


Entanglement-Ergodic Quantum Systems Equilibrate Exponentially Well

H. Wilming,¹ M. Goihl,² I. Roth,² and J. Eisert^{2,3}

¹*Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland*

²*Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*

³*Helmholtz Center Berlin, 14109 Berlin, Germany*

 (Received 18 March 2019; revised manuscript received 4 July 2019; published 13 November 2019)

One of the outstanding problems in nonequilibrium physics is to precisely understand when and how physically relevant observables in many-body systems equilibrate under unitary time evolution. General equilibration results show that equilibration is generic provided that the initial state has overlap with sufficiently many energy levels. But results not referring to typicality which show that natural initial states actually fulfill this condition are lacking. In this work, we present stringent results for equilibration for systems in which Rényi entanglement entropies in energy eigenstates with finite energy density are extensive for at least some, not necessarily connected, subsystems. Our results reverse the logic of common arguments, in that we derive equilibration from a weak condition akin to the eigenstate thermalization hypothesis, which is usually attributed to thermalization in systems that are assumed to equilibrate in the first place. We put the findings into the context of studies of many-body localization and many-body scars.

DOI: [10.1103/PhysRevLett.123.200604](https://doi.org/10.1103/PhysRevLett.123.200604)

Over recent years the study of the relaxation to equilibrium of complex many-body systems has attracted great attention. This interest can be motivated from at least two points of view. From a foundational viewpoint, it is desirable to understand how statistical equilibrium ensembles emerge within the framework of unitary quantum mechanics—without introducing any external probability measures. It is then necessary to first explain how systems undergoing unitary evolution attain equilibrium at all. A key ingredient to explain this behavior has been found to be the dynamical buildup of entanglement from low-entangled initial states and results showing equilibration under quite general conditions have been derived [1–12]. The increase of entanglement over time is a generic feature of complex quantum systems and leads to an increase of the entropy of subsystems over time reminiscent to the second law of thermodynamics.

From a more concrete perspective, the recent interest in the study of nonequilibrium dynamics is motivated by the fact that such dynamics can now be realized in well-controlled experiments, for example, in ion traps or optical lattices [13–18]. Moreover, the discovery of many-body localized systems [19], which equilibrate [20] but fail to thermalize [18], shows that there remains much to be understood about the equilibration behavior of complex quantum systems. Despite the great progress in understanding the equilibration behavior of many-body systems, rigorous results showing that systems with natural initial states equilibrate to high precision based on concrete physical properties have been lacking.

In this Letter we aim to fill this gap by taking a new perspective to the problem. To do this, we carefully

reconsider the entanglement content of energy eigenstates in complex, interacting many-body systems and devise a working definition of “entanglement-ergodic” systems whose energy eigenstates at finite energy density have a sufficient amount of entanglement between suitable subsystems. The condition we propose is very weak—yet we show that generically such systems equilibrate to exponential precision in the volume of the system if the initial state is given by a product state with finite, nonzero energy density. Commonly, one assumes equilibration and invokes the eigenstate thermalization hypothesis (ETH) [21–25] to make thermalization plausible. Here, we stringently derive equilibration from a highly plausible condition similar to, but we believe much weaker than the eigenstate thermalization hypothesis.

The main ingredient of our proof is a careful discussion of Rényi entanglement entropies in energy eigenstates with finite energy density. Combining this insight with the strongly peaked energy distribution of weakly correlated states and the monotonicity of Rényi entropies allows us to prove that experimentally accessible initial states are well smeared out over the energy spectrum, which implies high-precision equilibration for generic interacting Hamiltonians.

Formal setting.—We consider local Hamiltonians

$$H_\Lambda = \sum_{x \in \Lambda} h_x \quad (1)$$

on a regular lattice Λ in ν spatial dimensions with $N := |\Lambda|$ lattice sites. The Hilbert space is $\otimes_{x \in \Lambda} \mathcal{H}_x$ with $\dim \mathcal{H}_x = d$. Since we will be talking about the scaling of quantities with the lattice size, H_Λ should be seen as a

sequence of Hamiltonians, which is, for example, given by a family of translational invariant system on larger and larger square lattices with periodic boundary conditions. We call H_Λ l local if the diameter of the support of each h_x is at most l , strictly local if it is l local with l independent of the system size N , and uniformly bounded if $\|h_x\| \leq h$ for all $x \in \Lambda$ independent of the system size. Since we are mostly interested in energy densities instead of total energies later, we make the unusual choice to label eigenvectors of the Hamiltonian by their energy *densities* e_i as $|e_i\rangle_\Lambda = |E_i/N\rangle_\Lambda$, with $i = 1, \dots, d^N$ and E_i being the eigenvalue of H_Λ corresponding to $|e_i\rangle_\Lambda$. We always assume that the ground state has vanishing total energy. From now on, we will often drop the subscript Λ from states and Hamiltonians to simplify the notation.

Equilibration in closed systems.—We now briefly review general equilibration results that we will be using in the following. Consider any bounded observable A and any initial state ρ . Denote the time evolved states by $\rho(t)$. We want to study whether the expectation value $\langle A(t) \rangle_\rho = \text{Tr}[\rho(t)A]$ equilibrates over time. In a finite system, perfect equilibration, in the sense that $\langle A(t) \rangle$ becomes static for all times after the equilibration process, is impossible due to recurrences. However, it is perfectly possible that this value is very close to a stationary value \bar{A} for most of the time, with rare deviations. The value \bar{A} is then necessarily the infinite time average

$$\bar{A} := \overline{\langle A(t) \rangle} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{Tr}[\rho(t)A] = \text{Tr}(\omega A), \quad (2)$$

where ω denotes the time average $\overline{\rho(A)}$, which is again a valid density matrix. In the case where the Hamiltonian has no degenerate energy differences $G_{(i,j)} = E_i - E_j$, it has been shown that the time-averaged fluctuations around the equilibrium value are bounded by [5]

$$\text{Var}(A, H, \rho) := \overline{(\langle A(t) \rangle - \bar{A})^2} \leq \|A\|^2 e^{-S_2(\omega)}, \quad (3)$$

where S_α denotes the Rényi- α entropy

$$S_\alpha(\rho) := \frac{1}{1-\alpha} \log [\text{Tr}(\rho^\alpha)]. \quad (4)$$

We note for later use that in the limit $\alpha \rightarrow 1$ the von Neumann entropy is recovered and that Rényi entropies are monotonically decreasing in α . The condition of non-degenerate energy differences is generically fulfilled in interacting systems [26]. However, generalizations of the above result also exist if this condition is weakly violated [8]. It is also possible to show $\text{Var}(A, H, \rho) \leq 3\|A\|^2 e^{-S_\infty(\omega')}$, where ω' is the same operator as ω but with its largest eigenvalue replaced by zero [2], which sometimes gives a stronger bound. In particular, it also incorporates the case of energy eigenstates, which are always fully equilibrated. It is important to stress, however, that the bound Eq. (3) does

not lead to implications on the time it takes to observe equilibration.

Similarly to results in terms of bounded observables, one can also bound the distance of a local reduced state on a subsystem S from its time average as [3]

$$\|\overline{\text{Tr}_{S^c}[\rho(t)]} - \text{Tr}_{S^c}(\omega)\|_1 \leq 2d_S e^{-S_2(\omega)/2}. \quad (5)$$

Here, $\|\cdot\|_1$ denotes the trace norm, which bounds the difference in expectation value of all normalized observables as

$$\|\rho - \sigma\|_1 = \max_{A, \|A\|=1} |\text{Tr}(\rho A) - \text{Tr}(\sigma A)|. \quad (6)$$

Roughly speaking, the role of the norm of the observables in the previous bound is here taken by the dimension of the subsystem. We conclude that a large Rényi-2 entropy of the time-averaged state is a sufficient condition for a generic, closed quantum system to equilibrate eventually. Unfortunately, however, there are few general and rigorous results which show that natural initial states lead to time-averaged states whose energy distribution has a large Rényi-2 entropy [27,28]. Reference [28] makes it highly plausible from an operational point of view that states that can be prepared in experiments have large effective dimension, but does not show it for concrete states. Reference [27] shows that states with finite correlation length have Rényi-2 entropy of at least roughly the order $\log(N)$. While this formally leads to equilibration as $N \rightarrow \infty$, it is insufficient to obtain a finite entropy density at equilibrium (which is crucial from a thermodynamic point of view) and requires very large system sizes to explain equilibration. In the following, we show that in systems that have a sufficiently large amount of entanglement in energy eigenstates with finite energy density, finite entropy density and hence exponentially good equilibration in the system size follows for initial product states.

Entanglement and Rényi entropies.—Here, we are interested in ergodic, nonintegrable systems. Contrary to the case of classical mechanics, there is no generally agreed upon definition of what it exactly means for a quantum system to be ergodic or nonintegrable [29]. In recent years, it has been argued that a general characteristic of systems which can be considered ergodic is that energy eigenstates fulfill a so-called volume law in terms of their entanglement content. The condition we propose is inspired from this observation, but is much weaker. It is therefore useful to discuss volume laws before stating our condition. Consider an energy eigenvector with finite energy density $|e\rangle$ and denote by $\rho_A(e)$ the reduced density matrix on some (contiguous) subsystem A which is smaller than one-half of the total system, but still contains a finite fraction of the total system. A volume law means that the entanglement measured by a Rényi entropy S_α grows like the volume of A as

$$S_\alpha[\rho_A(e)] \sim |A|. \quad (7)$$

A natural question to ask is for which value of α this relation is supposed to hold. We now argue that this relation is a meaningful criterion only if $\alpha > 1$. This might come as a surprise since it is common to measure entanglement in terms of the von-Neumann entropy S_1 . This is due to the fact that the von-Neumann entropy describes the fraction of EPR pairs that can be distilled from asymptotically many copies of a state by local operations and classical communication [30]. However, the examples presented in the following proposition show that a volume law in terms of von-Neumann entropy is not a very useful criterion to determine whether a state of a many-body system deviates strongly from an unentangled state.

Proposition 1: (Counterexample) For any $1 > \epsilon > 0$ there exist state vectors $|\Psi^\epsilon\rangle$ on Λ with the following properties: (1) $|\Psi^\epsilon\rangle$ has overlap exponentially close to $1 - \epsilon$ in N with a product state vector $|\Psi\rangle$. (2) $|\Psi^\epsilon\rangle$ fulfills a volume law in the von Neumann entropy: There exist regions A with $|A| = N/2$ such that $S_1(\rho_A) \simeq \frac{\epsilon}{2} \log(d)N$. (3) All Rényi entropies with $\alpha > 1$ are bounded by a constant in the system size, $S_\alpha(\rho_A) \leq \text{const}$.

The state vectors that fulfill these conditions are simply of the form $|\Psi^\epsilon\rangle \propto \sqrt{1-\epsilon}|\Psi\rangle + \sqrt{\epsilon}|\Omega\rangle$, where $|\Omega\rangle$ is maximally entangled between A and its complement. The proof of these properties is given in the Supplemental Material [31], Sec. B. Given Properties 1 and 3, it cannot reasonably be said that the amount of entanglement in $|\Psi^\epsilon\rangle$ grows volumelike—even though it fulfills a volume law in terms of von-Neumann entropy. At the same time it suggests that we should require a volume law in terms of some Rényi entropy with $\alpha > 1$. Importantly, the inequality (proven in the Supplemental Material [31], Sec. A)

$$S_\alpha(\rho) \geq S_\infty(\rho) \geq \frac{\beta-1}{\beta} S_\beta(\rho), \quad \forall \alpha \geq 0, \quad (8)$$

which holds for any $\beta > 1$, shows that all the Rényi entropies with $\alpha > 1$ have the same scaling behavior. It therefore does not matter which one we consider and in the following we therefore mostly consider the case $\alpha = 2$.

Recently, there has been an increasing amount of numerical results and theoretical arguments that show that energy eigenstates of generic nonintegrable quantum systems with finite energy density have this property (see, for example, Refs. [43–49]). These results fit well to, and indeed are partly motivated by, the observation that certain properties of complex, strongly interacting systems at finite energy density can be well described by assuming that their Hamiltonians are random matrices [50–55], despite the Hamiltonians being local and thus belonging to a set of measure zero. It is well known that Haar random quantum states have extensive Rényi entanglement entropy on bipartite systems with very high probability [56–60].

Since eigenstates of random Hamiltonians are distributed according to the Haar measure, we expect that eigenstates of random Hamiltonians fulfill a volume law in terms of all Rényi entropies with very high probability [61].

Entanglement ergodicity.—In essence, in the following we will show that an extensive amount of entanglement entropy in energy eigenstates for Rényi entropies with $\alpha > 1$ is a sufficient criterion for exponentially good equilibration. Importantly, however, we will significantly weaken the assumption from volume laws—where the subsystem under consideration is assumed to be a contiguous region—by allowing the subsystem to be almost completely arbitrary. The only property that we demand from the subsystem is that it includes a finite fraction of the total system. In particular, it need not be connected and further may be chosen differently for every energy eigenstate. For example, it may have a fractal-like shape (up to the lattice spacing), or consist of a sublattice of spins that are far away from each other when compared to some natural length scale of the system. In the following, we call this (much weaker) form of a volume law a weak volume law. In particular, even states that have a finite correlation length, such as matrix product states (MPS), can be expected to generically follow a weak volume law—unless they are product states themselves.

We will now give a formal definition of what we demand from an entanglement-ergodic system. Notably, this definition is perfectly compatible with the mindset of the ETH that argues basically that eigenstates of local Hamiltonians in the bulk of the spectrum should be locally indistinguishable from Gibbs states due to their entanglement.

Definition 1.—We call a sequence of systems of increasing system size entanglement ergodic, if there exists a system size N_0 and a function $g: \mathbb{R} \rightarrow [0, \infty)$, such that for all system sizes $N \geq N_0$ it holds that (1) (Weak volume law) For every eigenvector $|e_i\rangle_\Lambda$ there exists *some* subsystem A_Λ such that the reduced state $\rho_{A_\Lambda}(e_i) := \text{Tr}_{A_\Lambda^c}(|e_i\rangle\langle e_i|_\Lambda)$ fulfills

$$S_2[\rho_{A_\Lambda}(e_i)] \geq g(e_i)N. \quad (9)$$

(2) The function g is sufficiently well behaved: It is Lipschitz continuous and positive for nonextremal energy densities, i.e., $g(e) > 0$ for $0 < e < e_{\max}$.

Some remarks on this definition are in order. (i) As shown by Eq. (8), we could have replaced the Rényi-2 entropy with any Rényi entropy with $\alpha > 1$ and would have obtained an equivalent definition. (ii) We only require that there is a lower bound on the entropy in the region A_Λ as a function of the energy density and not that the entropy is given by the function $e \mapsto g(e)$. In particular, we do not require that eigenstates with the same energy density also have the same Rényi entanglement entropy and allow for the possibility that states with vanishing energy density do not have a large amount of entanglement. For example, the

ground state of the system may be a product state. Similarly, we impose Lipschitz continuity of the function g for simplicity and concreteness. Similar conclusions as in the following can be reached by imposing different regularity assumptions on g . (iii) It would be perfectly fine for all what follows to allow for additional negative but subleading terms, e.g., in $O(\sqrt{N})$. These terms would only change the subleading behavior of our final results and may encode nonasymptotic information about the entropy in eigenstates. For simplicity, we omit such terms here.

(iv) We emphasize again that we only demand a weak volume law, i.e., that for any system size and any eigenstate there exists *some* finite fraction of the total system A_Λ whose entropy is sufficiently large. It is not required that this holds for all such subsystems nor is it required that this subsystem has any particular shape. For example, it could consist of every 10th site of the lattice. Nevertheless, in generic, strongly interacting systems, we expect that choosing A_Λ simply as one-half of the system is sufficient.

(v) Importantly, the weak assumption on the subsystem A_Λ allows even MPS-like eigenstates, which fulfill an area law [62] for contiguous regions, to be entanglement ergodic. As a concrete example, we prove the following statement in the Supplemental Material [31], Sec. C: States that are prepared by a translationally invariant, finite-depth, local quantum circuit and are not a product state have extensive in Rényi-2 entropy on a finite fraction of the system. Similarly, we expect an analogous result to hold more generally for generic MPS. A detailed discussion on the application of our framework to generic MPS is provided in the Supplemental Material [31], Sec. C. These results show that also systems featuring many-body localization [19], whose eigenstates are expected to be approximable by matrix product states [63,64], may fall within the framework of entanglement ergodicity. It is well known that such systems equilibrate, but fail to also thermalize. Thus even if we refer to the key property as entanglement ergodicity, it is a significantly weaker condition than what is commonly understood as ergodicity in the current literature.

(vi) Similarly, we expect the notion of entanglement ergodicity to be weaker than the ETH: if one assumes that the ETH applies to some subsystems containing a finite fraction of the total system, then we strongly believe that the ETH implies entanglement ergodicity and hence equilibration with high precision (see Supplemental Material [31], Sec. G).

(vii) With stronger assumptions on the regions A_Λ , we can extend the applicability of the definition of entanglement ergodicity to states related by quasilocal unitaries (those generated by time evolution under local Hamiltonians): For systems in which the regions A_Λ have an asymptotically vanishing surface-to-volume ratio, entanglement ergodicity is stable under quasilocal unitaries. The precise meaning and formal proof of this statement

is formulated in Supplemental Material [31], Sec. E. Intuitively it follows by observing that time evolution under a local Hamiltonian for a finite time can only decrease the entropy of a subregion by an amount that is proportional to its boundary, since the entropy has to “flow” out of the subregion through its boundary.

Consequences of entanglement ergodicity.—Let us now discuss the consequences of entanglement ergodicity. We first state a result on the diagonal entropy in entanglement-ergodic systems, to then turn to the implication that entanglement-ergodic systems equilibrate exponentially well.

Theorem 2: (Diagonal entropy in entanglement-ergodic systems) Consider an entanglement-ergodic system with strictly local, uniformly bounded Hamiltonian. Then for any energy density $e > 0$ there exists a constant $k(e) > 0$ and a system-size $N_0(e)$ such that for all system sizes $N > N_0(e)$ and for all product states $|\Psi\rangle_\Lambda$ with energy density e , we have

$$S_\alpha(\omega_\Lambda) \geq k(e)N, \quad (10)$$

where ω_Λ is the time average of $|\Psi\rangle\langle\Psi|_\Lambda$.

As a direct consequence of this result we obtain from Eq. (3) the following bounds on equilibration.

Corollary 3: (Equilibration in entanglement-ergodic systems) Under the same conditions as in Theorem 2 and the additional assumption of nondegenerate energy gaps of the Hamiltonian, there exists a constant $k(e) > 0$, such that

$$\text{Var}(A, H_\Lambda, |\Psi\rangle\langle\Psi|_\Lambda) \leq \|A\|^2 C e^{-k(e)N}. \quad (11)$$

Similar bounds hold for the reduced state on a small subsystem, as implied by Eq. (5). The proof of Theorem 2 is given in the Supplemental Material [31], Sec. D. It relies on recognizing that an extensive amount of Rényi entanglement entropy implies that a state has exponentially small overlap with all product states and to combine this statement with a recent central limit-type theorem for the energy distribution in product states [32]. The premises of Theorem 2 require the initial state to be a product state. In the Supplemental Material [31], Sec. F, we further extend our results to the case where the initial state is prepared from a product state by a quasilocal unitary under the additional assumption that the regions A_Λ have vanishing surface-to-volume ratio. We obtain equilibration bounds scaling as $O[\exp(-N^\beta)]$ for some $0 < \beta < 1$.

Quantum many-body scars.—Recently, it has been observed that sets of atypical energy eigenstates with small amounts of entanglement may rarely show up even at finite energy density in nonintegrable, kinetically constrained many-body systems—a phenomenon dubbed “quantum many-body scars,” which leads to exceedingly slow equilibration with long-lived oscillations from certain initial product states [65–69]. Even a toy model with complete

absence of equilibration for certain initial product states has been constructed [70]. This complete absence of equilibration can be explained due to a $SU(2)$ symmetry in the subspace of many-body scars [70], which also emerges approximately in more realistic Hamiltonians and breaks the assumption of nondegenerate energy gaps. Furthermore, many-body scars may have large squared overlap with the initial product state (of order $1/N$), which implies that such systems also violate entanglement-ergodicity within the subspace of many-body scars.

Conclusion.—We carefully formalized a notion of ergodicity based on extensive Rényi entanglement entropies in energy eigenstates and showed that this notion suffices to prove exponentially precise equilibration for Hamiltonians with nondegenerate energy gaps. Our condition is quite weak and we expect it to be fulfilled for generic interacting systems. The notion of ergodicity we introduced is connected to the eigenstate thermalization hypothesis (ETH), which asserts that local reduced states of energy eigenstates with finite energy density already resemble the reduced state of a corresponding Gibbs state. We thereby introduce a new perspective to the study of nonequilibrium quantum systems: We do not have to assume equilibration for systems to become apparently stationary and then turn to the ETH to show thermalization. Instead, we show that equilibration already follows from a weak ETH-like assumption. In fact, in this case the ETH implies much more, since our definition does, in general, not imply that the system also *thermalizes* (this apparent shortcoming is necessary when formulating a criterion that may also apply to many-body localized systems). Here, we did not discuss the timescales for relaxation to equilibrium, but were interested in the precision of equilibration after arbitrarily long times. While some progress in understanding equilibration timescales has been made recently, both in integrable [71–80] and generic, nonintegrable systems [8,81–88], finding rigorous arguments bounding equilibration timescales from reasonable assumptions remains an outstanding open problem.

H. W. would like to thank Rodrigo Gallego for fruitful discussions regarding the role of Rényi entropies in equilibration. We acknowledge financial support from the ERC (TAQ), the DFG (FOR 2724, CRC 183, EI 519/7-1, EI 519/9-1), the Templeton Foundation, and the Studienstiftung des Deutschen Volkes. H. W. further acknowledges contributions from the Swiss National Science Foundation via the NCCR QSIT as well as Project No. 200020_165843. This work has also received funding from the European Union’s Horizon 2020 research and innovation programme under Grant Agreement No. 817482 (PASQuanS).

[1] H. Tasaki, *Phys. Rev. Lett.* **80**, 1373 (1998).

[2] P. Reimann, *Phys. Rev. Lett.* **101**, 190403 (2008).

- [3] N. Linden, S. Popescu, A. J. Short, and A. Winter, *Phys. Rev. E* **79**, 061103 (2009).
- [4] S. Goldstein, J. L. Lebowitz, C. Mastrodonato, R. Tumulka, and N. Zanghi, *Proc. R. Soc. A* **466**, 3203 (2010).
- [5] A. J. Short, *New J. Phys.* **13**, 053009 (2011).
- [6] P. Reimann, *Phys. Scr.* **86**, 058512 (2012).
- [7] P. Reimann and M. Kastner, *New J. Phys.* **14**, 043020 (2012).
- [8] A. J. Short and T. C. Farrelly, *New J. Phys.* **14**, 013063 (2012).
- [9] L. Masanes, A. J. Roncaglia, and A. Acín, *Phys. Rev. E* **87**, 032137 (2013).
- [10] C. Gogolin and J. Eisert, *Rep. Prog. Phys.* **79**, 056001 (2016).
- [11] J. Eisert, M. Friesdorf, and C. Gogolin, *Nat. Phys.* **11**, 124 (2015).
- [12] B. N. Balz and P. Reimann, *Phys. Rev. Lett.* **118**, 190601 (2017).
- [13] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
- [14] I. Bloch, J. Dalibard, and S. Nascimbene, *Nat. Phys.* **8**, 267 (2012).
- [15] U. Schneider, L. Hackermüller, J. P. Ronzheimer, S. Will, S. Braun, T. Best, I. Bloch, E. Demler, S. Mandt, D. Rasch, and A. Rosch, *Nat. Phys.* **8**, 213 (2012).
- [16] S. Trotzky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, *Nat. Phys.* **8**, 325 (2012).
- [17] S. Braun, M. Friesdorf, S. S. Hodgman, M. Schreiber, J. P. Ronzheimer, A. Riera, M. del Rey, I. Bloch, J. Eisert, and U. Schneider, *Proc. Natl. Acad. Sci. U.S.A.* **112**, 3641 (2015).
- [18] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, *Science* **349**, 842 (2015).
- [19] R. Nandkishore and D. A. Huse, *Annu. Rev. Condens. Matter Phys.* **6**, 15 (2015).
- [20] M. Serbyn, Z. Papić, and D. A. Abanin, *Phys. Rev. B* **90**, 174302 (2014).
- [21] J. M. Deutsch, *Phys. Rev. A* **43**, 2046 (1991).
- [22] M. Srednicki, *Phys. Rev. E* **50**, 888 (1994).
- [23] M. Rigol, V. Dunjko, and M. Olshanii, *Nature (London)* **452**, 854 (2008).
- [24] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Rev. Mod. Phys.* **83**, 863 (2011).
- [25] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, *Adv. Phys.* **65**, 239 (2016).
- [26] In the case of global symmetries or superselection sectors, each sector should be considered separately.
- [27] T. Farrelly, F. G. S. L. Brandão, and M. Cramer, *Phys. Rev. Lett.* **118**, 140601 (2017).
- [28] R. Gallego, H. Wilming, J. Eisert, and C. Gogolin, *Phys. Rev. A* **98**, 022135 (2018).
- [29] J.-S. Caux and J. Mossel, *J. Stat. Mech.* (2011) P02023.
- [30] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996).
- [31] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.123.200604> for the technical proofs of inequality (8), Proposition 1, and Theorem 2 as well as further technical results and discussions on entanglement ergodicity, which includes Refs. [32–42].

- [32] A. Anshu, *New J. Phys.* **18**, 083011 (2016).
- [33] A. Uhlmann, *Rep. Math. Phys.* **9**, 273 (1976).
- [34] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
- [35] M. M. Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac, *Phys. Rev. Lett.* **100**, 070502 (2008).
- [36] M. Fannes, B. Nachtergaele, and R. F. Werner, *Commun. Math. Phys.* **144**, 443 (1992).
- [37] B. Nachtergaele, *Commun. Math. Phys.* **175**, 565 (1996).
- [38] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, *Quantum Inf. Comput.* **5–6**, 401 (2006).
- [39] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, *Quantum Inf. Comput.* **8**, 650 (2008).
- [40] X. Chen, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. B* **82**, 155138 (2010).
- [41] K. Van Acoleyen, M. Mariën, and F. Verstraete, *Phys. Rev. Lett.* **111**, 170501 (2013).
- [42] M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, and J. Eisert, *Phys. Rev. X* **4**, 031019 (2014).
- [43] J. R. Garrison and T. Grover, *Phys. Rev. X* **8**, 021026 (2018).
- [44] Y. O. Nakagawa, M. Watanabe, H. Fujita, and S. Sugiura, *Nat. Commun.* **9**, 1635 (2018).
- [45] T.-C. Lu and T. Grover, *Phys. Rev. E* **99**, 032111 (2019).
- [46] Y. Huang, *Nucl. Phys. B* **938**, 594 (2019).
- [47] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol, *Phys. Rev. Lett.* **119**, 020601 (2017).
- [48] L. Vidmar and M. Rigol, *Phys. Rev. Lett.* **119**, 220603 (2017).
- [49] J. Goold, C. Gogolin, S. R. Clark, J. Eisert, A. Scardicchio, and A. Silva, *Phys. Rev. B* **92**, 180202(R) (2015).
- [50] M. V. Berry, *Proc. R. Soc. A* **400**, 229 (1985).
- [51] T. Prosen, *Phys. Rev. E* **60**, 3949 (1999).
- [52] S. Müller, S. Heusler, P. Braun, F. Haake, and A. Altland, *Phys. Rev. Lett.* **93**, 014103 (2004).
- [53] C. Kollath, G. Roux, G. Biroli, and A. M. Läuchli, *J. Stat. Mech.* (2010) P08011.
- [54] J. P. Keating, N. Linden, and H. J. Wells, *Commun. Math. Phys.* **338**, 81 (2015).
- [55] P. Kos, M. Ljubotina, and T. Prosen, *Phys. Rev. X* **8**, 021062 (2018).
- [56] E. Lubkin, *J. Math. Phys. (N.Y.)* **19**, 1028 (1978).
- [57] S. Lloyd and H. Pagels, *Ann. Phys. (N.Y.)* **188**, 186 (1988).
- [58] D. N. Page, *Phys. Rev. Lett.* **71**, 1291 (1993).
- [59] C. Nadal, S. N. Majumdar, and M. Vergassola, *Phys. Rev. Lett.* **104**, 110501 (2010).
- [60] C. Nadal, S. N. Majumdar, and M. Vergassola, *J. Stat. Phys.* **142**, 403 (2011).
- [61] While it is immediate that each eigenstate of a random Hamiltonian fulfills a volume law with high probability, the distribution of all the basis states is not independent. It is thus not immediate that the probability that all d^N eigenstates fulfill a volume law is also arbitrarily close to unity for large systems. However, the probability for each of them not to follow a volume law is exponentially small in d^N according to Refs. [59,60], while there are only d^N states, which indicates that even this latter probability is very large.
- [62] J. Eisert, M. Cramer, and M. B. Plenio, *Rev. Mod. Phys.* **82**, 277 (2010).
- [63] B. Bauer and C. Nayak, *J. Stat. Mech.* (2013) P09005.
- [64] M. Friesdorf, A. H. Werner, W. Brown, V. B. Scholz, and J. Eisert, *Phys. Rev. Lett.* **114**, 170505 (2015).
- [65] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletić, and M. D. Lukin, *Nature (London)* **551**, 579 (2017).
- [66] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, *Nat. Phys.* **14**, 745 (2018).
- [67] S. Moudgalya, S. Rachel, B. A. Bernevig, and N. Regnault, *Phys. Rev. B* **98**, 235155 (2018).
- [68] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, *Phys. Rev. B* **98**, 155134 (2018).
- [69] S. Moudgalya, N. Regnault, and B. A. Bernevig, *Phys. Rev. B* **98**, 235156 (2018).
- [70] S. Choi, C. J. Turner, H. Pichler, W. W. Ho, A. A. Michailidis, Z. Papi, M. Serbyn, M. D. Lukin, and D. A. Abanin, *Phys. Rev. Lett.* **122**, 220603 (2019).
- [71] P. Calabrese and J. Cardy, *Phys. Rev. Lett.* **96**, 136801 (2006).
- [72] P. Calabrese and J.-S. Caux, *J. Stat. Mech.* (2007) P08032.
- [73] V. Eisler and I. Peschel, *J. Stat. Mech.* (2007) P06005.
- [74] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, *Phys. Rev. Lett.* **98**, 050405 (2007).
- [75] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, *Phys. Rev. Lett.* **100**, 030602 (2008).
- [76] M. Cramer, A. Flesch, I. P. McCulloch, U. Schollwöck, and J. Eisert, *Phys. Rev. Lett.* **101**, 063001 (2008).
- [77] T. Barthel and U. Schollwöck, *Phys. Rev. Lett.* **100**, 100601 (2008).
- [78] A. Flesch, M. Cramer, I. P. McCulloch, U. Schollwöck, and J. Eisert, *Phys. Rev. A* **78**, 033608 (2008).
- [79] P. Calabrese, F. H. L. Essler, and M. Fagotti, *Phys. Rev. Lett.* **106**, 227203 (2011).
- [80] J.-S. Caux and F. H. L. Essler, *Phys. Rev. Lett.* **110**, 257203 (2013).
- [81] S. Goldstein, T. Hara, and H. Tasaki, *Phys. Rev. Lett.* **111**, 140401 (2013).
- [82] A. S. L. Malabarba, L. P. García-Pintos, N. Linden, T. C. Farrelly, and A. J. Short, *Phys. Rev. E* **90**, 012121 (2014).
- [83] L. P. García-Pintos, N. Linden, A. S. L. Malabarba, A. J. Short, and A. Winter, *Phys. Rev. X* **7**, 031027 (2017).
- [84] S. Goldstein, T. Hara, and H. Tasaki, *New J. Phys.* **17**, 045002 (2015).
- [85] T. Farrelly, *New J. Phys.* **18**, 073014 (2016).
- [86] P. Reimann, *Nat. Commun.* **7**, 10821 (2016).
- [87] T. R. de Oliveira, C. Charalambous, D. Jonathan, M. Lewenstein, and A. Riera, *New J. Phys.* **20**, 033032 (2018).
- [88] H. Wilming, M. Goihl, C. Krumnow, and J. Eisert, arXiv: 1704.06291.