Fractional Quantum Hall States at Zero Magnetic Field

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We present a simple prescription to flatten isolated Bloch bands with a nonzero Chern number. We first show that approximate flattening of bands with a nonzero Chern number is possible by tuning ratios of nearest-neighbor and next-nearest-neighbor hoppings in the Haldane model and, similarly, in the chiral-π-flux square lattice model. Then we show that perfect flattening can be attained with further range hoppings that decrease exponentially with distance. Finally, we add interactions to the model and present exact diagonalization results for a small system at 1/3 filling that support (i) the existence of a spectral gap, (ii) that the ground state is a topological state, and (iii) that the Hall conductance is quantized.

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In a seminal paper, Haldane [1] has shown that noninteracting electrons hopping on a honeycomb lattice can exhibit the integer quantum Hall effect (IQHE) without the Landau levels induced by a uniform magnetic field, provided the system breaks time-reversal symmetry (TRS). In that model, electrons hop with a real-valued uniform nearest-neighbor (NN) amplitude of magnitude t_1 that preserves TRS, as well as complex-valued next-nearest-neighbor (NNN) amplitudes with the uniform magnitude t_2 that break TRS. A nonvanishing t_2 generically opens a band gap at the Fermi-Dirac points of graphene (half-filling). This band gap results in the Chern numbers taking opposite values of magnitude 1 on the upper and lower bands. Consequently, the model exhibits an IQHE at half-filling.

Given the fact that a band insulator can support the IQHE without a magnetic field, a natural question that we address in this Letter is whether a fractional quantum Hall effect (FQHE) is also possible in an interacting lattice model without a magnetic field. For the usual FQHE in an uniform magnetic field, all Landau levels share the same Chern number, ± 1 depending on the orientation of the uniform magnetic field. Moreover, in the absence of disorder, all Landau levels are flat (i.e., dispersionless) and thus can accommodate, when partially filled, an exponentially large number of Slater determinants, from which incompressible liquids are selected by interactions at some special filling fractions. Haldane's model fulfills the first ingredient for the FQHE: nonvanishing Chern numbers for the single-particle Bloch bands. We are going to construct two-dimensional lattice models without magnetic fields that also satisfy the second ingredient for the FQHE: band flattening.

There is a long history of flatband models. They have been studied since the 1970s in amorphous semiconductors [2–4], and understood using projection operators [5]. More recently they have been studied on kagome, honeycomb, and square lattices [6–11]. In Ref. [10] flatbands were isolated by gaps, and the question of whether it is possible to have a flatband with nonzero Chern number was raised. We shall answer this question affirmatively. We then add interactions and show evidence that the many-body state is a topological state with fractional Hall conductance at 1/3 filling.

Our starting point is two-dimensional local lattice models describing the hopping of spinless fermions. In the spirit of Haldane's model, we restrict the lattice models to those with only two Bloch bands and enforce locality by only allowing NN and NNN hoppings. We will show that, by varying the ratio of the NNN to NN hoppings, we can deform the bands to make them flatter. The characteristic measure for the flatness of a Bloch band is here the ratio of the bandwith to the band gap. We then show that this criterion for flatness can be saturated to the ideal limit of zero for the valence band by including arbitrary range hoppings. However, the flattened single-particle Hamiltonian still preserves locality in the sense that the hopping amplitudes decrease exponentially with the distance between any two lattice sites.

Consider the noninteracting two-band Bloch Hamiltonian of the generic form

$$H_0 := \sum_{\boldsymbol{k}\in \mathrm{BZ}} \psi_{\boldsymbol{k}}^{\dagger} \mathcal{H}_{\boldsymbol{k}} \psi_{\boldsymbol{k}}, \quad \mathcal{H}_{\boldsymbol{k}} := B_{0,\boldsymbol{k}}\sigma_0 + \boldsymbol{B}_{\boldsymbol{k}}\cdot\boldsymbol{\sigma}. \tag{1a}$$

Here, BZ stands for the Brillouin zone, $\psi_{\boldsymbol{k}}^{\dagger} = (c_{\boldsymbol{k},A}^{\dagger}, c_{\boldsymbol{k},B}^{\dagger})$, where $c_{\boldsymbol{k},s}^{\dagger}$ creates a Bloch states on sublattice $s = A, B$, and the 2×2 matrices σ_0 and $\boldsymbol{\sigma}$ are the identity matrix and the three Pauli matrices acting on the sublattice indices. If we define

$$\hat{\boldsymbol{B}}_{\boldsymbol{k}} := \frac{\boldsymbol{B}_{\boldsymbol{k}}}{|\boldsymbol{B}_{\boldsymbol{k}}|}, \qquad \tan\phi_{\boldsymbol{k}} := \frac{\hat{B}_{2,\boldsymbol{k}}}{\hat{B}_{1,\boldsymbol{k}}}, \qquad \cos\theta_{\boldsymbol{k}} := \hat{B}_{3,\boldsymbol{k}}, \tag{1b}$$

we can write the eigenvalues of Hamiltonian $\mathcal{H}_{\boldsymbol{k}}$ as $\varepsilon_{\pm,\boldsymbol{k}} = B_{0,\boldsymbol{k}} \pm |\boldsymbol{B}_{\boldsymbol{k}}|$ and for the corresponding orthonormal eigenvectors

$$\chi_{+,\boldsymbol{k}} = \begin{pmatrix} e^{-i\phi_{\boldsymbol{k}}/2} & \cos\frac{\theta_{\boldsymbol{k}}}{2} \\ e^{+i\phi_{\boldsymbol{k}}/2} & \sin\frac{\theta_{\boldsymbol{k}}}{2} \end{pmatrix}, \quad \chi_{-,\boldsymbol{k}} = \begin{pmatrix} e^{-i\phi_{\boldsymbol{k}}/2} & \sin\frac{\theta_{\boldsymbol{k}}}{2} \\ e^{+i\phi_{\boldsymbol{k}}/2} & -\cos\frac{\theta_{\boldsymbol{k}}}{2} \end{pmatrix}. \tag{1c}$$

Two examples of Hamiltonians of the form (1a) are the following.

Example 1: The honeycomb lattice.—We introduce the vectors $\boldsymbol{a}_1^t = (0, -1)$, $\boldsymbol{a}_2^t = (\sqrt{3}/2, 1/2)$, $\boldsymbol{a}_3^t = (-\sqrt{3}/2, 1/2)$ connecting NN and the vectors $\boldsymbol{b}_1^t = \boldsymbol{a}_2^t - \boldsymbol{a}_3^t$, $\boldsymbol{b}_2^t = \boldsymbol{a}_3^t - \boldsymbol{a}_1^t$, $\boldsymbol{b}_3^t = \boldsymbol{a}_1^t - \boldsymbol{a}_2^t$ connecting NNN from the honeycomb lattice depicted in Fig. 1(a). We denote with $\boldsymbol{k}$ a wave vector from the BZ of the reciprocal lattice dual to the triangular lattice spanned by $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, say. The model is then defined by the Bloch Hamiltonian [1]

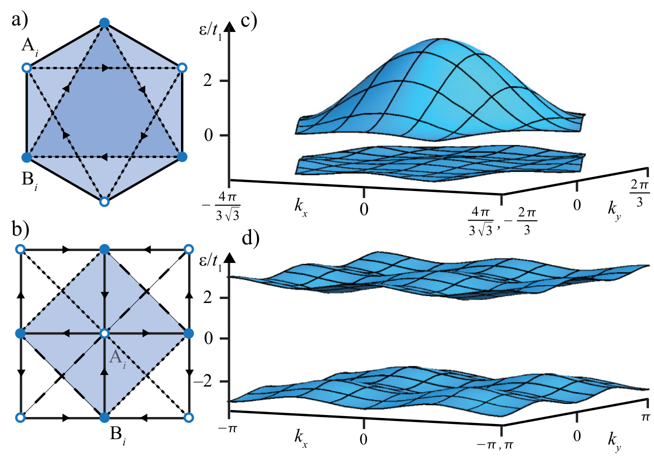

FIG. 1 (color online). (a) Unit cell of Haldane's model on the honeycomb lattice: the NN hopping amplitudes t_1 are real (solid lines) and the NNN hopping amplitudes are $t_2 e^{i2\pi\Phi/\Phi_0}$ in the direction of the arrow (dotted lines). The flux 3Φ and $-\Phi$ penetrate the dark shaded region and each of the light shaded regions, respectively. For $\Phi = \pi/3$, the model is gauge equivalent to having one flux quantum per unit cell. (b) The chiral-π-flux on the square lattice, where the unit cell corresponds to the shaded area. The NN hopping amplitudes are $t_1 e^{i\pi/4}$ in the direction of the arrow (solid lines) and the NNN hopping amplitudes are t_2 and $-t_2$ along the dashed and dotted lines, respectively. (c) The band structure of Haldane's model for $\cos\Phi = t_1/(4t_2) = 3\sqrt{3/43}$ with the flatness ratio $1/7$. (d) The band structure of the chiral-π-flux for $t_1/t_2 = \sqrt{2}$ with the flatness ratio $1/5$. The lower bands can be made exactly flat by adding longer range hoppings.

$$B_{0,\boldsymbol{k}} := 2t_2 \cos\Phi \sum_{i=1}^{3} \cos\boldsymbol{k}\cdot\boldsymbol{b}_i, \tag{2a}$$

$$\boldsymbol{B}_{\boldsymbol{k}} := \sum_{i=1}^{3} \begin{pmatrix} t_1 \cos\boldsymbol{k}\cdot\boldsymbol{a}_i \\ t_1 \sin\boldsymbol{k}\cdot\boldsymbol{a}_i \\ -2t_2 \sin\Phi \sin\boldsymbol{k}\cdot\boldsymbol{b}_i \end{pmatrix}, \tag{2b}$$

where $t_1 \geq 0$ and $t_2 \geq 0$ are NN and NNN hopping amplitudes, respectively, and the real numbers $\pm\Phi$ are the magnetic fluxes penetrating the two halves of the hexagonal unit cell. For $t_1 \gg t_2$, the gap $\Delta \equiv \min_{\boldsymbol{k}}\varepsilon_{+,\boldsymbol{k}} - \max_{\boldsymbol{k}}\varepsilon_{-,\boldsymbol{k}}$ is proportional to t_2. The width of the lower band is $\delta_- \equiv \max_{\boldsymbol{k}}\varepsilon_{-,\boldsymbol{k}} - \min_{\boldsymbol{k}}\varepsilon_{-,\boldsymbol{k}}$. The flatness ratio δ_-/Δ is extremal for the choice $\cos\Phi = t_1/(4t_2) = 3\sqrt{3/43}$, yielding an almost flat lower band with $\delta_-/\Delta = 1/7$ [see Fig. 1(c)].

Example 2: The square lattice.—We introduce the vectors $\boldsymbol{x}^t \equiv (1/\sqrt{2}, 1/\sqrt{2})$ and $\boldsymbol{y}^t \equiv (-1/\sqrt{2}, 1/\sqrt{2})$ connecting NNN from the square lattice as depicted in Fig. 1(b). We denote with $\boldsymbol{k}^t = (k_x, k_y)$ a wave vector from the BZ of the reciprocal lattice dual to the square lattice spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$. The model is then defined by the Bloch Hamiltonian [12]

$$B_{0,\boldsymbol{k}} := 0, \tag{3a}$$

$$B_{1,\boldsymbol{k}} + iB_{2,\boldsymbol{k}} := t_1 e^{-i\pi/4}[1 + e^{+i(k_y - k_x)}] + t_1 e^{+i\pi/4}[e^{-ik_x} + e^{+ik_y}], \tag{3b}$$

$$B_{3,\boldsymbol{k}} := 2t_2(\cos k_x - \cos k_y), \tag{3c}$$

where $t_1 \geq 0$ and $t_2 \geq 0$ are NN and NNN hopping amplitudes, respectively. The flatness ratio δ_-/Δ is extremal for the choice $t_1/t_2 = \sqrt{2}$, yielding two almost flat bands with $\delta_-/\Delta \approx 1/5$ [see Fig. 1(d)].

The Chern numbers for the bands labeled by $\pm$ in Eq. (1c) are given by

$$C_\pm = \mp \int_{\boldsymbol{k}\in\mathrm{BZ}} \frac{d^2\boldsymbol{k}}{4\pi} \epsilon_{\mu\nu}[\partial_{k_\mu}\cos\theta(\boldsymbol{k})][\partial_{k_\nu}\phi(\boldsymbol{k})]. \tag{4}$$

They have opposite signs if nonzero. All the information about the topology of the Bloch bands of a gaped system is encoded in the single-particle wave functions. For example, the Chern numbers depend solely on the eigenfunctions. Haldane's model (2) and the chiral-π-flux (3) are topologically equivalent in the sense that both have two bands with Chern numbers ± 1.

To enhance the effect of interactions, highly degenerate (i.e., flat) bands are desirable. It is always possible to deform the Bloch Hamiltonian (1a) so as to have one flatband with the energy -1, say, while preserving the eigenspinors $\chi_{\pm,\boldsymbol{k}}$ (1c). Indeed, this is achieved by turning the Bloch Hamiltonian (1a) into

$$\mathcal{H}_{\boldsymbol{k}}^{\mathrm{flat}} := \frac{\mathcal{H}_{\boldsymbol{k}}}{\varepsilon_{-,\boldsymbol{k}}}. \tag{5}$$

Note that whenever $B_{0,\boldsymbol{k}} \equiv 0$, the Hamiltonian (1a) has the spectral symmetry $\varepsilon_{+,\boldsymbol{k}} = -\varepsilon_{-,\boldsymbol{k}}$ so that both bands of

$\mathcal{H}^{\text{flat}}_{\boldsymbol{k}} = \hat{\boldsymbol{B}}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}$ are completely flat. This spectral symmetry applies to the chiral-π-flux (3) but not to Haldane's model (2) unless $\Phi = \pm\pi/2$.

Generically, $\mathcal{H}^{\text{flat}}_{\boldsymbol{k}}$ follows from a lattice model for which the hopping amplitudes are nonvanishing for arbitrary large separations. If, however, the hopping amplitudes decrease sufficiently fast with the separation, locality is preserved. To estimate the decay of the hopping amplitudes with the separation between any two sites in the flattened chiral-π-flux model (3), we calculate the decay of the Fourier coefficients $A_{n,n'}$ of

$$\frac{1}{|\varepsilon_{\pm,\boldsymbol{k}}|} = \sum_{n,n'=0}^{\infty} A_{n,n'} \cos nk_+ \cos n'k_-, \tag{6}$$

where $k_\pm \equiv k_x \pm k_y$. Because

$$\varepsilon^2_{\pm,\boldsymbol{k}} = (2t_1^2 - t_2^2)(2 + \cos k_+ + \cos k_-) + t_2^2(3 + \cos k_+ \cos k_-), \tag{7}$$

it is sufficient to consider the Fourier coefficients $\tilde{A}_n$ of

$$\frac{1}{\sqrt{1+\alpha\cos k}} = \sum_{n=0}^{\infty} \tilde{A}_n \cos nk, \quad \text{for } -1 < \alpha < 1. \tag{8}$$

In the limit $n \gg 1$, one finds that $\tilde{A}_n$ decays exponentially with n. We conclude that for any fixed n, the coefficients $A_{n,n'}$ decay exponentially with n' and vice versa for n, iff $|\alpha| < 1$. Flattening the energy bands preserves the locality of the chiral-π-flux model (3).

The fact that we have engineered single-particle wave functions in a flat Bloch band that support a Chern number ± 1 is one step in mimicking the FQHE. However, because of lattice effects, it is not a given that interactions lead to a many-body ground state supporting the FQHE. In fact, even when there is a uniform magnetic field, the combination of lattice effects and interactions is not well understood upon increasing the magnetic flux threading the elementary lattice unit cell. In Refs. [13,14], for example, the possibility of a FQHE induced by interactions for the Hofstadter problem, NN hopping with a uniform flux threading each elementary plaquette of the square lattice, was studied numerically. While the overlap between the Laughlin states on the torus and the lattice many-body ground states was close to unity when the plaquette flux is much smaller than the flux quantum, this overlap decreases when the plaquette flux becomes of the order of one quarter of the flux quantum. It is thus imperative to study how interactions lift the macroscopic degeneracy of a fractionally filled flat Bloch band and whether a gapped topological ground state emerges.

Two distinctive properties of such a ground state $|\Psi\rangle$ at filling fraction ν (where ν^{-1} is an odd integer) and with periodic boundary conditions (toroidal geometry) are (i) the ν^{-1}-fold topological degeneracy of the ground state manifold and (ii) the quantization of the Hall conductance σ_{xy} in units of $\nu e^2/h$. The Hall conductance is related to the Chern number C of the many-body ground state $|\Psi\rangle$ as $\sigma_{xy} = Ce^2/h$ [15]. Conventionally, the Chern number is evaluated using twisted boundary conditions $\langle \boldsymbol{r} + N_x\boldsymbol{x}|\Psi_{\boldsymbol{\gamma}}\rangle = e^{i\gamma_x}\langle \boldsymbol{r}|\Psi_{\boldsymbol{\gamma}}\rangle$ and $\langle \boldsymbol{r} + N_y\boldsymbol{y}|\Psi_{\boldsymbol{\gamma}}\rangle = e^{i\gamma_y}\langle \boldsymbol{r}|\Psi_{\boldsymbol{\gamma}}\rangle$, where $\boldsymbol{\gamma}^t = (\gamma_x, \gamma_y)$ is the twisting angle and $N_x \times N_y$ the number of unit cells. The Chern number is then given by [16]

$$C = \frac{1}{2\pi i}\int_{\boldsymbol{\gamma}\in[0,2\pi]^2} d^2\boldsymbol{\gamma} \boldsymbol{\nabla}_{\boldsymbol{\gamma}} \wedge \langle \Psi_{\boldsymbol{\gamma}}|\boldsymbol{\nabla}_{\boldsymbol{\gamma}}|\Psi_{\boldsymbol{\gamma}}\rangle. \tag{9a}$$

Here, we introduce

$$\tilde{C} = \frac{1}{2\pi i}\int_{\boldsymbol{k}\in\text{BZ}} d^2\boldsymbol{k} n_{-,\boldsymbol{k}}[\boldsymbol{\nabla}_{\boldsymbol{k}} \wedge (\chi^\dagger_{-,\boldsymbol{k}}\boldsymbol{\nabla}_{\boldsymbol{k}}\chi_{-,\boldsymbol{k}})] \tag{9b}$$

as a second way to calculate the Chern number, where $n_{-,\boldsymbol{k}} = \langle\Psi|c^\dagger_{\boldsymbol{k},-}c_{\boldsymbol{k},-}|\Psi\rangle$ is the occupation number of the single-particle Bloch state in the lower $(-)$ band with wave vector $\boldsymbol{k}$ evaluated in the many-body ground state. We can show that both formulas are equivalent, i.e., $C = \tilde{C}$. To this end, one expands the many-body wave function $|\Psi\rangle$ in a sum over Slater determinants and applies a gauge transformation to the single-particle states to remove the twist in the boundary conditions.

We close this Letter with an exact diagonalization study to show the existence of a gapped topological ground state for the chiral π-flux phase (3) in the presence of interactions. We consider an interaction defined by the repulsive two-body NN potential $V_{i,j}$ according to

$$H_{\text{int}} := \frac{1}{2}\sum_{i,j\in\Lambda}\rho_i V_{i,j}\rho_j \equiv V\sum_{\langle ij\rangle}\rho_i\rho_j, \qquad V > 0. \tag{10}$$

Directed NN bonds of the square lattice $\Lambda = A \cup B$ made of the open and filled circles of Fig. 1(b) are here denoted by $\langle ij\rangle$, while ρ_i is the occupation number on the site $i \in \Lambda$.

We also drive the model trough a topological phase transition to establish that a gapped topological many-body ground state results from the topological nature of the model. To this end, we add a sublattice-staggered chemical potential $4\mu_s$ to the single-particle Hamiltonian (1) by replacing $B_{3,\boldsymbol{k}} \to B_{3,\boldsymbol{k}} + 4\mu_s$ in Eq. (3). Then, the two noninteracting bands have a Chern number ± 1 for $|t_2/\mu_s| > 1$ and a vanishing Chern number for $|t_2/\mu_s| < 1$. The topological phase transition at $|t_2/\mu_s| = 1$ forces the single-particle spectral gap to close and the flattening of the bands is ill defined at that point [$\alpha = 1$ in Eq. (8)].

For a 3×6 sublattice A, we find a unique ground state that is separated by a gap of the order of the interaction strength V at filling $\nu = 1/3$ [see Fig. 2(a)]. This state loses its clear separation in energy from the other states at the topological phase transition $|t_2/\mu_s| = 1$ and another gapped ground state is obtained for $|t_2/\mu_s| < 1$. Based on (i) the topological degeneracy and (ii) the quantized Hall

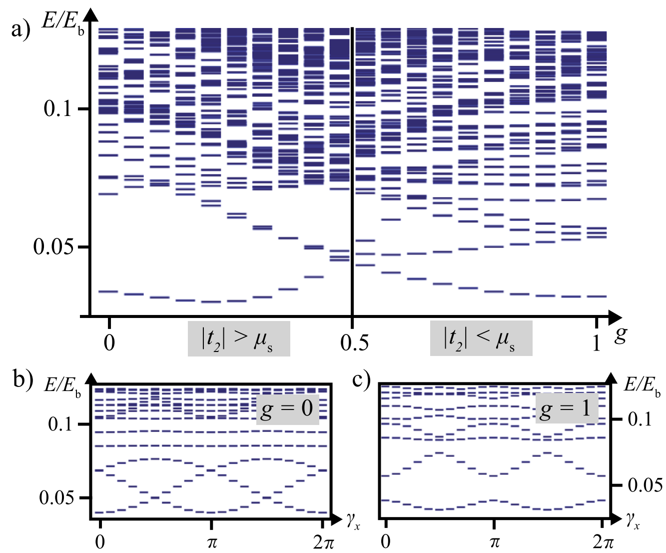

FIG. 2 (color online). (a) The lowest eigenvalues of $H_0^{\text{flat}} + H_{\text{int}}$ for the chiral π-flux phase obtained from exact diagonalization for 6 particles on a 3×6 sublattice A (1/3 filling), normalized by the bandwidth E_b. The parameters t_2 and μ_s of H_0^{flat} interpolate between topological ($|t_2| > |\mu_s|$) and nontopological ($|t_2| < |\mu_s|$) single-particle bands. Here, $g := (2/\pi) \times \arctan|t_2/\mu_s|$ and the energies are measured relative to the energy of the single-particle band. (b) The lowest eigenvalues in the center of mass momentum sector of the ground state of $H_0^{\text{flat}} + H_{\text{int}}$ with twisted boundary conditions as a function of the twisting angle γ_x for $\mu_s = 0$, $t_2 = t_1/\sqrt{2}$. The level crossings indicate the topological nontrivial nature of the three lowest states. (c) Same as (b), but for $\mu_s = t_1/\sqrt{2}$, $t_2 = 0$. The ground state is topologically trivial.

conductance, we will now argue that the first state is a topological many-body state while the latter state is topological trivial.

(i) Because of translational invariance, the Hamiltonian does not couple states with different center of mass momenta $\boldsymbol{Q} := \boldsymbol{k}_1 + \cdots + \boldsymbol{k}_N$, where $\boldsymbol{k}_i$, $i = 1, \ldots, N$ are the single-particle momenta of an N-particle state. At 1/3 filling of the 3×6 sublattice A, the particle number $N = 6$ is commensurate with the lattice dimensions and all three topological states have the same $\boldsymbol{Q}$. As a consequence, their topological degeneracy is lifted and a unique ground state appears. We can now use twisted boundary conditions to probe the topological nature of the ground state: varying γ_x between 0 and 2π is equivalent to the adiabatic insertion of a flux quantum in the system. During this process, a topological ground state with $C = 1/3$ should undergo two level crossings with the other two gapped topological states [17]. Indeed, we find these level crossings for the gapped ground state when the model has topological single-particle bands [Fig. 2(b)], whereas no level crossings are found otherwise [Fig. 2(c)].

(ii) We have also calculated the Chern number of the gapped ground state for $\mu_s = 0$, $t_2 = t_1/\sqrt{2}$ with the two equivalent formulas (9a) and (9b). We find $C = 0.29$ and $\tilde{C} = 0.30$ and attribute the deviations from $C = 1/3$ to the limitations of the small system size. For the topological trivial model with $\mu_s = t_1/\sqrt{2}$, $t_2 = 0$, we find that C and $\tilde{C}$ vanish to a precision of 10^{-6} and 10^{-3}, respectively.

In summary, we have proposed a simple recipe to deform any noninteracting lattice model so as to obtain flatbands, while preserving locality. We flattened the bands of the chiral π-flux phase and then lifted the resulting macroscopic ground state degeneracy with repulsive interactions. Via exact diagonalization, we have shown that a FQH-like topological ground state is obtained at 1/3 filling. This ground state, that is not well described by Laughlin-type wave functions, will be further studied in future works.

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Note added.—Recently, we became aware of Refs. [18,19] in which similar topological flatband models are discussed. Subsequently, Ref. [20] appeared with exact diagonalization results that are consistent with our findings.

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