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Quantum Charging Advantage Cannot Be Extensive without Global Operations

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Quantum batteries are devices made from quantum states, which store and release energy in a fast and efficient manner, thus offering numerous possibilities in future technological applications. They offer a significant charging speedup when compared to classical batteries, due to the possibility of using entangling charging operations. We show that the maximal speedup that can be achieved is extensive in the number of cells, thus offering at most quadratic scaling in the charging power over the classically achievable linear scaling. To reach such a scaling, a global charging protocol, charging all the cells collectively, needs to be employed. This concludes the quest on the limits of charging power of quantum batteries and adds to other results in which quantum methods are known to provide at most quadratic scaling over their classical counterparts.

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Introduction.—In recent years tremendous efforts have been devoted to developing quantum technologies, which are now coming to fruition in several fields of practical use. Among the largest successes is quantum metrology [1], which led to the detection of gravitational waves [2], quantum cryptography [3], which finds applications in communicating sensitive data [4,5], quantum computing, which promises to revolutionize chemistry [6] as well as to speed up or solve important problems in optimization, cybersecurity and data analysis [7], and nanoscale thermodynamic devices, which offer unprecedented precision in thermometry [8]. At large, society is moving toward quantum technologies, because they promise to offer faster, smaller, and more precise devices.

All of these achievements require an efficient way of storing and using energy, as well as fast charging and discharging. The necessity of charging and discharging goes well beyond the quantum world. Examples are electric vehicles where the charging time is one of the main bottlenecks in preventing the widespread use of such technology, or future fusion power plants, in which a large amount of energy needs to be pumped in a short amount of time and discharged in an instant to start the reaction. In the quantum world, nanoscale devices will require nanoscale batteries, with no energy to spare.

Outstanding successes of quantum technologies prompt a question whether quantum effects can also improve the energy storage to satisfy current and future demands. This leads to the notion of quantum battery, which is a quantum mechanical system acting as an energy storage, and in which quantum effects are expected to provide significant advantages over its classical counterpart (see [9,10] for reviews). Starting from the work of Alicki and Fannes [11], the possibility of using quantum effects (like coherence and entanglement) to increase the performance of a quantum battery has been heavily studied. These studies address several figures of merit, such as work extraction [11,12], energy storage [13–17], charging stability [18–21], available energy [22–24] (with the notion of ergotropy [25]), and charging power [26–37], the last being the actual focus of this Letter.

It has been shown [27] that quantum effects lead to a speedup in the charging power of a quantum battery. The source of this quantum speedup lies in the use of entangling operations, in which the cells are charged collectively as a whole. Those operations, where the number of cells that are being entangled together collectively scales with the system size (i.e., creating multipartite entanglement), are called global operations. In contrast, classical batteries are charged in parallel, meaning that each cell is charged independently of each other. The advantage of this collective versus parallel charging is measured by the ratio Γ, called the quantum charging advantage [27]. However, it is still not known how large the quantum advantage is in general. To this end, the best known result is [27]

$$\Gamma < \gamma[k^2(m-1)+k], \tag{1}$$

in which γ is a model-dependent constant, k is the maximum number of cells that are collectively charged, while m (called participation number) is the maximum number of parallel charging operations in which a single cell appears.

In principle, this bound allows for a superextensive scaling of the quantum advantage, meaning that the advantage can scale more than linearly with the number L of cells. For example, consider a charging protocol that has a finite and fixed value of k but having all-to-all couplings. In such a case, the participation number of a

given cell is of order $m = \binom{L-1}{k-1} \approx (L-1)^{k-1}/(k-1)!$, leading to a quantum advantage of order L^{k-1}.

This prompted a race toward finding the best possible scaling—the authors of [27] found that the scaling is of order L at most through an extensive numerical search, and proposed a conjecture that this extensive scaling cannot be surpassed. The search for scaling advantages continued in [13,14,28,29,32,38], which also showed at most extensive scaling, but it was later shown that some of these advantages were not caused by genuine quantum effects [39]. A genuine, extensive, quantum advantage was found in [33], in a setup including both global charging operations and all-to-all couplings. The conjecture still held, but remained unproven, together with uncertain role as to which all-to-all interactions play in determining the quantum advantage.

In this Letter, we prove this conjecture, showing that a quantum battery provides at most extensive advantage over classical batteries. Furthermore, we show that this scaling is achievable only via global charging operations, i.e., we show that all-to-all interactions, and more generally, the participation number, does not provide any scaling advantage.

We first provide a general bound (Theorem 1), constraining the maximum charging power achievable with a general quantum battery with any general Hamiltonian, not necessarily realized by L identical cells, thus including also more general cases described in the literature [28]. The conjecture is proven as a consequence of this theorem (Corollary 1), applied to the battery made of identical cells. Together with examples showing extensive advantage [26,33], already found and discussed in the literature, this result concludes the quest for the best possible scaling which can be obtained by quantum batteries.

Setup.—We consider quantum batteries made out of a time independent initial Hamiltonian $\hat{H}$, having discrete spectrum. At time $t = 0$ a possibly time-dependent driving Hamiltonian $\hat{V}(t)$ is turned on and the initial state $\hat{\rho}_0$ is evolved according to the quench

$$\frac{d\hat{\rho}_t}{dt} = -i[\hat{V}(t), \hat{\rho}_t]. \tag{2}$$

The energy stored in the battery, measured by the initial Hamiltonian, changes from $E(0) = \mathrm{tr}(\hat{H}\hat{\rho}_0)$ to $E(t) = \mathrm{tr}(\hat{H}\hat{\rho}_t)$ during time evolution. Charging the battery means reaching large values of $E(t) - E(0)$.

An important figure of merit is the instantaneous charging power of the battery. It is defined as the instantaneous change in the energy stored per unit of time:

$$P(t) = \mathrm{tr}\left(\hat{H}\frac{d\hat{\rho}_t}{dt}\right), \tag{3}$$

where we used that $\hat{H}$ is time independent.

Generally, the instantaneous power is bounded in terms of the commutator between $\hat{H}$ and the driving term

$$|P(t)| \leq \|[\hat{H}, \hat{V}(t)]\| \leq 2\|\hat{H}\|\|\hat{V}(t)\| \tag{4}$$

through the operator norm [40].

However, the driving is often limited in realistic situations. For example, in lattice systems, the interaction couples only nearby sites and therefore $\hat{V}$ transfers energy only between not-too-distant energy levels of the initial Hamiltonian $\hat{H}$. Taking the spectral decomposition of the initial Hamiltonian to be $\hat{H} = \sum_j E_j|E_j\rangle\langle E_j|$, where we assume the energy levels E_j being ordered, we express the driving Hamiltonian as $\hat{V} = \sum_{j,m=1}^{N} V_{jm}|E_j\rangle\langle E_m|$. The limiting property is formalized as follows: we define ΔE as the minimum value such that for all j and m,

$$\text{when } |E_j - E_m| > \Delta E, \quad \text{then } V_{jm} = 0. \tag{5}$$

Thus, it is natural to look for a more precise bound than Eq. (4), taking this common property into account.

Main result.—In the conditions of Eq. (5), we now show that a more stringent bound can be derived.

Theorem 1.—For driving that couples energy levels with at most ΔE energy difference, as expressed by Eq. (5), the instantaneous charging power is bounded as

$$|P(t)| \leq \Delta E\|\hat{V}(t) - v_{\min}(t)\|/2, \tag{6}$$

where $v_{\min}(t)$ is the smallest eigenvalue of $\hat{V}(t)$, and $\|\ \|$ denotes the operator norm.

Hence, the operator norm of the initial Hamiltonian, central in inequality (4), is not the relevant figure of merit. Instead, the crucial quantity is the maximal value of energy (as measured by $\hat{H}$) that can be transferred by $\hat{V}$ in a single time step. While nontrivial to prove, this result is very intuitive. The charging power is the amount of the energy stored in the battery in a single time step. Thus, this change in energy must be bounded by the maximum amount of energy that the driving term can transfer to the system during that time.

The fact that the bound is not given by the operator norm of $\hat{H}$ has a far reaching consequence. As outlined in the Introduction, it has been a matter of active research which combination of initial and driving Hamiltonians can reach a charging power scaling with $\|\hat{H}\|$. Theorem 1 shows that to reach this scaling one needs to consider driving terms having nonvanishing matrix elements between the ground state and the highly excited states of $\hat{H}$. This latter property defines the so-called global charging operations which we discuss more extensively later. Another point of this bound is that it applies to any Hamiltonian $\hat{H}$, even with interacting cells.

Sketch of the proof.—The full proof of the theorem is technically involved. Here, we sketch the main idea, while we refer the reader to the Supplemental Material [41] for details.

We express the commutator between $\hat{H}$ and $\hat{V}$ as an integral of commutators which are more easily and directly bounded. In particular, we define certain operator functions $\hat{h}(e)$ and $\hat{v}(e)$, depending on a continuous parameter e and satisfying

$$[\hat{H},\hat{V}]=\int_{e=0}^{\Delta E}[\hat{h}(e),\hat{v}(e)]de, \tag{7}$$

as well as the following properties:

$$\|\hat{h}(e)\|=\frac{1}{2},\qquad \|\hat{v}(e)\|=\|\hat{V}\|. \tag{8}$$

We apply triangle inequality to Eq. (7) to derive bound $\|[\hat{H},\hat{V}]\|\leq\Delta E\|\hat{V}(t)\|$. Since any number λ commutes with $\hat{H}$, and $\hat{V}'_\lambda=\hat{V}-\lambda$ also satisfies Eq. (5), we can make this bound tighter by minimizing over λ,

$$\begin{aligned}\|[\hat{H},\hat{V}]\|&=\inf_\lambda\|[\hat{H},\hat{V}'_\lambda]\|\leq\inf_\lambda\Delta E\|\hat{V}-\lambda\|\\&=\frac{\Delta E}{2}\|\hat{V}-v_{\min}\|.\end{aligned} \tag{9}$$

The theorem then follows from Eq. (4).

Lattice case.—In case of a battery made by cells, each of them given, for example, by a qubit, Theorem 1 provides a much more stringent bound than other known bounds [27].

We consider a battery composed of L identical cells, having initial Hamiltonian

$$\hat{H}=\sum_{l=1}^{L}\hat{H}^{(l)}, \tag{10}$$

where $\hat{H}^{(l)}=\hat{I}\otimes\cdots\otimes\hat{I}\otimes\hat{H}_s\otimes\hat{I}\otimes\cdots\otimes\hat{I}$ and $\hat{H}_s$ is the single site Hamiltonian at the lth place. We charge this battery by turning on the driving Hamiltonian,

$$\hat{V}(t)=\sum_{\mathbf{i}\in K(L,k)}\hat{V}_{\mathbf{i}}(t), \tag{11}$$

where, by definition, each term in the summation couples together at most k cells. Expressed mathematically,

$$\begin{aligned}&K(L,k)=\bigcup_{n=1}^{k}C(L,n),\\&C(L,n)=\{(i_1,\ldots,i_n)|i_1<\cdots<i_n\quad\text{and}\\&\qquad i_j\in\{1,\ldots,L\}\},\end{aligned} \tag{12}$$

where $C(L,n)$ is a set of all combinations of n sites, and $V_{\mathbf{i}}$ acts as an identity on the site which does not appear in the index, i.e., for any local matrix $\hat{M}^{(l)}=\hat{I}\otimes\cdots\otimes\hat{I}\otimes\hat{M}\otimes\hat{I}\otimes\cdots\otimes\hat{I}$, where $\hat{M}$ is at the lth place, if $l\notin\mathbf{i}=(i_1,\ldots,i_n)$, then $[\hat{M}^{(l)},\hat{V}_{\mathbf{i}}(t)]=0$. The cases with $k\propto L$ are called global operations. The corollary follows.

Corollary 1.—For initial and driving Hamiltonians (10) and (11), the instantaneous charging power is bounded as

$$|P(t)|\leq k\|\hat{H}_s-E_{s\,\min}\|\|\hat{V}(t)-v_{\min}(t)\|/2, \tag{13}$$

where $E_{s\,\min}$ is the single cell ground state energy.

The result is proven by showing that the maximum energy jump ΔE in this case is given by $k\|\hat{H}_s\|$. As shown in the Supplemental Material [41], the corollary then follows directly from Theorem 1.

The consequences of the bound in Eq. (13) are remarkable. In particular, it rules out the possibility of having extensive quantum charging advantage without global charging operations.

To show that, we need to discuss each of these terms separately: as explained above, k is the number of cells being coupled together by the driving, and thus $k\leq L$. $\|\hat{H}_s-E_{s\,\min}\|$ is a number that depends on particulars of a single cell but does not scale with L. The last term, $\|\hat{V}(t)-v_{\min}(t)\|$, which we call potential (in analogy with electric circuits), can be in principle made arbitrarily large. Physically, this would correspond to investing a very large or infinite energy into the driving. With larger driving energy, the charging is faster. Thus, to compute the quantum charging advantage, we need to compare the parallel and quantum scaling on an equal footing, by assuming that the energy scale that is invested into the driving is the same in both cases. We do that by fixing the potential of the quantum driving to be at most equal to the potential of the parallel driving, $\|\hat{V}-v_{\min}\|\leq\|\hat{V}^{\|}-v^{\|}_{\min}\|$. This is the constraint $C0$, introduced and argued for in Ref. [27].

Parallel charging is given by $k=1$ in driving Hamiltonian (11), while the initial state is assumed to be a product state $\hat{\rho}=\hat{\rho}_s^{\otimes L}$. Thus, in the parallel charging scenario the driving affects each cell independently. From this, we easily calculate that both the potential $\|\hat{V}^{\|}-v^{\|}_{\min}\|=L\|\hat{V}^{\|}_s-v^{\|}_{s\,\min}\|$ and the charging power $P^{\|}=LP^{\|}_s$ scale linearly with L. $\hat{\rho}_s$, $\|\hat{V}^{\|}_s-v^{\|}_{s\,\min}\|$, and $P^{\|}_s$ denote the state, potential, and charging power of a single cell, respectively.

Combining Eq. (13), constraint $C0$, and the results for parallel charging, we bound the quantum advantage as

$$|\Gamma|=\frac{|P|}{|P^{\|}|}\leq\frac{k\|\hat{H}_s-E_{s\,\min}\|\|\hat{V}^{\|}_s-v^{\|}_{s\,\min}\|L}{2|P^{\|}_s|L}=\gamma k, \tag{14}$$

where γ is L and k independent. Thus, the quantum advantage scales with the maximum number k of cells that are coupled together by $\hat{V}$. If this number does not scale with the lattice size L, then the quantum advantage cannot scale with L, as extensively foreshadowed in the

Introduction. The extensive scaling is possible only for global interactions, $k \propto L$. By showing that the only source of the quantum advantage comes from the global entangling operations, we showed that the advantage comes from genuine quantum effects. This addresses the discussion of the role of quantumness posed in relation to the bound on charging power found in [39].

Finally, we ask what is the maximal scaling of power with L that a quantum charging protocol can achieve. Clearly, using Eq. (13), the maximum charging power is given by the product of k, and by whatever scaling can be constructed from $\|\hat{V} - v_{\min}\|$ (for now leaving constraint $C0$ behind). It is possible to artificially construct some driving Hamiltonians that scale superextensively, i.e., with higher powers of L [28,29]. However, such models are unphysical [33,39], because they would lead to a free energy that is superextensive in the thermodynamic limit. Therefore, for any physical model, considering extensive energy $\|\hat{V} - v_{\min}\| \sim L$, the maximal charging power scales at most quadratically $P \sim L^2$, for global operations $k \propto L$. (Compare with the linear scaling of parallel charging.)

As an illustrative example, we study charging of a quantum battery by means of a driving Hamiltonian obtained via a simple generalization of the celebrated Sachdev-Ye Hamiltonian [42], i.e., a random, two-local, all-to-all Hamiltonian

$$\hat{V} = C \sum_{i<j}^{L} \sum_{\alpha=x,y,z} J_{ij}^{\alpha} \hat{\sigma}_i^{\alpha} \hat{\sigma}_j^{\alpha}, \tag{15}$$

where the coupling constants J_{ij}^{α} are randomly extracted from a normal distribution and the normalization factor C is chosen such that $\|\hat{V} - v_{\min}\| = 2$, to ensure a fair comparison between different realizations (instances). ($C \propto L^{-3/2}$, which for the Sachdev-Ye Hamiltonian follows from the replica formalism [42,43]. We numerically confirm this scaling in the Supplemental Material [41].) The results are shown in Fig. 1(a). We clearly see that the power is bounded by the degree of k locality and not by the participation number. As a result, we do not find any extensive charging advantage for this model as expected. Interestingly, we observe that both the maximum power as well as the maximum value of the commutator norm

$$\|[\hat{H}, \hat{V}]\|$$

slightly decreases with the system size L. This is a finite size effect which sensitively reduces by further increasing the system size. We present an analysis of this phenomenon in the Supplemental Material [41].

Does global charging always lead to an extensive quantum advantage?—The presence of a global charging term in $\hat{V}$ does not guarantee an extensive charging advantage.

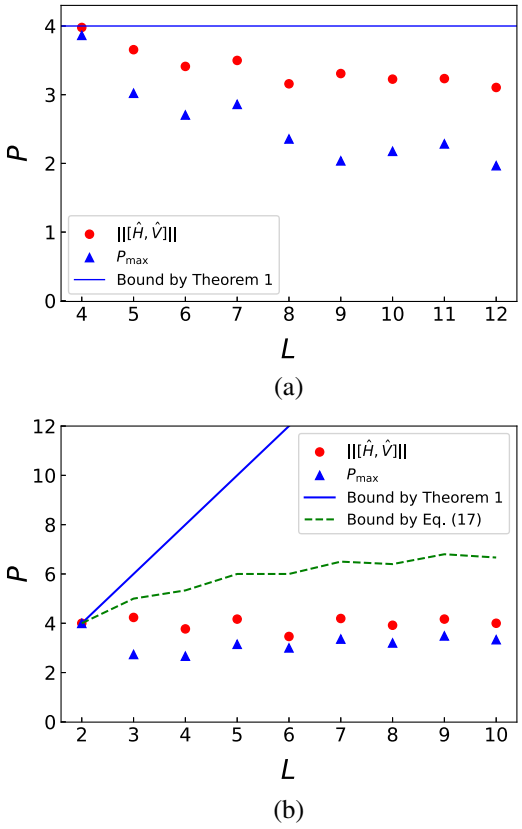

FIG. 1. (a) Maximum power $P_{\max}$ and maximum operator norm of the commutator $\|[\hat{H}, \hat{V}]\|$ as a function of L (maximized over all time and 500 realizations of disorder for each value of L), starting from the ground state of initial Hamiltonian $\hat{H} = \sum_{l=1}^{L} h\hat{\sigma}_l^z$ and charged by driving Hamiltonian as in Eq. (15). For this example, we fixed $h = 1$. We observe a slight decrease in $P_{\max}$ and $\|[\hat{H}, \hat{V}]\|$ with growing L, which is a result of increasing dimensionality of the system, resulting in the lower chance of the of the initial state to be optimal. While keeping the number of realizations of different driving fixed at 500, which means that upper the bound is more difficult to reach. (b) The same as (a) but for a driving given by Eq. (16). For this example, we additionally fixed $V = 1$.

As an example, consider a battery composed of L qubits having initial Hamiltonian $\hat{H} = \sum_{l=1}^{L} h\hat{\sigma}_l^z$ and charged via the following driving

$$\hat{V} = \frac{V}{\lfloor L/2 \rfloor + 1} \left(\sum_{l=\text{odd}} \hat{\sigma}_l^x \otimes \hat{\sigma}_{l+1}^x + \bigotimes_{l=1}^{L} \hat{\sigma}_l^x \right), \tag{16}$$

with V being a constant. From Theorem 1, we obtain $P \leq 2Lh\|\hat{V}\| = 2LhV$ (using $\|\hat{H}_s - E_{s\,\min}\| = 2h$), due to the second term representing a global operation, which couples all of the sites at the same time. Presence of this global charging term suggests possibility of an extensive charging advantage.

However, in this case an extensive quantum advantage is not reached. This is because the nearest-neighbor terms $\sum_{l=\text{odd}} \hat{\sigma}_l^x \otimes \hat{\sigma}_{l+1}^x$, which provide nonextensive advantage, dominate the interaction. They contribute $V/(1+1/\lfloor L/2 \rfloor)$ while the global term contributes only $V/(\lfloor L/2 \rfloor + 1)$ to the total norm $\|\hat{V}\|$. This subextensive scaling is confirmed by the following, alternative, inequality, which is derived in the Supplemental Material [41],

$$|P(t)| \leq \sum_{k=1}^{L} k \|\hat{V}_k\| \|\hat{H}_s - E_{s\,\min}\|, \tag{17}$$

where $\hat{V}_k$ is the k-local part of $\hat{V}$. From inequality (17) we obtain $P \leq 4\|\hat{V}_2\|h + 2L\|\hat{V}_L\|h = \{[4/(1+1/\lfloor L/2 \rfloor)] + [2L/(\lfloor L/2 \rfloor + 1)]\}Vh \approx 8Vh$ for the present example, which indeed confirms that the power does not display an extensive advantage. As a further confirmation, we explicitly computed the maximum charging power for this driving Hamiltonian (16), reported in Fig. 1(b). We clearly see that the charging power stays well-below the threshold given by Eq. (17).

Discussion and conclusions.—We found a bound on the maximum charging power which can be achieved by charging a quantum battery via an external quench protocol.

This bound shows that the maximum charging power is not dependent on the operator norm of the battery Hamiltonian, by which the amount of charged energy is measured. Instead, it is governed by the maximum energy difference ΔE between two eigenstates of the battery Hamiltonian for which the driving Hamiltonian has a nonvanishing matrix element. In other words, the charging power is limited by the amount of energy that the driving Hamiltonian can add into the battery in a single step, a result which *a posteriori* seems very natural. This bound can be applied to a general quantum battery, described by any Hamiltonian, even those of interacting quantum cells.

When applied to quantum batteries made of L identical cells, this bound provides a limit on how fast they can be charged as compared to classical batteries. The maximum speed by which a quantum battery can be charged depends only on the number k of cells interacting together in a single term. It does not depend on the participation number, which is the number of independent terms in the driving Hamiltonian in which a single cell appears. For example, pairwise interactions can provide a quantum speedup by at most a factor of 2, even in the case of all-to-all couplings, where every cell is connected to every other cell. For a speedup of a factor of k, one needs to consider k-cell interactions, while the maximal speedup of L is achieved for L-particle interactions. While charging power of classical batteries scales linearly with the number of cells ($\propto L$), quantum batteries provide at most quadratic scaling in charging power ($\propto L^2$). This quadratic scaling cannot be reached without global operations. However, the mere presence of global charging operations does not always guarantee an extensive charging advantage, as we demonstrated on an explicit example.

This Letter adds to other results, in which quantum systems provide at most quadratic improvement over the known classical method, like the Heisenberg limit in sensitivity scaling in quantum metrology over the classically achievable shot-noise limit [1,44,45], and Grover's search algorithm [46], which is known to be asymptotically optimal [47].

The bound specifies, for a given battery Hamiltonian and for a given driving, the maximum instantaneous charging power achievable in that particular setup. It does not give any information about the quantum state for which such a power can be achieved. This constitutes an interesting question for future research.

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[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.128.140501 for the explicit proofs of the main theorem and its corollaries, as well as for more details on the all-to-all Hamiltonian analyzed in Eq. (15).

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