

Microscopic Origin of the Quantum Mpemba Effect in Integrable Systems

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The highly complicated nature of far from equilibrium systems can lead to a complete breakdown of the physical intuition developed in equilibrium. A famous example of this is the Mpemba effect, which states that nonequilibrium states may relax faster when they are further from equilibrium or, put another way, hot water can freeze faster than warm water. Despite possessing a storied history, the precise criteria and mechanisms underpinning this phenomenon are still not known. Here, we study a quantum version of the Mpemba effect that takes place in closed many-body systems with a $U(1)$ conserved charge: in certain cases a more asymmetric initial configuration relaxes and restores the symmetry faster than a more symmetric one. In contrast to the classical case, we establish the criteria for this to occur in arbitrary integrable quantum systems using the recently introduced entanglement asymmetry. We describe the quantum Mpemba effect in such systems and relate the properties of the initial state, specifically its charge fluctuations, to the criteria for its occurrence. These criteria are expounded using exact analytic and numerical techniques in several examples, a free fermion model, the Rule 54 cellular automaton, and the Lieb-Liniger model.

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Introduction.—Consider a system in thermal equilibrium with a bath. If we wish to lower the temperature of the system we can gradually change the temperature of the bath to the desired final value with the system following suit. Naturally, the time taken for this process depends on the temperature difference: the greater the initial temperature the longer the time to effect the change. Therefore, it should come as a surprise that if instead the bath temperature is quenched, i.e., changed suddenly, the opposite can happen, an initially hotter system can cool faster. This counterintuitive phenomenon, equally likely to be found in the lab or at home, is known as the Mpemba effect [1]. Originally described in the context of freezing water, the effect boasts a long history stretching back millennia and has since been extended to encompass a number of other systems [2–11]. Despite this, however, debate remains over its precise origin and predictability [12,13]. A general criterion for its occurrence and a full understanding of the phenomenon remain elusive.

In this Letter, we investigate such anomalous relaxation dynamics of a nonequilibrium state, not in the context of a classical, open system as described above but, instead, using an isolated, many-body quantum system possessing a conserved $U(1)$ charge. In this context a natural analog of the Mpemba effect is observed when an initial

configuration that is more charge asymmetric, and hence more out of equilibrium, relaxes faster than a more symmetric one [14]. We note that Mpemba effects have been studied previously in quantum systems; however, these have used thermal states and/or open systems [15–22]. In contrast, herein we study the quench dynamics of zero temperature states in a closed system with the effect being driven instead by quantum rather than thermal fluctuations. We establish two criteria for its occurrence in integrable systems, which are then expressed in terms of the charge probability distribution of the initial state. This, therefore, imbues the quantum Mpemba effect with a level of predictability absent in the classical counterpart. We illustrate our treatment through a number of representative examples including both free and interacting models as well as cases where the Mpemba effect is present and absent.

Setting.—We consider a closed quantum system whose dynamics are governed by an integrable Hamiltonian H and which possesses a $U(1)$ conserved charge Q , i.e., $[H, Q] = 0$. The system is initialized in a broken symmetry state $\rho = |\Psi_0\rangle\langle\Psi_0|$ such that $[\rho, Q] \neq 0$, which is not an eigenstate of H , $[\rho, H] \neq 0$, and then allowed to evolve unitarily according to e^{-iHt} . Since the total system is closed, $\rho(t)$ will never relax to a stationary state.

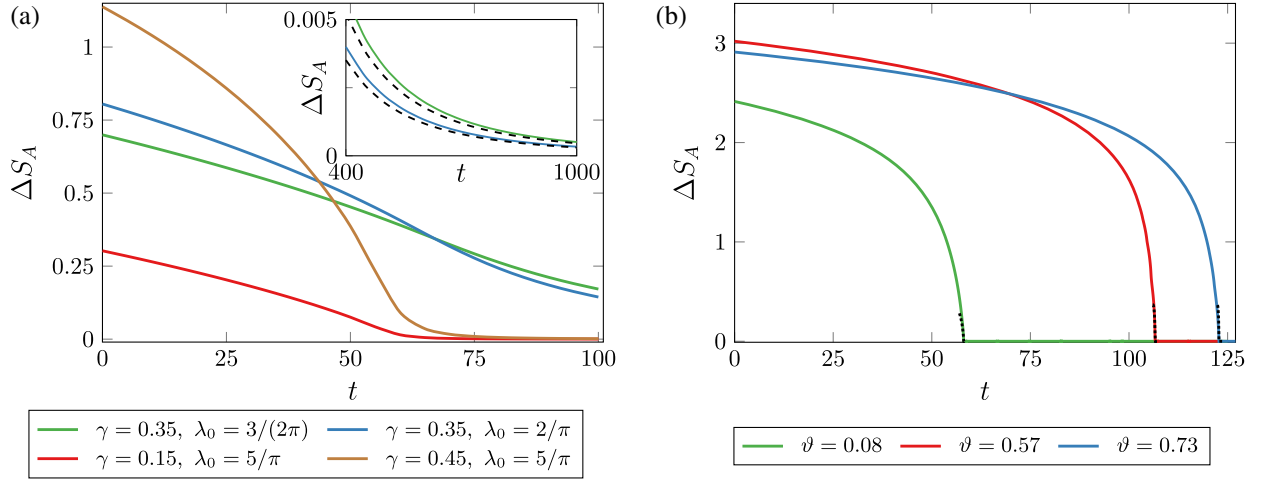


FIG. 1. Evolution of the entanglement asymmetry ΔS_A as a function of time for a subsystem of $\ell = 100$ sites. Different curves correspond to different initial states and the *crossings signal the occurrence of QME*. The figure reports the results for a free fermionic system (a) and for the interacting Rule 54 quantum cellular automaton (b). The results are obtained using Eq. (8). The value of the parameters that determine the occurrence of QME (see main text) are for the curves in (a) $(\mathcal{X}, \vartheta'') = (6 \times 10^{-3}, 0.55)$, $(7 \times 10^{-3}, 0.35)$, $(1.4 \times 10^{-2}, 8.1 \times 10^{-4})$, $(1.7 \times 10^{-3}, 9 \times 10^{-5})$ (in clockwise order according to the legend) and, in (b), $\vartheta = 0.07, 0.25, 0.20$ (from left to right in the legend).

Instead, we study a subsystem A of length ℓ described by the reduced density matrix $\rho_A(t) = \text{tr}_{\bar{A}}[\rho(t)]$ with \bar{A} being the complement of A . We denote the restriction of Q to the subsystem by Q_A , and its average by $q_0 = \text{tr}[\rho_A(0)Q_A]$. This restriction to the quench dynamics of a subsystem results in nontrivial behavior of the charge fluctuations throughout A due to the broken $U(1)$ symmetry in the initial state. Under generic circumstances, however, we expect that locally the system relaxes to a stationary state [23–29], and, barring some exotic instances [30–32], that this stationary state restores the $U(1)$ symmetry, i.e., $\lim_{t \rightarrow \infty} [\rho_A(t), Q_A] = 0$. This symmetry restoration can therefore serve as a proxy, albeit a coarse-grained one, of the relaxation to the stationary state and thus allows us to study the quantum Mpemba effect (QME) in a closed quantum system. One can compare different broken symmetry states and determine if states with “more” symmetry breaking can relax faster.

To quantify the notion of symmetry breaking and restoration we use the entanglement asymmetry, $\Delta S_A(t)$, a recently introduced measure that distills the interplay between the spreading of entanglement and the dynamics of a conserved charge [14,32–37]. The entanglement asymmetry is defined as the relative entropy between two different reduced density matrices: $\rho_A(t)$ and its symmetrized version $\rho_{A,Q} = \sum_q \Pi_q \rho_A(t) \Pi_q$ where Π_q is the projector onto the eigenspace of Q_A with eigenvalue q , namely

$$\Delta S_A(t) = \text{tr}[\rho_A(t)(\log \rho_A(t) - \log \rho_{A,Q}(t))]. \quad (1)$$

Being a relative entropy, $\Delta S_A(t) \geq 0$, and it can be shown that equality is achieved only when $[\rho_A(t), Q_A] = 0$, that is, when the symmetry is restored locally. Given the expected relaxation of the state at long times this translates to $\lim_{t \rightarrow \infty} \Delta S_A(t) = 0$. Using $\Delta S_A(t)$ we can quantify how far a state is from being symmetric and, therefore, also how close it is to the stationary state. For instance, for two states $\rho_{A,j}$, $j = 1, 2$ we say that the symmetry is broken more in $\rho_{A,1}$ than $\rho_{A,2}$ if $\Delta S_{A,1} > \Delta S_{A,2}$ and accordingly the former is further from equilibrium than the latter.

With this in mind the QME is defined as occurring if two conditions are met: (i) $\Delta S_{A,1}(0) - \Delta S_{A,2}(0) > 0$, and (ii) $\Delta S_{A,1}(\tau) - \Delta S_{A,2}(\tau) < 0$, $\forall \tau > t_M$, where t_M is the Mpemba time. That is, while $\rho_{A,1}$ is initially further from equilibrium than $\rho_{A,2}$ it relaxes to the stationary state faster. Figure 1 shows examples of this phenomenon occurring (or not) in both a free and an interacting system.

General treatment.—We aim to compute the entanglement asymmetry and relate conditions (i) and (ii) to properties of the initial state $|\Psi_0\rangle$ thereby obtaining a predictive criteria for the QME. To this end, we start by employing the replica trick and defining the Rényi entanglement asymmetry as

$$\Delta S_A^{(n)} = \frac{1}{1-n} (\log \text{tr}(\rho_{A,Q}^n) - \log \text{tr}(\rho_A^n)), \quad (2)$$

whose limit $n \rightarrow 1$ gives (1). Using the Fourier representation of the projector $\Pi_q = \int_{-\pi}^{\pi} (d\alpha/2\pi) e^{i\alpha(Q_A - q)}$, the moments of $\rho_{A,Q}$ are

$$\text{tr}[\rho_{A,Q}^n(t)] = \int_{-\pi}^{\pi} \frac{d\alpha}{(2\pi)^{n-1}} \delta_p\left(\sum_j \alpha_j\right) Z_n(\alpha, t), \quad (3)$$

where $Z_n(\alpha, t) = \text{tr}[\prod_{j=1}^n (\rho_A(t) e^{i\alpha_j Q_A})]$ are the charged moments and $\delta_p(x)$ is the 2π -periodic Dirac delta function. This form is particularly useful, as the charged moments can be efficiently calculated: both for interacting [38,39] and noninteracting dynamics [14]. In particular, as we show in the Supplemental Material (SM) [40], these quantities take a particularly simple form for $n \rightarrow 1$, which is our limit of interest. For clarity we focus on the case of a single quasiparticle species, although the extension to many is straightforward. In this case, using an emergent quasiparticle picture we find [40,42]

$$\frac{\text{tr}[\rho_{A,Q}^n]}{\text{tr}[\rho_A^n]} = I_n + \mathcal{O}[(n-1)^2], \quad (4)$$

with the definition

$$I_n = \int_{-\pi}^{\pi} \frac{d\alpha}{(2\pi)^{n-1}} \delta\left(\sum_j \alpha_j\right) e^{\ell \sum_{j=1}^n \int d\lambda x_{\zeta}(\lambda) f_{\alpha_j}(\lambda)}, \quad (5)$$

where $x_{\zeta}(\lambda) = \max[1 - 2|v(\lambda)|\zeta, 0]$, $v(\lambda)$ is the quasiparticle velocity, $\zeta = t/\ell$ is the rescaled time variable, and $f_{\alpha}(\lambda)$ specifies the contribution of the mode λ . The precise form of f_{α} is unimportant but we note that it satisfies $f_{\alpha}^*(\lambda) = f_{-\alpha}(\lambda)$ and $f_{\alpha}(\lambda) = f_{\alpha+\pi}(\lambda)$. By using the Poisson summation formula [43] to rewrite the periodic delta function, we obtain the series representation of I_n

$$I_n = \sum_{k=-\infty}^{\infty} J_k^n, \quad J_k = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{i k \alpha} \exp\left[\ell \int d\lambda x_{\zeta}(\lambda) f_{\alpha}(\lambda)\right]. \quad (6)$$

To express the asymmetry ΔS_A , we need to perform the analytic continuation $n \rightarrow z \in \mathbb{C}$. Since the resulting function has to be real for $z \in \mathbb{R}$, we make the minimal choice

$$I_n \rightarrow I_z, \quad J_k^n \rightarrow \frac{1}{2} (e^{z \log J_k} + e^{z \log J_k^*}). \quad (7)$$

With this choice we obtain our first main result: a completely general expression for the entanglement asymmetry

$$\Delta S_A = -\lim_{z \rightarrow 1} [\partial_z I_z] = -\sum_{k=-\infty}^{\infty} \text{Re}[J_k(t) \log J_k(t)]. \quad (8)$$

This formulation allows us to rewrite the conditions (i) and (ii) stated above in terms of properties of the initial state. Namely, by expanding $\Delta S_A(t)$ for large ℓ both at $t = 0$ and $t \gg \ell$ we can express the QME conditions as [40] (i) $J_{q_{0,1}}(0) - J_{q_{0,2}}(0) < 0$, and (ii) $J_{0,1}(t) - J_{0,2}(t) > 0$,

$t \gg \ell$, where $q_{0,j}$ denotes the expectation value of the charge in $\rho_{A,j}(0)$. These conditions have a clean physical interpretation, which we now explain. Originally our first condition required that $\rho_{A,1}(0)$ be more asymmetric than $\rho_{A,2}(0)$ and so it is natural to expect that the fluctuations of charge in $\rho_{A,1}(0)$ be larger than in $\rho_{A,2}(0)$. This expectation is indeed borne out: $J_k(0)$ can be shown [40] to take the following form in the leading order in ℓ :

$$J_k(0) \simeq \frac{1}{\sqrt{\pi\sigma_0^2}} e^{-\frac{(k-q_0)^2}{2\sigma_0^2}}, \quad (9)$$

where q_0 is the expectation value of the charge in the initial state, and σ_0 is the variance of the charge. In other words, $J_{q_{0,j},j}(0)$ is proportional to the probability of measuring the charge equal to $q_{0,j}$ in $\rho_{A,j}(0)$. Therefore, (i) is equivalent to there being a lower probability of measuring the expectation value of the charge in the more asymmetric state. That is, the more asymmetric the state the less peaked its charge probability is about the average. For the second condition, the original statement is that after some time the asymmetry—and therefore the charge fluctuations—is greater in $\rho_{A,2}(t)$ than $\rho_{A,1}(t)$. However, the only way for the subsystem to suppress its charge fluctuations is by transporting charge through its boundaries. Thus, one can expect that in the more asymmetric case the charge fluctuations are transported predominantly by the faster modes and hence allow for faster relaxation. This is encapsulated in the rewritten condition. One can interpret $J_{0,j}(t)$ at long times, $t \gg \ell$, as the probability that slowest modes transport no charge, and hence produce no charge fluctuations. To see this we note that $J_0(0)/\sqrt{2}$ gives the probability of measuring charge $q = 0$ in the initial state [cf. Eq. (9)]. For $t \neq 0$ however the presence of $x_{\zeta}(\lambda)$ in Eq. (6) has the effect of filtering out the contribution of fast modes. In particular, for large times we have

$$\exp\left[\ell \int d\lambda x_{\zeta}(\lambda) f_{\alpha}(\lambda)\right] \approx \text{tr}[\rho_A e^{i\alpha Q_{\text{sl}}}], \quad (10)$$

where Q_{sl} is the charge of the slow modes. Accordingly, since the slowest modes are the last to relax, the charge distribution relaxes faster if they are less involved in the charge transport.

Combining these conditions, we can state that the QME occurs for a class of initial states if, by increasing the charge fluctuations in a subsystem, one simultaneously has the slowest modes carrying less charge and thereby resulting in faster charge transport. The above discussion demystifies the QME, connecting it with the charge distribution of the initial state among the transport modes and constitutes the second main result of this Letter.

Examples.—In the following, we will illustrate and test our predictions in three concrete examples, which are

ordered according to the amount of explicit results we can obtain. In particular, we show how our results allow one to predict all the crossings in Fig. 1.

Example 1.—Let us first consider the simplest possible setting, i.e., free fermionic systems. For concreteness, we focus on a simple model with Hamiltonian $H = \int_{-\pi}^{\pi} d\lambda \epsilon(\lambda) \eta_{\lambda}^{\dagger} \eta_{\lambda}$, where η_{λ}^{\dagger} and η_{λ} are canonical fermionic creation and annihilation operators and $\epsilon(\lambda)$ is the single-particle dispersion relation. H has a single $U(1)$ conserved charge, the total particle number, $Q = \int d\lambda \eta_{\lambda}^{\dagger} \eta_{\lambda}$. The initial configuration is given by a squeezed state, which breaks this symmetry and can be written in terms of eigenstates of the postquench Hamiltonian as $|\Psi_0\rangle = \exp[-\int_0^{\pi} d\lambda \mathcal{M}(\lambda) \eta_{\lambda}^{\dagger} \eta_{-\lambda}^{\dagger}] |0\rangle$. Here, $\mathcal{M}(\lambda)$ is an arbitrary real and odd function and $\eta_{\lambda} |0\rangle = 0$ for all λ . In this case, Eq. (4) applies exactly for any $n \in \mathbb{N}$ with $v(\lambda) = \epsilon'(\lambda)$ and $f_{\alpha}(\lambda) = \log(1 - \vartheta(\lambda) + \vartheta(\lambda) e^{2i\alpha}) / (4\pi)$, where $\vartheta(\lambda) = \mathcal{M}(\lambda)^2 / (1 + \mathcal{M}(\lambda)^2)$ is the quasiparticle occupation function. Given the simple form of $f_{\alpha}(\lambda)$, the integral over α in Eq. (6) can be evaluated (see Sec. III of the SM [40]), the analytic continuation to take the limit $n \rightarrow 1$ is unambiguous and the full time evolution of ΔS_A can be studied with Eq. (8). The latter is computed numerically by truncating the series to produce the plots in Fig. 1(a) and can be used to verify the asymptotic expansions used to obtain (i) and (ii) (explicitly reported with a dotted line in the figure's inset). In particular, for the figure we chose $\epsilon(\lambda) = \sqrt{\lambda^2 + 1}$ and $\mathcal{M}(\lambda) = \gamma(g(\lambda + \lambda_0) + g(-\lambda - \lambda_0) - g(\lambda - \lambda_0) - g(-\lambda + \lambda_0))$ with $g(x) = 1/(1 + e^{-4(x-1/\pi)})$.

The conditions for QME are easily specialized to the noninteracting case. Specifically, we have $2\sigma_0^2/\ell = \mathcal{X} \equiv \int_{-\pi}^{\pi} d\lambda \chi(\lambda)$, where $\chi(\lambda) = \vartheta(\lambda)[1 - \vartheta(\lambda)] / (2\pi)$ is the charge susceptibility per mode. Given the specific form of $v(\lambda)$ and $\mathcal{M}(\lambda)$ we obtain

$$J_0 \simeq 1 - \frac{\vartheta''(0)}{48\pi} \ell \Lambda_{\zeta}^3, \quad (11)$$

where $\Lambda_{\zeta} = 1/(2\zeta v'(0))$ determines the window of the slowest modes, i.e., $x_{\zeta}(\lambda) \neq 0$ for $\lambda \in [-\Lambda_{\zeta}, \Lambda_{\zeta}]$. This means that QME occurs if $\mathcal{X}_1 > \mathcal{X}_2$ and $\vartheta_2''(0) > \vartheta_1''(0)$.

Example 2.—Let us now move on to consider what can be regarded as the natural next step from free systems, i.e., the quantum cellular automaton Rule 54 [44]. This is a locally interacting chain of L spin-1/2 variables (or qubits) where the time evolution happens in discrete time steps. Rule 54 is Bethe ansatz integrable [45,46] and, therefore, supports stable quasiparticles. Although these quasiparticles undergo nontrivial elastic scattering, the latter is simple enough to allow a wealth of explicit results that are not available for “generic” interacting integrable models [46–60]; see Ref. [61] for a recent review.

As time is discrete, the dynamics are generated by the evolution operator for a time step rather than by a Hamiltonian. Alternatively, one can think of it as the Floquet operator for a

periodically driven system in continuous time. Its explicit form reads as $\mathbb{U} = \Pi^{\dagger} \mathbb{U}_e \Pi \mathbb{U}_e$, where Π is the periodic one-site shift and $\mathbb{U}_e = \prod_{j \in \mathbb{Z}_L} U_{2j}$. The local operator U_j acts nontrivially only on site j , implementing a deterministic update that depends on the states of the neighboring sites, i.e., $\langle s'_1 s'_2 s'_3 | U_2 | s_1 s_2 s_3 \rangle = \delta_{s'_1, s_1} \delta_{s'_2, s_1 + s_2 + s_3 \pmod{2}} \delta_{s'_3, s_3}$.

Remarkably, this system admits a class of *solvable* states with a completely accessible quench dynamics written as $|\Psi_0\rangle = (\sqrt{1 - \vartheta} |00\rangle + \sqrt{\vartheta} |01\rangle)^{\otimes L/2}$ [59]. Here, $\{|0\rangle, |1\rangle\}$ is the computational basis for one qubit and the parameter $0 < \vartheta < 1$ fixes the quasiparticle occupations. Specifically, recalling the quasiparticles in Rule 54 are specified by a “discrete rapidity” $\nu = \pm$ [46], after a quench from a solvable state we have $\vartheta_+ = \vartheta_- = \vartheta$ [59].

The solvable states are not eigenstates of the $U(1)$ charge $Q = \sum_j (1 - \sigma_{j+1}^z)(2 + \sigma_{j+2}^z + \sigma_j^z \sigma_{j+2}^z) / 4$; therefore, one can study the restoration of the latter, and the possible emergence of QME, by computing the entanglement asymmetry. In particular, employing a quasiparticle picture we find that Eq. (4) holds in Rule 54 provided that the integral over λ is replaced by the sum over the discrete rapidities ν [40]. In this case we obtain $f_{\alpha, \nu} = \log(1 - \vartheta + \vartheta e^{2i\alpha}) / 2$. Performing then the analytic continuation in Eq. (7) we arrive at Eq. (8) where, however, J_k are computed explicitly as

$$J_k = \begin{cases} \binom{\ell x_{\zeta}}{k/2} \bar{\vartheta}^{k/2} (1 - \bar{\vartheta})^{\ell x_{\zeta} - k/2} & k/2 \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}. \quad (12)$$

Here, $\bar{\vartheta} = \min(\vartheta, 1 - \vartheta)$ and $x_{\zeta} = \max[1 - 2v_{\vartheta}\zeta, 0]$ with $v_{\vartheta} = 1/(1 + 2\vartheta)$. Plugging this explicit expression in Eq. (8) we can easily compute the full time evolution of the asymmetry: some representative examples are reported in Fig. 1(b). This can be again used to check that our asymptotic analysis agrees with the exact values (the asymptotic predictions of SM [40] are explicitly shown by the dotted lines).

Moreover, combining (12) with (i) and (ii), we can again find simple conditions for the occurrence of QME. In particular, we have that the variance in Eq. (9) is again written in terms of the charge susceptibility per mode, i.e., $2\sigma_0^2/\ell = \mathcal{X} \equiv \vartheta(1 - \vartheta)$, while Eq. (12) gives $J_0 \simeq 1 + \ell x_{\zeta} \log(1 - \bar{\vartheta})$. A sufficient condition for QME is then $\mathcal{X}_1 > \mathcal{X}_2$ and $v_{\vartheta_1} > v_{\vartheta_2}$. Recalling the definition of the effective velocity the second condition becomes $\vartheta_1 < \vartheta_2$.

Note that in the case of Rule 54 one can independently test the validity of the asymptotic analysis but not of the analytic continuation Eq. (7). Indeed, the ratios $\text{tr}[\rho_{A,Q}^n] / \text{tr}[\rho_A^n]$ are only accessible for $\zeta \leq 1/2$ and $\zeta \geq 3/2$ for generic n [62] and one cannot use the quasiparticle picture to interpolate between them [63].

Example 3.—Lastly, we consider the Lieb-Linger model of interacting bosons. The Hamiltonian is given

by $H = \int_0^L dx b^\dagger(x)[- \partial_x]^2 b(x) + c[b^\dagger(x)b(x)]^2$, where $b^\dagger(x), b(x)$ obey canonical bosonic commutation relations and we take $c > 0$. Again it has a single $U(1)$ charge, the particle number, $Q = \int b^\dagger(x)b(x)$. We quench the system from a coherent state that breaks this symmetry, i.e., $|\Psi_0\rangle = e^{-\frac{1}{2}Ld + \sqrt{db_0}}|0\rangle$, where $b_0 = \int_0^L dx b^\dagger(x)$, $|0\rangle$ is the vacuum and d is the average charge density. The model is integrable and many analytic formulas for the quench from this state have been previously obtained [64] and are collected in the SM along with the expression for f_α [40]. We find that $2\sigma_0^2 = \ell d$, and that the QME always occurs between states with different values of d . Indeed, to check condition (ii) we perform an analysis of J_0 that at long times has a form similar to the free case [40]

$$J_0(\mu) \propto 1 - \frac{\ell \Lambda_\zeta^3}{24} |\vartheta''(0)| \rho'(0), \quad (13)$$

where $\rho'(0)$ is the density of states of the slowest quasiparticles. Unlike the free case, Λ_ζ acquires a state dependence via the interactions and the behavior is governed by $|\vartheta''(0)|$, which is a rapidly decreasing function of d thus verifying that condition (ii) holds at long times.

Conclusions.—In this Letter we provided a microscopic characterization of the quantum Mpemba effect in integrable models by linking it to the properties of the charge distribution in the initial state. Namely, we showed that the effect occurs for two initial states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ if the first state has a broader charge distribution, i.e., is more asymmetric, but stores less charge in the slower modes, making the charge transport faster.

In fact, the connection that we established between QME and the transport properties of the system should not rely on integrability, since the QME does not require the latter to occur [14,37]. In the presence of weak integrability breaking terms resulting in a finite quasiparticle lifetime, or in chaotic systems that have no quasiparticles, one could still relate the QME to a faster transport of charge. More generally, relating the anomalous relaxation to transport properties could also prove to be a fruitful direction in the characterization of the classical Mpemba effect.

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