

Projectively Enriched Symmetry and Topology in Acoustic Crystals

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Symmetry plays a key role in modern physics, as manifested in the revolutionary topological classification of matter in the past decade. So far, we seem to have a complete theory of topological phases from internal symmetries as well as crystallographic symmetry groups. However, an intrinsic element, i.e., the gauge symmetry in physical systems, has been overlooked in the current framework. Here, we show that the algebraic structure of crystal symmetries can be projectively enriched due to the gauge symmetry, which subsequently gives rise to new topological physics never witnessed under ordinary symmetries. We demonstrate the idea by theoretical analysis, numerical simulation, and experimental realization of a topological acoustic lattice with projective translation symmetries under a \mathbb{Z}_2 gauge field, which exhibits unique features of rich topologies, including a single Dirac point, Möbius topological insulator, and graphenelike semimetal phases on a rectangular lattice. Our work reveals the impact when gauge and crystal symmetries meet together with topology and opens the door to a vast unexplored land of topological states by projective symmetries.

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The concepts of symmetry and topology have permeated into almost all branches of physics [1–6]. In topological matters, the “topology” refers to the nontrivial global structures in the phase winding of wave functions in momentum space. Symmetries constrain the possible topological structures [7], and it is clear that their significance lies in how the symmetries act on the wave functions or, in mathematical terms, how they are represented in the wave function space.

There are two important points that have not been appreciated in the study of topological states of matter. First, under a gauge field, crystal symmetries will be projectively represented (see Supplemental Material for a brief introduction on projective symmetry [8]), and this impacts the algebra of symmetry operations [10–12]. This can be understood from the analogy with Aharonov-Bohm effect [13]: The phase of wave function for different paths is modified by the gauge field, which, in turn, revises the representation of spatial symmetries. Second, the gauge fields can actually be intrinsic and ubiquitous for real physical systems, not necessarily the applied magnetic field as in the Aharonov-Bohm effect. Particularly, the \mathbb{Z}_2 gauge field is intrinsic to artificial periodic systems with time-reversal symmetry T —i.e., the hopping amplitudes are real numbers that can take either positive or negative signs—and, moreover, it can be precisely engineered with current technology [14–19].

To demonstrate the idea, let us consider the simple 2D lattice model in Fig. 1. Here, each rectangular plaquette carries a π gauge flux. Figure 1(a) shows a specific gauge configuration, where each red (blue) colored bond has a negative (positive) hopping amplitude. Note that one is free to choose a gauge configuration for technical convenience, since all physics depends only on the flux configuration. The fundamental symmetries for the 2D lattice are the two primitive translations L_x and L_y . Without the gauge field, the translations commute with each other, which is the algebra forming the foundation of solid state physics. However, one notes that, under the \mathbb{Z}_2 gauge field, this fundamental algebra is modified to

$$\{L_x, L_y\} = 0, \quad (1)$$

because moving around a plaquette will endow the wave function with a π phase. An immediate consequence is that each band is twofold degenerate for a generic momentum, since Eq. (1) resembles the algebra of the Pauli matrices. More interestingly, at point M (π, π) of the Brillouin zone (BZ), L_x , L_y , and T together with the imaginary unit i generate a real Clifford algebra $C^{0,4}$, which has a unique four-dimensional representation [8]. This means that the system must have a fourfold degenerate Dirac point at M [see Fig. 1(d)], described by the Dirac model:

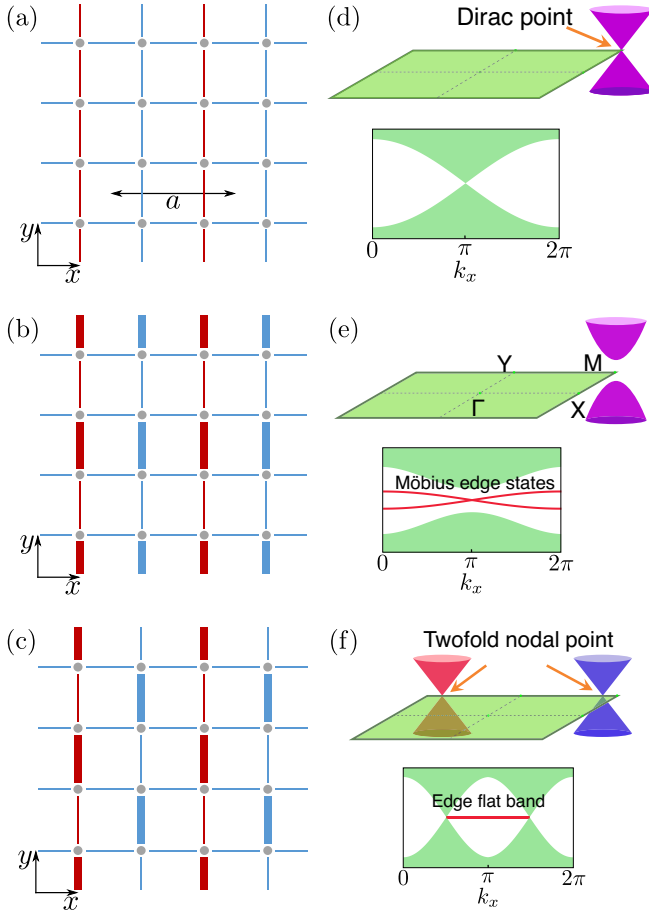


FIG. 1. Illustration of the models. The red (blue) bonds denote negative (positive) hopping amplitudes. Thus, each plaquette has a π flux. (a) Both primitive translational symmetries $L_{x,y}$ are preserved. The band structure features a fourfold degenerate Dirac point as shown in (d). (b) The staggered dimerization pattern which preserves L_x . The edge bands with the Möbius twist are illustrated in (e). (c) The alternative dimerization pattern with both primitive translation symmetries broken. It generates a graphenelike semimetal phase as illustrated in (f). The bottom panel of (d)–(f) illustrates the spectra for an open edge along x .

$$h_D(\mathbf{q}) = q_x \Gamma_1 + q_y \Gamma_2. \quad (2)$$

Here, Γ_μ with $\mu = 1, 2, \dots, 5$ are the five Hermitian 4×4 Dirac matrices satisfying $\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu} 1_4$. A concrete representation may be given by $\Gamma_1 = \tau_3 \otimes \sigma_2$, $\Gamma_2 = \tau_2 \otimes \sigma_0$, $\Gamma_3 = \tau_1 \otimes \sigma_0$, $\Gamma_4 = \tau_3 \otimes \sigma_1$, and $\Gamma_5 = \tau_3 \otimes \sigma_3$, with τ 's and σ 's being two sets of the Pauli matrices. It is important to note that this Dirac point is enabled solely by the projective translation symmetries and T , and there is only a single Dirac point in the BZ, which contrasts with all previous cases where Dirac points must require additional point group symmetries and they cannot exist as a single Fermi point in 2D T -invariant systems [20].

Breaking the primitive translation such as L_y by dimerization will destroy the Dirac point and drive a topological phase transition. Two representative configurations are

shown in Figs. 1(b) and 1(c). In the Dirac model in Eq. (2), the two dimerization patterns correspond to perturbation terms $m_1 \Gamma_3$ and $m_2 i \Gamma_2 \Gamma_5$, respectively.

Interestingly, the case in Fig. 1(b) realizes a Möbius topological insulator. In the bulk, a band gap opens, and the band structure is characterized by a Möbius \mathbb{Z}_2 topological invariant enabled by L_x and the sublattice symmetry Γ_5 . In the eigenspace of L_x , the Hamiltonian $H(\mathbf{k})$ is diagonalized into two blocks: $H(\mathbf{k}) = \text{diag}[h_1(\mathbf{k}), h_2(\mathbf{k})]$, which are connected by the sublattice symmetry Γ_5 . The topological invariant is given by [11]

$$\nu = \frac{1}{2\pi} \int_{(0,2\pi) \times S^1} d^2 k \mathcal{F} + \frac{1}{\pi} \gamma(0) \pmod{2}. \quad (3)$$

Here, \mathcal{F} is the Berry curvature, and $\gamma(0)$ is the Berry phase on the $k_x = 0$ path in the BZ. Both are defined for the valence bands of $h_1(\mathbf{k})$. The hallmark of this insulator is that its edge parallel to the x direction will have a Möbius edge band; i.e., the band has a twisted structure similar to the edge of a Möbius strip [see Fig. 1(e)]. Furthermore, the band is completely detached from the bulk bands, distinct from the usual topological insulators where the edge bands must connect the bulk bands. We note that similar Möbius states were discussed in a few limited cases, but all require complicated nonsymmorphic symmetries [21–24], in contrast to the case here based on the projective translation symmetry.

The alternative dimerization in Fig. 1(c) splits the fourfold Dirac point into two twofold nodal points along the k_y direction. While both primitive translational symmetries L_x and L_y are broken, the sublattice symmetry Γ_5 is preserved. Hence, there is a topological charge defined by the winding number $w = (1/4\pi i) \oint_C d\mathbf{k} \cdot \text{tr} \Gamma_5 H^{-1}(\mathbf{k}) \nabla H(\mathbf{k})$ on a circle C surrounding a Fermi point. The nontrivial topological charge leads to a flat edge band on the edge parallel to x , connecting the projections of the two Fermi points [Fig. 1(f)]. One may note that this phase is similar to graphene, but there are actually important differences. First, twofold linear nodal points are known to be common to hexagonal lattices (including graphene), but here they occur in a rectangular lattice. Second, the points in graphene are pinned at the high-symmetry points, whereas the points here are unpinned; i.e., they can freely move on the Y - M path without breaking any symmetry.

Without dimerization [Fig. 1(a)], the primitive unit cell actually consists of two sites. Then, there are two twofold nodal points at $[\pm(\pi/2), \pi]$, and the fourfold Dirac point for the doubled unit cell is actually folded from the two twofold nodal points. We have chosen the doubled unit cell, because we wanted to discuss the criticality of topological phases from two dimerization patterns.

One may wonder, as the primitive unit cell has already corresponded to a twofold Dirac semimetal phase, why we make the alternative dimerization. The answer is without dimerization the twofold nodal points have no topological

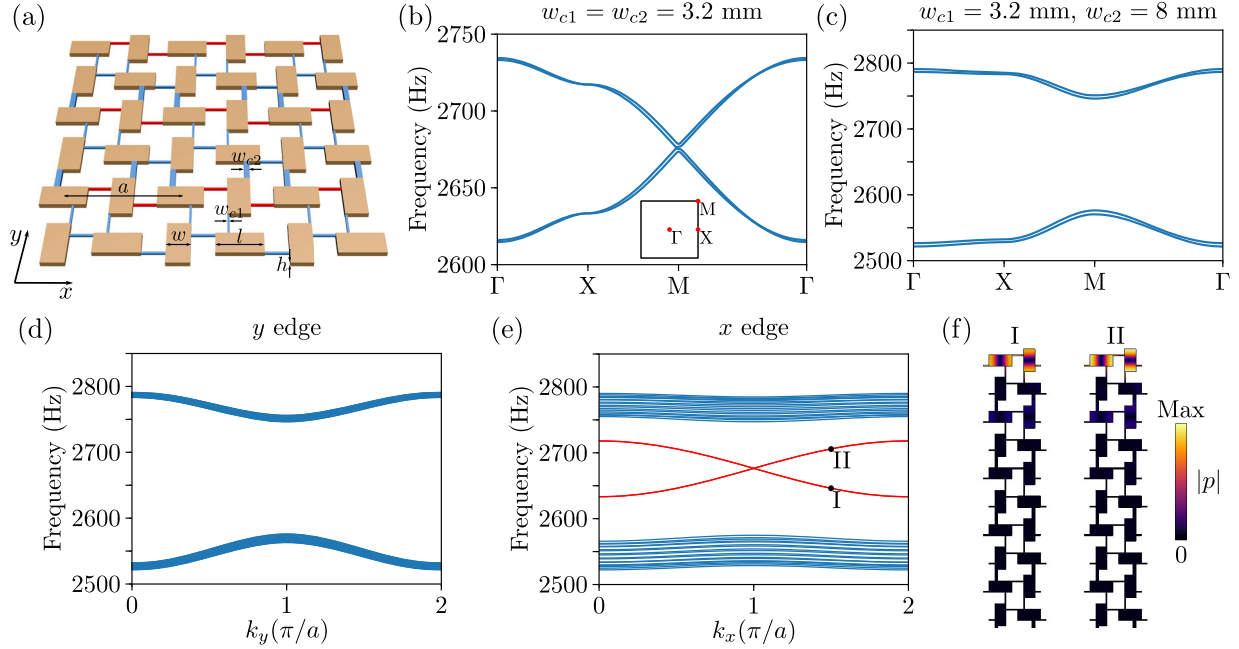


FIG. 2. (a) Schematic of designed acoustic lattice, which is made by cuboid resonators (orange) and coupling tubes (red and blue). The lattice constant $a = 160$ mm. Other parameter values are given in the main text. (b),(c) Simulated bulk bands for $w_{c1} = w_{c2} = 3.2$ mm (b) and $w_{c1} = 3.2$ mm and $w_{c2} = 8$ mm (c), respectively. (d),(e) Simulated dispersions for structures with the open boundary condition along the x direction but the periodic boundary condition along the y direction, and vice versa, respectively. Bands colored in blue (red) correspond to bulk (edge) modes. In both simulations, the lattice consists of ten unit cells. (f) Eigenmode patterns for the two edge modes marked by “I” and “II” in (e).

charge, and, therefore, there is no edge flat band. Thus, it is remarkable that the graphenelike topological semimetal can be realized on a rectangular lattice. Meanwhile, it is noteworthy that here the projectively represented translation symmetries play an essential role, but for graphene the twofold degeneracy is inherited from the D_3 symmetry.

Now we proceed to demonstrate the above phenomena in an acoustic crystal. Our design, as depicted in Fig. 2(a), consists of cuboid acoustic resonators (colored in orange) and coupling tubes (colored in red and blue). This kind of design has been used to construct various topological tight-binding Hamiltonians in acoustics, such as Weyl semimetals [25–29] and higher-order topological insulators [17–19,30–33]. However, the underlying projective crystal symmetries and their resultant topological physics were never revealed. Here, the cuboid resonator has a size of $64 \text{ mm} \times 32 \text{ mm} \times 8 \text{ mm}$, supporting a dipolar mode at around 2680 Hz. Couplings between the resonators are enabled by thin tubes with square cross sections. The whole structure is hollow, filled with air, and surrounded by hard walls.

Here, positive and negative couplings are realized by placing the coupling tubes at different sides of the dipolar mode’s nodal line. In Fig. 2(a), tubes that enable positive and negative couplings are colored in blue and red, respectively. In such a configuration, this acoustic lattice carries π flux per plaquette, as required. Moreover, the coupling strengths can be engineered by tuning the widths

of the coupling tubes. Thus, the tight-binding model with both primitive translation symmetries $L_{x,y}$, as well as two aforementioned dimerization patterns, can all be realized.

In the absence of coupling dimerization (i.e., all the coupling tubes have the same width), the bulk bands are all nearly twofold degenerate over the entire BZ except for the M point, where a fourfold Dirac point approximately appears, as can be seen from Fig. 2(b). Then, we proceed to transit the fourfold degenerate Dirac criticality into other topological phases by introducing dimerization. We impose a staggered dimerization pattern [see Fig. 2(a)] by letting $w_{c1} = 3.2$ mm and $w_{c2} = 8$ mm. Since the primitive translation symmetry L_y is broken, the degeneracy at M is lifted and a band gap is opened, as shown in Fig. 2(c). Next, we look into the boundary modes in this gapped phase. On the open edge parallel to x , one clearly observes two crossing edge bands inside the bulk band gap and, particularly, fully detached from the bulk bands [see Fig. 2(e)]. These edge bands, as discussed previously, actually represent a single band forming a Möbius twist. In contrast, for the open edge along the y direction, L_x is broken, and, therefore, no in-gap edge modes are observed [see Fig. 2(d)]. All these simulation results agree well with the predicted phenomena [8].

We then conducted experiments to probe the signatures of the Möbius topological insulator. As shown in Fig. 3(a), a sample with 20×10 resonators was fabricated through 3D printing. We first measure the bulk transmission to confirm the existence of the band gap. The measured

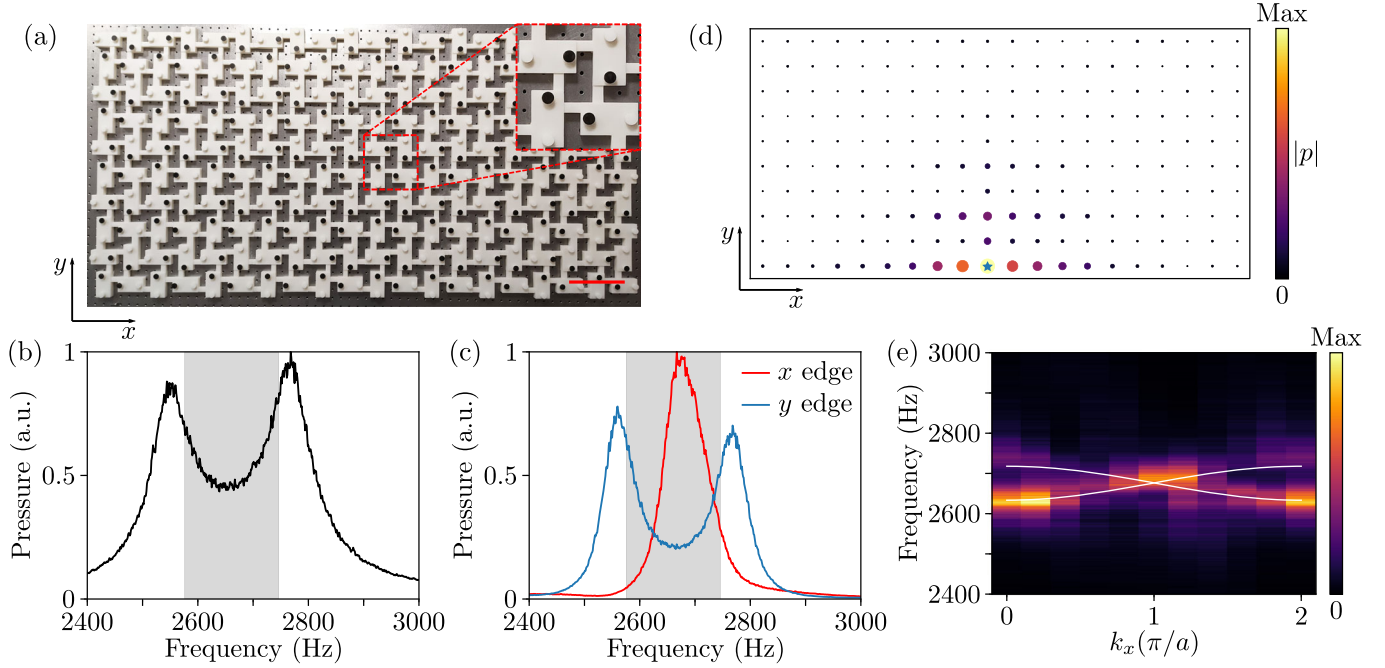


FIG. 3. (a) A photo of fabricated sample with 20×10 resonators. The inset shows one unit cell. The scale bar is 160 mm. (b) Measured acoustic pressure when the speaker and microphone are placed at two different resonators inside the bulk. The gray region denotes the simulated bulk band gap. (c) Measured acoustic pressure when the speaker and microphone are placed at two different resonators on the x edge (red curve) and the y edge (blue curve). (d) Measured acoustic pressure distribution at 2680 Hz. Both the color and size of the circles denote measured amplitude. (e) Measured edge dispersion. The white lines indicate simulated edge bands.

acoustic pressure, as plotted in Fig. 3(b), shows two peaks separated by a gap. The frequency range of the measured gap matches well with the band gap found in simulation [shaded in gray in Fig. 3(b)]. Then, we measure the transmission on the edges of the sample. For the y edge, the measured spectrum [blue curve in Fig. 3(c)] is similar to the bulk transmission spectrum. We observed only two separated peaks lying at frequencies corresponding to bulk bands, which is consistent with the simulation given in Fig. 2(d) showing that there are no in-gap edge modes. In contrast, the transmission spectrum obtained on the x edge is significantly different from the previous two. As given by the red curve in Fig. 3(c), there is only one peak in the spectrum. Furthermore, this peak is inside the band gap, consistent with the consequence of the Möbius-twisted edge band on the x edge.

We also mapped out the field distribution to further confirm the existence of the Möbius edge band. In this experiment, a speaker was fixed at the center of the bottom edge [denoted by the blue star in Fig. 3(d)], and a microphone scanned the whole sample to obtain both the amplitude and phase of sound at each resonator [8]. Figure 3(d) shows the measured pressure at 2680 Hz [corresponding to the peak frequency of the x -edge spectrum in Fig. 3(c)]. As can be seen, the sound wave is mainly localized at the edge. The pressure at both sides of the source is found to be similar, which reflects the fact that edge bands have both positive group velocity and negative

velocity branches. We note that the excited edge modes decay fast along the edge due to the background loss caused by material absorption, which is a common issue in coupled acoustic resonator lattices. Furthermore, the edge dispersion can be directly visualized by performing Fourier transformation on the measured field distribution data. As can be seen from Fig. 3(e), the Fourier spectrum (color map) agrees well with the simulated edge dispersion (solid white lines). These experiment results in both real space and momentum space, together with the transmission spectra, confirm the existence of the Möbius edge band.

We also implemented a different dimerization pattern, following that in Fig. 1(c), which leads to a graphenelike semimetal phase. The acoustic unit cell with this alternative dimerization is shown in Fig. 4(a), where the coupling tubes enabling weak and strong couplings have widths $w_{c1} = 3.2$ mm and $w_{c2} = 5$ mm, respectively. The corresponding bulk dispersion, as given in Fig. 4(b), clearly shows the emergence of two twofold nodal points on $M'-M$. As discussed above, these two Fermi points, similar to the ones in graphene, carry nontrivial topological charges that lead to edge bands connecting the projections of the two Fermi points. To see this, we simulated a strip with the periodic boundary condition along x and the open boundary condition along y . As shown in Fig. 4(c), we indeed observe edge bands (red curves) connecting the projections of the Fermi points. We again conducted acoustic pressure measurement on the edge resonators under an edge

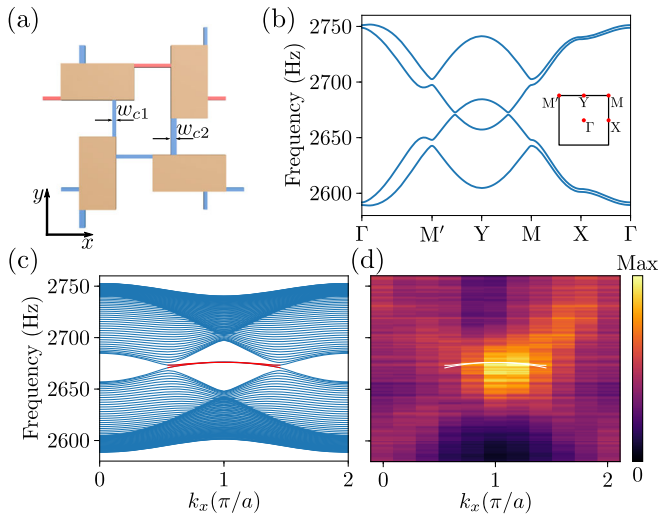


FIG. 4. (a) Unit cell of the acoustic lattice with alternative dimerization. (b) Simulated bulk bands for $w_{c1} = 3.2$ mm and $w_{c2} = 5$ mm. (c) Simulated dispersion for a lattice with the open boundary condition along the y direction but the periodic boundary condition along the x direction. There are edge states (colored in red) connecting the projections of the Fermi points. (d) Experimentally measured edge dispersion. Here, the white lines denote simulated edge bands.

excitation. The resulting Fourier spectrum is plotted in Fig. 4(d). As can be seen, the frequencies and momenta of high-intensity regions match well with the edge band in Fig. 4(c).

In conclusion, we demonstrated a new class of topological phases protected by projective crystal symmetries in an acoustic system. A Dirac semimetal enforced by projective translation symmetries was shown to give rise to a Möbius insulator and a graphenelike semimetal under different perturbations. Our results point to a promising yet unexplored direction to discover novel topological phases. There are various ways to generalize the results in this work. First, while our demonstration is in acoustics, the idea can also be realized in other classical systems utilizing electromagnetic waves and elastic waves [14–16] or, similarly, in electric circuits [34]. Second, it would be interesting to explore more possibilities by increasing the system dimension and considering other types of lattices and projective symmetries [10,12]. Besides, our system also provides a natural platform to study the effect of non-Hermiticity through introducing deliberately designed dissipation [35].

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Note added.—Recently, we became aware of Ref. [36].

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