\ln^2}$	$+\frac{304094}{225}$
$c_{8.5}^{c}$	$\frac{17411624626943}{22632825600}\pi$
c_9^{c}	$-\frac{2325452157955686875}{36614882304}\ln(5)-\frac{189182288}{3675}\gamma^2+\frac{9958568909678}{38201625}\gamma-\frac{35883263448213399}{448448000}\ln(3)-\frac{904011369824}{19845}\gamma\ln(2)$
	$+\frac{17956170280520566}{343814625}\ln(2)-\frac{13247132039065189}{69526957500}+\frac{919213293396729}{85899345920}\pi^4-\frac{45448745981842837}{1109812838400}\pi^2+\frac{3089591667}{1400}\ln(3)^2$
	$+\frac{262937890625}{31752}\ln(5)^2-\frac{1255868873488}{14175}\ln(2)^2+\frac{2825264}{35}\zeta(3)+\frac{1975634469}{4900}\ln(2)\ln(3)$
	$+\frac{3089591667}{700}\gamma\ln(3)+\frac{262937890625}{15876}\ln(2)\ln(5)+\frac{16842023587039315}{213497856}\ln(7)+\frac{262937890625}{15876}\gamma\ln(5)$
$c_9^{\ln}$	$-\frac{452005684912}{19845}\ln(2)+\frac{4964431663039}{38201625}+\frac{3089591667}{1400}\ln(3)-\frac{189182288}{3675}\gamma+\frac{262937890625}{31752}\ln(5)$
$c_9^{\ln^2}$	$-\frac{47295572}{3675}$
$c_{9.5}^{c}$	$-\frac{151427301903}{98000}\pi\ln(3)-\frac{768417611}{113400}\pi^3-\frac{609707863599642191}{6590678814720}\pi+\frac{82220684377}{3969000}\pi\gamma$
	$+\frac{25820141287513}{3969000}\pi\ln(2)-\frac{111806640625}{63504}\pi\ln(5)$
$c_{9.5}^{\ln}$	$+\frac{82220684377}{7938000}\pi$

analytical knowledge at 6.5PN for δU^{e^2} [49] and at 4PN for δU^{e^4} [23,49]. Higher-order terms in the eccentricity expansion have been obtained in Refs. [50,66] up to the order $O(e^{20})$, but at the 4PN level of approximation only, by combining the 4PN results of Ref. [23] with the first law for eccentric orbits [44]. The $O(e^4)$ 9.5PN-accurate results of Ref. [50] have also been transcribed there in terms of the corresponding EOB potentials $\bar{d}(u)$ and $q(u)\equiv q_4(u)$.

We have extended here the calculation of Ref. [50] by including contributions of sixth order in eccentricity through the same, 9.5PN, level. Our analytical computation of the conservative SF effects along an eccentric orbit in a Schwarzschild background follows the same approach as in Ref. [50], to which we refer for a full account of intermediate steps. We work with the redshift function $z_1=U^{-1}$ and its first-order SF perturbation $\delta z_1=-\delta U/U_0^2$ (with U_0 denoting the corresponding background value). The small-eccentricity expansion of the time-averaged value $\langle\delta z_1\rangle$, expressed in terms of the (Schwarzschild-background) inverse parameter $u_p\equiv 1/p$ and eccentricity e, reads

TABLE XIII. Coefficients entering the expression for $q_6(u_p)$, Eq. (A3), in terms of the redshift function and its derivatives.

Coefficient	Value
$B(u_p)$	$-\frac{1}{1024}(3276u_p^4-7371u_p^3+6212u_p^2-2312u_p+320)\frac{u_p(1-2u_p)^3}{(1-3u_p)^{9/2}}$
$C_0^{e^0}(u_p)$	$\frac{3}{1024}(207u_p^2-216u_p+56)\frac{u_p(1-2u_p)^3}{(1-3u_p)^{7/2}}$
$C_1^{e^0}(u_p)$	$-\frac{1}{1280}(-2048-568038697344u_p^{12}+169966688256u_p^{13}-223561417224u_p^8+73260864684u_p^7$ $+3259361067u_p^5-17888024322u_p^6+42346416u_p^3+501408678672u_p^9-802643130720u_p^{10}$ $+867902879808u_p^{11}-2782272u_p^2-438377232u_p^4+111360u_p)\frac{(1-2u_p)^3}{u_p^2(1-6u_p)^8(1-3u_p)^{5/2}}$
$C_2^{e^0}(u_p)$	$-\frac{1}{1280}(291931776u_p^{11}-1340770752u_p^{10}+2510150688u_p^9-2598166704u_p^8+1680719416u_p^7$ $-721475988u_p^6+211137750u_p^5-42247977u_p^4+5663472u_p^3-483984u_p^2$ $+23808u_p-512)\frac{(1-2u_p)^3}{u_p(1-6u_p)^7(1-3u_p)^{3/2}}$
$C_3^{e^0}(u_p)$	$-\frac{1}{160}(1918080u_p^8-4203792u_p^7+3933936u_p^6-2053224u_p^5+656676u_p^4-132441u_p^3+16376u_p^2$ $-1112u_p+32)\frac{(1-2u_p)^4}{(1-3u_p)^{1/2}(1-6u_p)^6}$
$C_4^{e^0}(u_p)$	$-\frac{1}{960}(45720u_p^5-50316u_p^4+20554u_p^3-4349u_p^2+552u_p-24)\frac{u_p(1-2u_p)^5(1-3u_p)^{1/2}}{(1-6u_p)^5}$
$C_5^{e^0}(u_p)$	$-\frac{3}{80}\frac{u_p^3(1-2u_p)^6(1-3u_p)^{3/2}}{(1-6u_p)^4}$

(Table continued)

TABLE XIII. *(Continued)*

Coefficient	Value
$C_6^{e^0}(u_p)$	$-\frac{1}{720}\frac{u_p^3(1-2u_p)^6(1-3u_p)^{5/2}}{(1-6u_p)^3}$
$C_0^{e^2}(u_p)$	$-\frac{1}{320}(779683968u_p^9-1886037696u_p^8+1975861728u_p^7-1178577360u_p^6+442967544u_p^5$ $-109412372u_p^4+17816962u_p^3-1833219u_p^2+106288u_p-2672)\frac{(1-2u_p)^3}{u_p(1-6u_p)^7(1-3u_p)^{3/2}}$
$C_1^{e^2}(u_p)$	$\frac{1}{120}(16132608u_p^8-33912864u_p^7+30156192u_p^6-14824092u_p^5+4434916u_p^4-837157u_p^3$ $+98524u_p^2-6600u_p+192)\frac{(1-2u_p)^4}{u_p^2(1-6u_p)^6(1-3u_p)^{1/2}}$
$C_2^{e^2}(u_p)$	$\frac{1}{120}(26640u_p^5-10776u_p^4-6970u_p^3+4001u_p^2-552u_p+24)\frac{(1-2u_p)^5(1-3u_p)^{1/2}}{u_p(1-6u_p)^5}$
$C_3^{e^2}(u_p)$	$-\frac{1}{30}(4-45u_p+72u_p^2)\frac{(1-2u_p)^6(1-3u_p)^{3/2}}{(1-6u_p)^4}$
$C_4^{e^2}(u_p)$	$\frac{1}{60}\frac{u_p(1-2u_p)^6(1-3u_p)^{5/2}}{(1-6u_p)^3}$
$C_0^{e^4}(u_p)$	$-\frac{1}{15}(49032u_p^5-43812u_p^4+11586u_p^3-609u_p^2-88u_p+8)\frac{(1-2u_p)^5(1-3u_p)^{1/2}}{u_p^3(1-6u_p)^5}$
$C_1^{e^4}(u_p)$	$\frac{4}{15}(8-81u_p+144u_p^2)\frac{(1-2u_p)^6(1-3u_p)^{3/2}}{u_p^2(1-6u_p)^4}$
$C_2^{e^4}(u_p)$	$-\frac{4}{15}\frac{(1-2u_p)^6(1-3u_p)^{5/2}}{u_p(1-6u_p)^3}$
$C_0^{e^6}(u_p)$	$\frac{16}{5}\frac{(1-2u_p)^6(1-3u_p)^{5/2}}{u_p^3(1-6u_p)^3}$

TABLE XIV. Coefficients entering the PN expansion of $q_6(u)$, Eq. (A4).

Coefficient	Value
b_2^{c}	$-\frac{827}{3}+\frac{1399437}{50}\ln(3)-\frac{2358912}{25}\ln(2)+\frac{390625}{18}\ln(5)$
b_3^{c}	$\frac{2613083}{1050}+\frac{6875745536}{4725}\ln(2)-\frac{23132628}{175}\ln(3)-\frac{101687500}{189}\ln(5)$
$b_{3.5}^{\mathrm{c}}$	$-\frac{2723471}{756000}\pi$
b_4^{c}	$\frac{153776136875}{23328}\ln(5)+\frac{447248}{1575}\gamma-\frac{9678652821}{5600}\ln(3)+\frac{96889010407}{116640}\ln(7)$ $-\frac{41589250561}{7938000}-\frac{9733841}{327680}\pi^2-\frac{211076833264}{14175}\ln(2)$
$b_4^{\ln}$	$+\frac{223624}{1575}$
$b_{4.5}^{\mathrm{c}}$	$+\frac{1783458013}{56448000}\pi$
b_5^{c}	$\frac{3651910996}{86625}\gamma-\frac{7733712492302375}{201180672}\ln(5)+\frac{912077147376081}{15680000}\ln(3)-\frac{211655031897463}{9331200}\ln(7)$ $+\frac{50431773773997 16}{81860625}\ln(2)+\frac{38342542739}{7864320}\pi^2+\frac{15438788608}{875}\ln(2)^2$ $-\frac{1061386821}{875}\ln(3)^2-\frac{830563821453539}{1746360000}$ $-\frac{208984375}{189}\ln(5)^2-\frac{417968750}{189}\gamma\ln(5)-\frac{2122773642}{875}\ln(2)\ln(3)-\frac{2122773642}{875}\gamma\ln(3)$ $+\frac{70193205248}{7875}\gamma\ln(2)-\frac{417968750}{189}\ln(2)\ln(5)$
$b_5^{\ln}$	$-\frac{208984375}{189}\ln(5)+\frac{35096602624}{7875}\ln(2)+\frac{1825955498}{86625}-\frac{1061386821}{875}\ln(3)$
$b_{5.5}^{\mathrm{c}}$	$-\frac{375333092211461}{905313024000}\pi$
b_6^{c}	$-\frac{54126285229417}{73573500}\gamma-\frac{315130937024}{2025}\gamma\ln(2)+\frac{7843492521}{2450}\ln(2)\ln(3)+\frac{39285904041}{2450}\gamma\ln(3)+\frac{448936953125}{7938}\ln(2)\ln(5)$ $+\frac{448936953125}{7938}\gamma\ln(5)-\frac{11892972284088646293}{31391360000}\ln(3)-\frac{1686162964063105097}{12770257500}\ln(2)-\frac{2314158285520063375}{36614882304}\ln(5)$ $-\frac{192}{7}\zeta(3)+\frac{1878836255027762051}{6065280000}\ln(7)+\frac{4253856}{1225}\gamma^2-\frac{262462223346649}{10737418240}\pi^4-\frac{375306539275861}{23121100800}\pi^2$ $+\frac{39285904041}{4900}\ln(3)^2+\frac{448936953125}{15876}\ln(5)^2-\frac{21523313234464}{70875}\ln(2)^2$ $+\frac{384973167765003181159}{58736373696000}$
$b_6^{\ln}$	$-\frac{54126285229417}{147147000}+\frac{4253856}{1225}\gamma-\frac{157565468512}{2025}\ln(2)+\frac{39285904041}{4900}\ln(3)+\frac{448936953125}{15876}\ln(5)$
$b_6^{\ln^2}$	$+\frac{1063464}{1225}$
$b_{6.5}^{\mathrm{c}}$	$-\frac{102893846003}{19845000}\pi\gamma+\frac{431653923653437}{19845000}\pi\ln(2)$ $\frac{3047518189388380479 6413}{144994933923840000}\pi-\frac{2758233739833}{490000}\pi\ln(3)+\frac{961624729}{567000}\pi^3-\frac{22361328125}{3969}\pi\ln(5)$
$b_{6.5}^{\ln}$	$-\frac{102893846003}{39690000}\pi$

$$\langle \delta z_1 \rangle = \delta z_1^{e^0}(u_p) + e^2 \delta z_1^{e^2}(u_p) + e^4 \delta z_1^{e^4}(u_p) + e^6 \delta z_1^{e^6}(u_p) + O(e^8). \tag{A1}$$

New with this work is the computation of the 9.5PN-accurate $O(e^6)$ contribution, namely,

$$\begin{aligned}\delta z_1^{e^6} = {} & c_3^{\rm c} u_p^3 + c_4^{\rm c} u_p^4 + (c_5^{\rm c} + c_5^{\ln} \ln(u_p)) u_p^5 \\ & + (c_6^{\rm c} + c_6^{\ln} \ln(u_p)) u_p^6 + c_{6.5}^{\rm c} u_p^{13/2} \\ & + (c_7^{\rm c} + c_7^{\ln} \ln(u_p)) u_p^7 + c_{7.5}^{\rm c} u_p^{15/2} \\ & + (c_8^{\rm c} + c_8^{\ln} \ln(u_p) + c_8^{\ln^2} \ln^2(u_p)) u_p^8 + c_{8.5}^{\rm c} u_p^{17/2} \\ & + (c_9^{\rm c} + c_9^{\ln} \ln(u_p) + c_9^{\ln^2} \ln^2(u_p)) u_p^9 \\ & + (c_{9.5}^{\rm c} + c_{9.5}^{\ln} \ln(u_p) u_p^{19/2} + O(u_p^{10}),\end{aligned} \tag{A2}$$

with coefficients listed in Table XII.

The improved knowledge of the redshift function can then be converted into the EOB potential $q_6(u) p_r^6 \in \hat{Q}(u, p_r)$ by using the relation (obtained in extending to the $O(e^6)$ level the $O(e^4)$-level results of Ref. [44])

$$q_6(u_p) = B(u_p) + \sum_{k=0}^{3} \left(\sum_{n=0}^{6-2k} C_n^{e^{2k}}(u_p) \frac{d^n}{du_p^n} \delta z_1^{e^{2k}}(u_p) \right), \tag{A3}$$

where we have used the notation $\frac{d^0}{du_p^0} f = f$. The coefficients $B(u_p)$ and $C_n^{e^{2k}}(u_p)$ are listed in Table XIII.

The PN expansion of $q_6(u)$ then reads

$$\begin{aligned}q_6(u) = {} & b_2^{\rm c} u^2 + b_3^{\rm c} u^3 + b_{3.5}^{\rm c} u^{7/2} + (b_4^{\rm c} + b_4^{\ln} \ln(u)) u^4 \\ & + b_{4.5}^{\rm c} u^{9/2} + (b_5^{\rm c} + b_5^{\ln} \ln(u)) u^5 + b_{5.5}^{\rm c} u^{11/2} \\ & + (b_6^{\rm c} + b_6^{\ln} \ln(u) + b_6^{\ln^2} \ln^2(u)) u^6 \\ & + b_{6.5}^{\rm c} u^{13/2} + O(u^7),\end{aligned} \tag{A4}$$

with coefficients listed in Table XIV.

APPENDIX B: TRANSFORMING THE ENERGY-GAUGE 2PM Q TERM, $q_{2\rm EG}(H_S)u^2$, INTO ITS (CLOSED-FORM) p_r-GAUGE VERSION VIA AN ABEL TRANSFORM

In the energy gauge, the 2PM EOB Q potential reads $\hat{Q}_{\rm EG}^{\rm 2PM} = q_{2\rm EG}(\gamma) u^2$ where $\gamma = H_S$. We want to transform it in a p_r-dependent one, say, $\hat{Q}_{p_r}^{\rm 2PM} = q_2^{(p_r)}(p_r) u^2$, that leads to the same scattering angle. Using Eq. (4.22) of Ref. [28], this means that the two functions must yield the same integral $\int_{-\infty}^{+\infty} d\sigma Q$, where $d\sigma = dR/P^R$. Writing this condition at the 2PM level [neglecting any $O(G^3)$ correction] is easily seen to lead to the condition

$$q_{2\rm EG}(\gamma) = \frac{2}{\pi} \int_0^{\sqrt{\gamma^2-1}} dp_r \frac{q_2^{(p_r)}(p_r)}{\sqrt{\gamma^2 - 1 - p_r^2}}. \tag{B1}$$

Reexpressing this condition (and the two functions) in terms of the variables $c \equiv \gamma^2 - 1$ and $x \equiv p_r^2$ yields

$$q_{2\rm EG}(c) = \frac{1}{\pi} \int_0^c dx \frac{q_2^{(p_r)}(x)/\sqrt{x}}{\sqrt{c - x}}. \tag{B2}$$

The latter condition expresses the fact that the function $q_{2\rm EG}(c)$ is the (usual) Abel transform of the function $q_2^{(p_r)}(x)/\sqrt{x}$. But the Abel transform (with inverse square root kernel) is just (in the sense of Marcel Riesz's integral operators) a derivative of order $-\frac{1}{2}$. Therefore, the inverse transform (a derivative of order $+\frac{1}{2}$) can simply be written as the composition of a derivative and an Abel transform. Hence, we have the following formula for the inverse of Eq. (B2):

$$q_2^{(p_r)}(x) = \sqrt{x} \frac{d}{dx} \int_0^x \frac{q_{2\rm EG}(c)}{\sqrt{x - c}} dc \equiv \sqrt{x} \frac{d}{dx} I(x). \tag{B3}$$

The function $q_{2\rm EG}(c)$ to be inserted in this formula is [after expressing γ in terms of $c \equiv \gamma^2 - 1$ in $q_{2\rm EG}(\gamma)$, Eq. (11.5)]

$$q_{2\rm EG}(c) = \frac{3}{2}(4 + 5c) \left(1 - \frac{1}{h(c)} \right), \tag{B4}$$

with $h(c) = \sqrt{1 - 2\nu + 2\nu(1 + c)^{1/2}}$.

One can first easily obtain the all-order PN expansion of the function $q_2^{(p_r)}(x)$ (where we recall that $x = p_r^2$) by expanding $q_{2\rm EG}(c)$, Eq. (B4), in powers of c and then inserting this expansion in Eq. (B3). The result reads

$$\begin{aligned}q_2^{(p_r)}(x) = {} & 6\nu x + (8\nu - 6\nu^2) x^2 + \left(-\frac{9}{5}\nu - \frac{27}{5}\nu^2 + 6\nu^3 \right) x^3 + \left(\frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4 \right) x^4 \\ & + \left(-\frac{11}{21}\nu - \frac{11}{7}\nu^2 - \frac{20}{7}\nu^3 - \frac{5}{3}\nu^4 + 6\nu^5 \right) x^5 + \left(\frac{4}{11}\nu + \frac{12}{11}\nu^2 + \frac{170}{77}\nu^3 + \frac{30}{11}\nu^4 - 6\nu^6 \right) x^6 \\ & + \left(-\frac{3}{11}\nu - \frac{9}{11}\nu^2 - \frac{250}{143}\nu^3 - \frac{35}{13}\nu^4 - \frac{315}{143}\nu^5 + \frac{21}{13}\nu^6 + 6\nu^7 \right) x^7 + O(x^8).\end{aligned} \tag{B5}$$

It is also possible to obtain a closed-form expression for the function $q_2^{(p_r)}(x)$ by computing the integral $I(x)$ entering the inverse Abel transform, Eq. (B3).

To compute the integral $I(x)$, Eq. (B3), we change the variable $c = \gamma^2 - 1$ back into γ. This yields

$$I(x) = \frac{3}{2}\int_1^{\sqrt{1+x}} \frac{(5\gamma^2-1)2\gamma d\gamma}{\sqrt{1+x-\gamma^2}}\left(1 - \frac{1}{\sqrt{1+2\nu(\gamma-1)}}\right). \tag{B6}$$

We introduce then the notation

$$\gamma_r \equiv \sqrt{1+x}, \qquad \gamma_\nu \equiv \frac{1}{2\nu} - 1 \geq 1, \tag{B7}$$

so that

$$\begin{aligned} I(x) &= 3\int_1^{\gamma_r} d\gamma \frac{(5\gamma^2-1)\gamma}{\sqrt{\gamma_r^2-\gamma^2}}\left(1 - \frac{1}{\sqrt{2\nu}\sqrt{\gamma_\nu+\gamma}}\right) \\ &= 2(1+5\gamma_r^2)\sqrt{\gamma_r^2-1} \\ &\quad - \frac{3}{\sqrt{2\nu}}\int_1^{\gamma_r} d\gamma \frac{(5\gamma^2-1)\gamma}{\sqrt{(\gamma_r^2-\gamma^2)(\gamma_\nu+\gamma)}} \\ &\equiv 2(1+5\gamma_r^2)\sqrt{\gamma_r^2-1} - \frac{3}{\sqrt{2\nu}}J, \end{aligned} \tag{B8}$$

where we introduced

$$\begin{aligned} J &\equiv \int_1^{\gamma_r} d\gamma \frac{(5\gamma^2-1)\gamma}{\sqrt{(\gamma_r^2-\gamma^2)(\gamma_\nu+\gamma)}} \\ &\equiv \int_1^{\gamma_r} d\gamma \frac{Q_3(\gamma)}{\sqrt{P_3(\gamma)}}. \end{aligned} \tag{B9}$$

Here, P_3 and Q_3 denote the cubic polynomials in γ entering the integrand of the integral J.

At this stage, it is already clear that the original integral $I(x)$ is the sum of an elementary term and of an elliptic integral given by J. To get an explicit form of the elliptic integral J, we need to perform the Legendre reduction of J. This means writing the identity

$$[2(d_0+d_1\gamma)\sqrt{P_3}]' = \frac{2d_1P_3 + (d_0+d_1\gamma)P_3'}{\sqrt{P_3}} \tag{B10}$$

and determining the coefficients d_0 and d_1 so as to reduce the integral $\int d\gamma Q_3(\gamma)/\sqrt{P_3(\gamma)}$ to an integral whose numerator is a polynomial of degree 1. Indeed, the choice

$$d_0 = \frac{4}{3}\gamma_\nu, \qquad d_1 = -1, \tag{B11}$$

implies

$$\begin{aligned} &2d_1P_3 + (d_0+d_1\gamma)P_3' \\ &\quad = Q_3 + \left(-3\gamma_r^2 - \frac{8}{3}\gamma_\nu^2 + 1\right)\gamma - \frac{2}{3}\gamma_\nu\gamma_r^2 \\ &\quad = Q_3 + \left(-3\gamma_r^2 - \frac{8}{3}\gamma_\nu^2 + 1\right)(\gamma+\gamma_\nu) + \frac{7}{3}\gamma_\nu\gamma_r^2 \\ &\qquad + \frac{8}{3}\gamma_\nu^3 - \gamma_\nu, \end{aligned} \tag{B12}$$

that is,

$$2d_1P_3 + (d_0+d_1\gamma)P_3' \equiv Q_3 + C_1(\gamma+\gamma_\nu) + C_2, \tag{B13}$$

where

$$\begin{aligned} C_1 &= -3\gamma_r^2 - \frac{8}{3}\gamma_\nu^2 + 1, \\ C_2 &= \frac{7}{3}\gamma_\nu\gamma_r^2 + \frac{8}{3}\gamma_\nu^3 - \gamma_\nu. \end{aligned} \tag{B14}$$

Thereby, the identity (B10) becomes

$$[2(d_0+d_1\gamma)\sqrt{P_3}]' = \frac{Q_3 + C_1(\gamma+\gamma_\nu) + C_2}{\sqrt{P_3}}, \tag{B15}$$

so integrating both sides gives

$$\begin{aligned} -2\left(\frac{4}{3}\gamma_\nu - 1\right)\sqrt{P_3(1)} &= J + C_1\int_1^{\gamma_r} d\gamma \frac{\gamma+\gamma_\nu}{\sqrt{P_3}} \\ &\quad + C_2\int_1^{\gamma_r}\frac{d\gamma}{\sqrt{P_3}}, \end{aligned} \tag{B16}$$

where

$$P_3(1) = (\gamma_r^2-1)(1+\gamma_\nu). \tag{B17}$$

This yields the following expression for J:

$$\begin{aligned} J &= -2\left(\frac{4}{3}\gamma_\nu - 1\right)\sqrt{(\gamma_r^2-1)(1+\gamma_\nu)} \\ &\quad - C_1\int_1^{\gamma_r} d\gamma \frac{\gamma+\gamma_\nu}{\sqrt{P_3(\gamma)}} \\ &\quad - C_2\int_1^{\gamma_r}\frac{d\gamma}{\sqrt{P_3(\gamma)}}. \end{aligned} \tag{B18}$$

The remaining integrals are then explicitly expressible in terms of complete Legendre elliptic integrals, namely,

$$\mathbb{I}_1 = \int_1^{\gamma_r} \frac{d\gamma}{\sqrt{P_3(\gamma)}} = \frac{2}{\sqrt{a-c}} \text{EllipticF}\left(\arcsin\sqrt{\frac{a-1}{a-b}}, \sqrt{\frac{a-b}{a-c}}\right),$$
$$\mathbb{I}_2 = \int_1^{\gamma_r} d\gamma \frac{\gamma+\gamma_\nu}{\sqrt{P_3(\gamma)}} = -2\sqrt{a-c}\,\text{EllipticE}\left(\arcsin\sqrt{\frac{a-1}{a-b}}, \sqrt{\frac{a-b}{a-c}}\right), \quad \text{(B19)}$$

where $a = \gamma_r$, $b = -\gamma_r$, $c = -\gamma_\nu$, and

$$\sqrt{a-c} = \gamma_r + \gamma_\nu, \qquad \frac{a-1}{a-b} = \frac{\gamma_r - 1}{2\gamma_r}, \qquad \frac{a-b}{a-c} = \frac{2\gamma_r}{\gamma_r+\gamma_\nu}. \quad \text{(B20)}$$

Here, we got $\mathbb{I}_1$ from Ref. [67], Eq. (6), p. 254, Sec. 3.131, and $\mathbb{I}_2$ from Ref. [67], Eq. (5), p. 255, Sec. 3.132, using in the latter case $(x-c)$ in the numerator of the integrand and simplifying the final result. The minus sign in $\mathbb{I}_2$ corresponds to a general prefactor a/b, which is -1 in the present case.

Inserting the latter elliptic-integral representation of J in the above expression of $I(x)$ and then inserting $I(x)$ in Eq. (B3) finally gives a closed-form expression for the 2PM-level p_r-gauge function $q_2^{(p_r)}(x)$ (with $x = p_r^2$). This exercise shows, however, that the energy-gauge expression of the 2PM dynamics, involving the algebraic function $q_{\rm 2EG}(\gamma)$, Eq. (11.5), is drastically simpler than its p_r-gauge retranscription.

APPENDIX C: RADIAL ACTION AND PERIASTRON ADVANCE IN THE TEST-MASS LIMIT

We recall the notations $\gamma = \hat{\mathcal{E}}_{\rm eff}$,

$$\gamma^2 - 1 \equiv p_\infty^2 \equiv -|p|^2, \quad \text{(C1)}$$

and

$$e^2 \equiv 1 + p_\infty^2 j^2. \quad \text{(C2)}$$

The exact radial action in the test-mass (or Schwarzschild, or $\nu \to 0$) limit reads

$$I_r^S(\gamma, j) = \frac{1}{2\pi} \oint du \frac{\sqrt{P_3(u)}}{u^2(1-2u)}, \quad \text{(C3)}$$

where $P_3(u)$ is the following cubic polynomial in $u = \frac{1}{r}$,

$$P_3(u) = \gamma^2 - (1-2u)(1+j^2u^2) = \gamma^2 - (1 - 2u + j^2u^2 - 2j^2u^3) = \gamma^2 - 1 + 2u - j^2u^2 + 2j^2u^3. \quad \text{(C4)}$$

Here, we are interested in ellipticlike motions with $0 < e^2 < 1$, i.e., with $-1 < p_\infty^2 j^2 < 0$. The inequality $|p|j < 1$ does not *a priori* allow us (contrary to the scattering-motion case) to straightforwardly use a PM expansion in powers of $\frac{1}{j} \propto G$ at a fixed value of γ (or p_∞). The standard expansion technique for ellipticlike motions is the PN expansion. A useful way to formalize the PN expansion is to introduce a PN scaling, say, with the bookkeeping parameter η introduced in the scaling relations (13.11). The main geometrical effect of this scaling is to introduce a parametric separation between the two roots of the cubic polynomial $P_3(u)$ that are close to the roots,

$$u_\pm = \frac{1 \pm e}{j^2}, \quad \text{(C5)}$$

of

$$P_2(u) = \gamma^2 - 1 + 2u - j^2u^2 = p_\infty^2 + 2u - j^2u^2, \quad \text{(C6)}$$

and the third root of $P_3(u)$. It is easily seen that this is formally equivalent to introducing a related PN bookkeeping parameter, say, ϵ, and to write $P_3(u)$ as

$$P_3(u) = p_\infty^2 + 2u - j^2u^2 + \epsilon(2j^2u^3). \quad \text{(C7)}$$

One can then expand the radial integral (C3) in powers of ϵ, using the technique explained in Ref. [57].

From the general result given in Eqs. (3.8) and (3.9) of Ref. [57], one can see that the PN expansion of the sum $I_r^S + j$ defines, when considered at a fixed (negative) value of p_∞^2, an analytic function of the variable $\frac{1}{j}$ having an expansion in powers of $\frac{1}{j}$ of the form

$$I_r^S(\gamma, j) + j = I_0^S(\gamma) + \sum_{n\geq 0} \frac{I_{2n+1}^S(\gamma)}{j^{2n+1}}. \quad \text{(C8)}$$

We wish to algorithmically compute the coefficients $I_0^S(\gamma)$ and $I_{2n+1}^S(\gamma)$ entering the Laurent expansion (C8).This expansion shows that, when keeping fixed p_∞^2 (with $p_\infty^2 < 0$), one can analytically continue $I_r^S(\gamma, j)$ down to $\frac{1}{j} \to 0$. To be able to use the integral definition (C3) of $I_r^S(\gamma, j)$ in the limit $\frac{1}{j} \to 0$, one must (following the method of Sommerfeld used in Ref. [57]) interpret the integral $\oint du$ as a contour integral in the complex u plane, along a closed contour $\mathcal{C}$ circling around the two roots of $P_3(u)$ close to (C5). When $\frac{1}{j} \to 0$, the latter two roots become complex

(because $e \approx \pm i\sqrt{-p_\infty^2}j$) and tend toward $\pm i\frac{\sqrt{-p_\infty^2}}{j}$. The important point is that, in this limit, these two roots tend toward zero, and therefore remain well separated from the third root which tends toward $\frac{1}{2}$ [indeed, the sum of the three roots of $P_3(u)$ is equal to $\frac{1}{2}$]. One can technically see the possibility of expanding the contour integral defining $I_r^S(\gamma, j)$ in this limit by introducing the scaled integration variable y such that $u = \frac{y}{j}$. In terms of this variable, we have the contour integral

$$I_r^S(\gamma, j) = \frac{j}{2\pi}\oint_{\mathcal{C}} dy \frac{\sqrt{p_\infty^2 - y^2 + \frac{2}{j}(y + y^3)}}{y^2(1 - \frac{2}{j}y)}. \tag{C9}$$

As the contour $\mathcal{C}$ circles around the roots $\pm i\frac{\sqrt{-p_\infty^2}}{j}$ of $P_2'(y) = p_\infty^2 - y^2$ (while avoiding them), it is allowed to expand the integrand in powers of $\frac{1}{j}$. The latter expansion leads to well-defined complex-contour integral expressions for the looked-for coefficients $I_0^S(\gamma)$ and $I_{2n+1}^S(\gamma)$. One can then contract the complex contour $\mathcal{C}$ down to the (doubled) interval $[-i\sqrt{-p_\infty^2}, +i\sqrt{-p_\infty^2}]$ along the imaginary axis, and thereby reduce the integrals to real integrals in the variable $x = y/(i\sqrt{-p_\infty^2})$. The latter real integrals on the interval $x \in [-1, +1]$ can then be evaluated by using Hadamard's partie finie [57]. Using this technique, we computed the exact expressions of the test-mass coefficients $I_0^S(\gamma)$ and $I_{2n+1}^S(\gamma)$ given in the text.

Let us also note that Ref. [57] [see Eq. (A.8) there] has explicitly computed the (simpler) complete elliptic integral giving the test-particle periastron advance $K_{\rm Sch} = \Phi_{\rm Sch}/(2\pi)$, i.e., the j derivative of $I_r^{\rm Sch}(\hat{\mathcal{E}}_{\rm eff}, j)$. The authors expressed the result in the simplified form

$$K_{\rm Sch}(\hat{\mathcal{E}}_{\rm eff}, j) = K_{\rm Sch,circ}(j)(1+\xi)^{1/4} F\left[\frac{1}{4}, \frac{3}{4}, 1, \frac{\xi}{3}\right], \tag{C10}$$

where the prefactor

$$K_{\rm Sch,circ}(j) = \left(1 - \frac{12}{j^2}\eta^2\right)^{-1/4} \tag{C11}$$

corresponds to the circular-orbit limit and where the argument ξ is defined as

$$\xi = \tan^2\left(\frac{1}{3}\arcsin(\sqrt{x})\right), \tag{C12}$$

in terms of

$$x \equiv \frac{108}{j^2}\eta^4\left(1 - \frac{12}{j^2}\eta^2\right)^{-3}\left(2\tilde{E} + \frac{1}{j^2} - 36\frac{\tilde{E}}{j^2}\eta^2(1 + 3\tilde{E}\eta^2) - \frac{16}{j^4}\eta^2\right). \tag{C13}$$

Here, $\tilde{E}$ denotes

$$\tilde{E} \equiv \frac{\hat{\mathcal{E}}_{\rm eff}^2 - 1}{2} \equiv \bar{E}_{\rm eff}\left(1 + \frac{1}{2}\bar{E}_{\rm eff}\eta^2\right), \tag{C14}$$

where we introduced the further notation

$$\gamma = \hat{\mathcal{E}}_{\rm eff} \equiv 1 + \bar{E}_{\rm eff}\eta^2. \tag{C15}$$

Inserting the PN expansion of ξ in terms of x, i.e.,

$$\xi = \frac{1}{9}x + \frac{11}{243}x^2 + \frac{169}{6561}x^3 + \frac{1009}{59049}x^4 + O(x^5), \tag{C16}$$

with

$$x = 108\frac{(2\tilde{E}j^2 + 1)}{j^4}\eta^4 + 432\frac{(9\tilde{E}j^2 + 5)}{j^6}\eta^6 - 3888\frac{(-12\tilde{E}j^2 - 8 + 3\tilde{E}^2j^4)}{j^8}\eta^8 - 46656\frac{(-8\tilde{E}j^2 - 8 + 9\tilde{E}^2j^4)}{j^{10}}\eta^{10} + O(\eta^{12}), \tag{C17}$$

in the expression of $K_{\rm Sch}(\hat{\mathcal{E}}_{\rm eff}, j)$, then yields

$$K_{\rm Sch}(\bar{E}_{\rm eff}, j) = 1 + \frac{3}{j^2}\eta^2 + \left(\frac{105}{4j^4} + \frac{15}{2j^2}\bar{E}_{\rm eff}\right)\eta^4 + \left(\frac{15}{4j^2}\bar{E}_{\rm eff}^2 + \frac{315}{2j^4}\bar{E}_{\rm eff} + \frac{1155}{4j^6}\right)\eta^6 + \left(\frac{225225}{64j^8} + \frac{4725}{16j^4}\bar{E}_{\rm eff}^2 + \frac{45045}{16j^6}\bar{E}_{\rm eff}\right)\eta^8 + \left(\frac{765765}{16j^8}\bar{E}_{\rm eff} + \frac{2909907}{64j^{10}} + \frac{3465}{16j^4}\bar{E}_{\rm eff}^3 + \frac{315315}{32j^6}\bar{E}_{\rm eff}^2\right)\eta^{10} + O(\eta^{12}), \tag{C18}$$

where we used the energy variable $\bar{E}_{\rm eff} = (\gamma - 1)/\eta^2$. The latter expression is easily checked to agree with (minus) the j derivative of the $\nu \to 0$ limit of our 5PN-expanded radial action above, as given in Table XI.

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