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Can We Make Sense of Dissipation without Causality?

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(Received 22 November 2021; revised 10 July 2022; accepted 10 August 2022; published 3 October 2022)

Relativity opens the door to a counterintuitive fact: A state can be stable to perturbations in one frame of reference and unstable in another one. For this reason, the job of testing the stability of states that are not Lorentz invariant can be very cumbersome. We show that two observers can disagree on whether a state is stable or unstable only if the perturbations can exit the light cone. Furthermore, we show that, if a perturbation exits the light cone and its intensity changes with time due to dissipation, then there are always two observers that disagree on the stability of the state. Hence, "stability" is a Lorentz-invariant property of dissipative theories if and only if the principle of causality is respected. We present 14 applications to physical problems from all areas of relativistic physics ranging from theory to simulation.

DOI: 10.1103/PhysRevX.12.041001

Subject Areas: Astrophysics, Fluid Dynamics
Particles and Fields

I. INTRODUCTION

Deterministic field theories (such as hydrodynamics, classical electrodynamics, and general relativity) find application in all areas of physics ranging from condensed matter physics to string theory. Recently, the whole area of classical field theory has been receiving a new boost due to experimental advances, such as the discovery of the quark-gluon plasma at the RHIC and LHC [1] and the now commonplace detection of gravitational-wave mergers from compact objects by LIGO, Virgo, and KAGRA [2], which have driven the development of an ever-increasing number of fluidlike theories to describe exotic phenomena of all kinds [3–7]. Most notably, relativistic dissipative hydrodynamics is becoming a standard tool in the study of a host of physical problems, from high-energy physics [8] to astrophysics [9–11].

The search for the "correct" field theory for describing a given phenomenon typically involves formulating a large number of alternative candidate theories, many of which are then ruled out or proven to be equivalent to others. Usually, there is so much freedom in the construction of a phenomenological theory, that it is easy to get lost in the landscape of alternative formulations. For example, there are at least 11 different formulations of relativistic viscous hydrodynamics [12–22], seven formulations of superfluid hydrodynamics [23–29], and six formulations of radiation hydrodynamics [30–35]. However, in a relativistic setting, all this freedom comes at a price: Most of the theories that one can formulate lead to completely unphysical predictions [36]. For example, since flow of energy equals density of momentum, in some (unphysical) theories, a fluid can spontaneously accelerate, departing from equilibrium, and pushing heat in the opposite direction to conserve the total momentum [37,38]. Pathologies of this kind constitute a serious problem for numerical simulations, because unphysical artifacts cannot be separated from physical effects.

Luckily, there is a standard procedure that allows us to the test the reliability of a relativistic theory and rule out a considerable fraction of candidate theories: the causality-stability assessment. The idea is simple: A theory can be considered reliable only if signals do not propagate faster than light (causality [39]) and if the state of thermodynamic equilibrium (or the vacuum, for zero-temperature theories) is stable against (possibly large [49]) perturbations. For decades, there has been a whole line of research devoted to assessing these two properties [50–61]. Unfortunately, the assessment procedure is complicated (especially for what concerns stability), and the proposed theories are much more numerous than those that are, then, effectively tested. It is clear that a universal and easily applicable criterion that can be used to quickly assess if a theory is stable or not would be a breakthrough for the field (which is exactly what this paper provides).

One aspect of the assessment is particularly problematic. When we study the dynamics of small perturbations around the vacuum state, the linearized field equations are the same in all reference frames, because we are linearizing a

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Lorentz-convariant theory around a Lorentz-invariant state. On the other hand, if the unperturbed state has finite temperature (or chemical potential), its total four-momentum defines a preferred reference frame, so that the linearized field equations look different in different frames. This opens the door to a counterintuitive fact: At finite temperature, the equilibrium state may be stable in one reference frame but unstable in another one. This paradox is possible only because different observers impose their initial data on different constant-time hypersurfaces and hence deal with different initial-value problems [38,62,63]. The result is that one needs to test the stability of the equilibrium in all reference frames to be sure that a theory really makes sense. This is unfortunate because the stability analysis in a reference frame in which the system is moving can be very cumbersome (the background is anisotropic).

The goal of this paper is to finally resolve the paradox of systems that are stable in one reference frame and unstable in others. We prove that this can happen only if the principle of causality is violated. The intuition behind this fact is that two observers can disagree on whether a perturbation is growing or decaying only if (by relativity of simultaneity [64]) the perturbation can be chronologically reordered so that the two observers disagree on which part of the perturbation is in the past and which is in the future. Since this can happen only if the perturbation propagates outside the light cone, it follows that one needs to violate causality if one wants to have two observers disagreeing on a stability assessment. This simple idea, once formulated in mathematical terms, will result in two theorems, according to which causal theories that are stable in one reference frame are also stable in any other frame.

In the following, we first describe the physical setup of the problem and the general physical mechanisms at the origin of the instability of dissipative theories. We then rigorously prove the main result in Sec. III. A reader not interested in the technical details may, however, skip to Sec. IV, where we provide a simple argument that summarizes the essence of the whole paper, or directly to Sec. V, where we present 14 examples of concrete applications of our results to theories that are commonly used in a number of fields.

In case some readers wish to have a brief summary of how the relativistic stability assessment usually works, they can see Appendix A for a quick overview. Particular emphasis is given to the differences with the nonrelativistic case. The mathematical and logical foundations of the method are laid out in Ref. [36].

Throughout the paper, we adopt the signature $(-,+,+,+)$ and work in natural units $c=1$. The spacetime is Minkowski, with metric g; we use global inertial coordinates generically denoted by x^a (so that $\nabla_a = \partial_a$). Finally, all observers are inertial observers; i.e., they do not accelerate and they do not rotate.

II. SOME PERTINENT CONTEXT

The idea that there could be a connection between causality violations and instabilities has a long history, which may be summarized in the words of Israel [65]: "If the source of an effect can be delayed, it should be possible for a system to borrow energy from its ground state, and this implies instability." This argument is a restatement of the Hawking-Ellis vacuum-conservation theorem [40], according to which, if energy can enter an empty region faster than the speed of light, then the dominant energy condition is violated, and the energy density may become negative in some reference frame. Unfortunately, these ideas are not applicable to our case, because we are not studying the stability of the vacuum state, but that of a finite-temperature equilibrium state. More importantly, causality violations can occur even in systems that obey the dominant energy condition. For example, take a barotropic perfect fluid with equation of state [66]

$$P(\rho) = \frac{\rho}{3}[1+\sin(\rho^2)], \tag{1}$$

where P is the pressure, and ρ is the energy density, in some fixed units. This fluid is consistent with the dominant energy condition ($\rho > |P|$), but its equations are acausal because the speed of sound $dP/d\rho$ is unbounded above.

Luckily, it is not so hard to modify the idea of Israel, adapting it to our case of interest: We need only to replace "energy" with "entropy" and "ground state" with "equilibrium state" [61]. Let us see in more detail how this works with a simple qualitative argument.

A. Acausality + dissipation = instability?

Imagine that a signal travels between two events p and q, which are spacelike separated, i.e., $g(p-q, p-q) > 0$. By relativity of simultaneity [64], we know that there are some reference frames in which p happens before q, and other reference frames in which q happens before p. Hence, in some reference frames the signal is traveling superluminally from p to q, while in other frames it travels superluminally from q to p.

Now, imagine repeating this experiment, placing between p and q a dissipative medium, which absorbs the signal along the way. Then, the signal is emitted from, say, p. It travels in the direction of q, but it decays before reaching q. But in those reference frames in which q happens before p, we observe that the signal is spontaneously generated in the middle of the medium, it grows without any external influence (nothing happens at q), and travels to p. Thus, the medium is unstable to the spontaneous generation of perturbations. One may argue that this type of perturbation is not really spontaneous, because still we need an emitter or receiver at p for it to occur. However, the argument still works if we send p at spacelike infinity, so that we are left with a medium that absorbs or emits a spacelike beam, which travels from or to infinity.

The idea of the argument above is the same as that of Israel [65]: If the cause of a signal (i.e., p) can be delayed, then the system can spontaneously generate a perturbation, borrowing entropy from the equilibrium state, and reversing the dissipative processes that should, instead, damp the perturbation. This implies instability.

In addition to this qualitative argument, what are the concrete indications that causality and instabilities may be related? Let us provide a brief summary of the present understanding of the causality-stability problem.

B. Breakdown of causality and stability in infrared theories

For deterministic field theories, the principle of causality reduces to a mathematical condition on the field equations: A variation of the initial data in a region of space $\mathcal{R}$ cannot affect the solution outside the future light cone of $\mathcal{R}$ [40–42]; see Fig. 1. If the equations are linear, causality also means that the retarded Green's function has support within the future light cone [43,44]. It turns out that many phenomenological equations in physics are not consistent with this causality criterion and, therefore, allow for superluminal propagation of signals. The best-known example is the diffusion equation $\partial_t T = D\partial_x^2 T$, whose Green's function is

$$\mathcal{G}(t,x) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right), \tag{2}$$

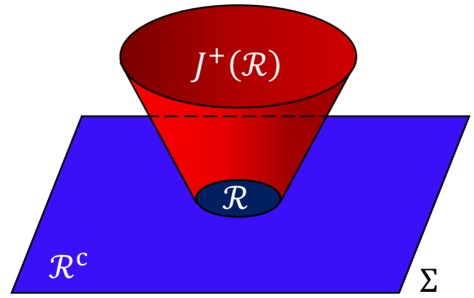

FIG. 1. Geometric visualization of the principle of causality. Take an arbitrary spacelike Cauchy 3D surface Σ. The simplest example of such a surface is the hyperplane $\{t=0\}$. Divide Σ into two regions: $\mathcal{R}$ (dark blue) and $\mathcal{R}^c$ (light blue). "Paint in red" all the timelike and lightlike curves that originate from $\mathcal{R}$ and propagate toward the future. The red paint will cover a set $J^+(\mathcal{R})$ called the "domain of influence of $\mathcal{R}$" or "causal future of $\mathcal{R}$" or "future light cone of $\mathcal{R}$." Causality demands the following: If we compare two arbitrary solutions (of the field equations) whose initial data differ on $\mathcal{R}$ but coincide on $\mathcal{R}^c$, such solutions can differ only inside $J^+(\mathcal{R})$. This is equivalent to saying that information coming from $\mathcal{R}$ can never exit $J^+(\mathcal{R})$.

whose tails extend far beyond the future light cone, propagating energy and information at infinite speeds. Such unphysical violations of the principle of causality usually occur in theories that are the low-frequency limit of some "more complete" causal theories [15,63,67]. This is indeed the case of the diffusion equation that is (at least in ideal gases [14,20,68]) the low-frequency limit of the telegraph equation [69–71], which is known to be causal. The same is true for the Schrödinger equation, which is the acausal low-frequency limit of the Klein-Gordon equation (with the field redefinition $\phi = e^{-imt}\psi$ [72]):

$$\begin{array}{lll} \text{Telegraph:} & (\tau\partial_t^2+\partial_t)T = D\partial_x^2 T & \\ & & \xrightarrow{\partial_t\to 0} \\ \text{Klein-Gordon:} & (\partial_t^2+m^2)\phi = \partial_x^2\phi & \end{array} \quad \begin{array}{ll} \text{Diffusion:} & \partial_t T = D\partial_x^2 T \\ \text{Schrödinger:} & i\partial_t\psi = -\partial_x^2\psi/2m \end{array} \tag{3}$$

For the reason above, causality violations usually occur only on very short timescales, where the predictions of the acausal equation differ from those of its causal progenitor. In other words, causality violations usually happen outside the regime of validity of the "infrared approximation," upon which the acausal equation is built. Hence, one may argue that as long as we manage to keep the high-frequency part of the solutions small, the predictions of the acausal equation should be reliable and the causality violations negligible [73–76].

Unfortunately, in a relativistically covariant context, keeping the acausal high-frequency part of the solutions small is almost impossible (at least in some reference frames) if the equation is acausal and dissipative. The first authors who noticed this issue were Hiscock and Lindblom [36], who verified that any Fick-type diffusion law becomes unstable in some reference frame, due to the fast growth of some unphysical high-frequency modes (see Appendix A for a quick overview of their methodology). A similar mechanism has been observed in several other systems of equations [51–56]: If causality is violated, and the system is dissipative, there is some reference frame in which the system becomes unstable, due to the appearance of fast-growing modes.

The fact that these instabilities usually depend on the frame of reference (i.e., the growing modes exist in some reference frames but not in others) is deeply counterintuitive. Hence, it seemed natural to regard the unphysical growing modes as a mere "mathematical pathology" of the equations. Indeed, acausal field equations often do not present a good Cauchy problem for arbitrary data on spacelike 3D surfaces [43]; hence, it is not surprising that there is some reference frame in which an acausal theory "misbehaves" [19]. However, this does not explain why dissipative systems are so exceptionally problematic: While nondissipative acausal theories (like that considered by Aharonov *et al.* [43]) are singular only when the initial data are imposed on a characteristic surface, dissipative acausal systems are usually unstable in a continuum of reference frames [36]. Hence, one may wonder whether acausality and dissipation are fundamentally incompatible. This is what we aim to understand here.

III. CAUSALITY-STABILITY RELATIONS

We finally reach the central part of the paper. This section is arranged into three subsections, each of which is a separate, stand-alone, result. In particular,

(1) In Sec. III A, we present a more rigorous version of the argument given in Sec. II A, according to which, if a system is acausal and dissipative, then there is a reference frame in which it is unstable. Although linearity of the equations is never invoked explicitly, this argument is expected to be particularly useful for linear stability analyses (we also provide a concrete example in the Supplemental Material [77]).

(2) In Sec. III B, we present the following theorem: If a localized deviation from equilibrium decays over time uniformly in one reference frame, and its support does not exit the light cone, then it decays over time in all reference frames. This theorem is valid for both linear and nonlinear field equations.

(3) In Sec. III C, we present another theorem: If (in the linear regime) a causal theory predicts the existence of a growing sinusoidal plane-wave solution in one reference frame, then this theory is linearly unstable in all reference frames.

Combined, these results should lead us to a simple stability criterion: A dissipative theory which is stable in one reference frame is causal if and only if it is stable in all reference frames. Note that this "causality-stability relation" is strongly corroborated by all the explicit stability analyses that have been performed till now (which the author is aware of) for many different theories, including the Israel-Stewart theory (both in the Eckart [51] and in the Landau [52] flow frame), divergence-type theories [53], Geroch-Lindblom theories [54], inviscid theories for heat conduction [78], first-order viscous hydrodynamics [18,56], second-order viscous hydrodynamics [55], third-order viscous hydrodynamics [59], and Carter's multifluid theory [79].

Since the three arguments presented in this section are stand-alone, in each subsection we work under slightly different assumptions (e.g., in Sec. III B, we deal with nonlinear deviations from equilibrium with compact support, whereas in Sec. III C, we study a linear plane wave with infinite support). However, there are three fundamental ideas that remain the same across the whole paper:

(i) "Causality": Information cannot exit the light cone [40–44].

(ii) "Instability": There is a reference frame in which deviations from equilibrium can grow in time [36].

(iii) "Dissipation": There is a reference frame in which deviations from equilibrium decay in time.

Eventually, this fact will allows us to construct a simple "unified argument" (in Sec. IV), which combines the three main results of this section.

A. Acausal dissipative systems are not covariantly stable

We consider a small perturbation that is traveling superluminally across a medium, disturbing the equilibrium state and violating causality. We assume that such a perturbation can be modeled as a localized wave packet (like a sound pulse), which moves along a spacelike worldline. If the wave packet is highly oscillating (ultraviolet limit), such a worldline is a characteristic of the field equations. Let us also assume that there is an observer A (say, Alice), in whose reference frame the system exhibits a dissipative behavior. Since the unperturbed state is the equilibrium state, a reasonable definition of "dissipative behavior" is that all localized perturbations eventually decay to zero for large times. Hence, we can require that, in the reference frame of Alice, the intensity of the perturbation is a decreasing function of time. The Minkowski diagram of this process is presented in Fig. 2 (left panel).

Now we immediately see the problem: Since the perturbation is traveling along a spacelike path, which part of this path happens "earlier" and which happens "later" depends on the frame of reference. Hence, we can surely

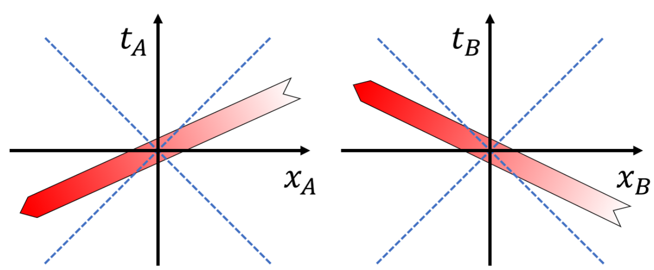

FIG. 2. Minkowski diagrams of the argument outlined in Sec. III A. Reference frame of Alice (left panel): the perturbation moves superluminally from the left to the right, and its intensity decreases with time as a result of dissipation. Reference frame of Bob (right panel): the perturbation moves from the right to the left, and its intensity grows with time. The two points of view are connected by a Lorentz boost. The shades of red are a color map of the intensity of the perturbation (red large, white small); the arrows have the orientation induced by φ (see the main text); the blue dashed lines are the light cone.

find a second observer B (say, Bob) in motion with respect to Alice, in whose reference frame the perturbation is growing in time (Fig. 2, right panel). Let us show it analytically.

At each point p of the spacelike worldline drawn by the center of the wave packet, we may quantify the intensity of the perturbation using a Lorentz scalar $\varphi(p)$ [80]. The inverse of the relation $\varphi(p)$ defines a Lorentz-invariant parametrization on the worldline: $p(\varphi)$. Using this parametrization and approximating the worldline to a straight line passing through the origin, we can write a relation of the form $x_A(\varphi) = wt_A(\varphi)$, with $w > 1$ (spacelike condition). If we boost this relation to Bob's frame, we obtain

$$t_B(\varphi) = \gamma(1 - vw)t_A(\varphi), \tag{4}$$

where v and γ are the boost's velocity and Lorentz factor. Taking the derivative of Eq. (4) and inverting the result, we find

$$\frac{d\varphi}{dt_B} = \frac{1}{\gamma(1 - vw)} \frac{d\varphi}{dt_A}. \tag{5}$$

We see that if $w^{-1} < v < 1$, then the sign of $d\varphi/dt_B$ is opposite that of $d\varphi/dt_A$. Thus, if the perturbation is damped in the reference frame of Alice ($d\varphi/dt_A < 0$), it grows in the reference frame of Bob ($d\varphi/dt_B > 0$), meaning that the equilibrium state is unstable in Bob's frame.

We can draw several conclusions from the argument above. First of all, we see that the instability can occur only if the system is both acausal and dissipative. In fact, if it were causal, then $w \leq 1$, and the factor $1 - vw$ would always be positive; if it were nondissipative, then $\varphi = \text{const}$, and Eq. (5) would reduce to the identity $0 = 0$. It is also immediately explained why the reference frames in which the system is unstable form a continuum: They are all those reference frames in which the chronological order of the events inside the perturbation is inverted with respect to the chronological order perceived by Alice. Finally, by looking at Eq. (5), we see that the instability is most violent close to $v = w^{-1}$, namely, at the unstable-to-stable transition frame, where one has $d\varphi/dt_B = \infty$. This is a well-known feature of this kind of instability: Rather than the growth rate, it is the growth time (the inverse of the rate) that changes sign smoothly as we move from an unstable to a stable frame of reference [36,38,63]. In the Supplemental Material [77], we apply this argument to the superluminal telegraph equation, showing that one can correctly predict the onset and the quantitative aspects of the instability without performing the whole stability analysis explicitly.

We can also make some additional comments:

(i) When $v > w^{-1}$, the perturbation grows with time in Bob's frame ($d\varphi/dt_B > 0$); hence, we may say that the system looks "antidissipative" in Bob's frame. On the other hand, the obedience to the second law of thermodynamics ($\nabla_a s^a \geq 0$, where s^a is the entropy current) is a Lorentz-invariant property of the system. This implies that the entropy grows also in the reference frame of Bob ($dS_B/dt_B \geq 0$). It follows that, in Bob's frame, the entropy is an increasing function of the intensity of the perturbation:

$$\frac{dS_B}{d\varphi} = \frac{dS_B}{dt_B} \frac{dt_B}{d\varphi} \geq 0. \tag{6}$$

In other words, the equilibrium state is not the maximum entropy state in Bob's frame [81]. The recently discovered connection between instabilities and violations of the maximum entropy principle [38,60,61] can be understood in light of this simple argument.

(ii) It is evident from Fig. 2 that for the argument to be rigorous, the whole shape of the perturbation, and not just its peak, must be drifting superluminally. Hence, our argument cannot be extended to causal systems whose group velocity happens to be superluminal for some specific frequency (like those studied in Refs. [84–86], which can be stable [55]). Only genuinely acausal systems [43] are affected by the present instability mechanism.

(iii) Since the high-frequency wave packets travel on the "acoustic cone" (also known as the characteristic cone) of the field equations [45], we can conclude that the instability appears whenever the hyperplane $\{t_B = \text{const}\}$ is more sloping than the acoustic cone, so that part of the future acoustic cone sinks below the hyperplane. Therefore, if the material is isotropic in the reference frame of Alice, the acausal dissipative theory is unstable in Bob's frame if the hyperplane $\{t_B = \text{const}\}$ is "timelike" with respect to the acoustic metric

$$\tilde{g}^{ab} = g^{ab} + (1 - w^2)u_A^a u_A^b \qquad (u_A^a = \text{Alice's four-velocity}). \tag{7}$$

We explore this point in more detail in Sec. IV.

(iv) The instability mechanism described here differs profoundly from the condensation instability of the tachyon field. In fact, the tachyon field is a causal system [43], which is unstable in all reference frames, whereas here we are dealing with acausal systems, which are stable in some reference frames and unstable in others.

B. Lorentz invariance of dissipation

We have seen that causality violations lead to instabilities. Now we prove that frame-dependent instabilities (namely, deviations from equilibrium that grow in Bob's frame while they decay in Alice's frame) are forbidden if

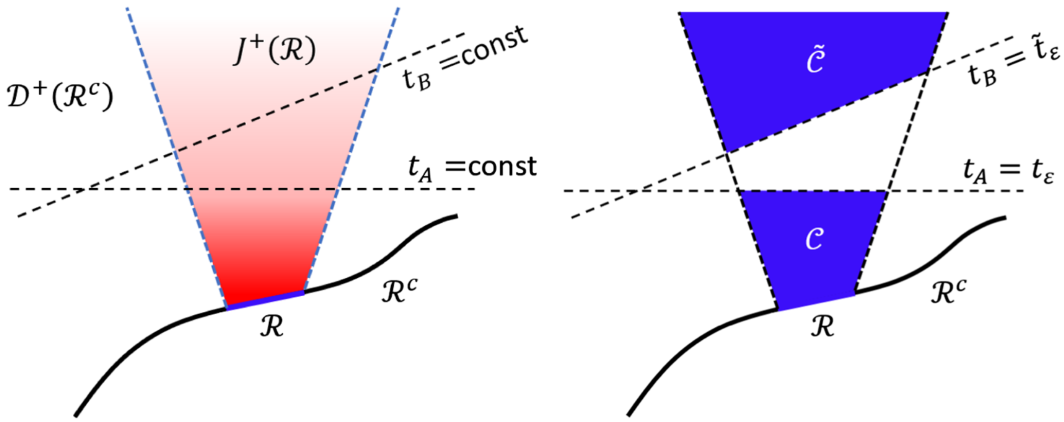

FIG. 3. Left panel: Minkowski diagram of a subluminal perturbation (in Alice's coordinates). The blue segment is $\mathcal{R}$, where the perturbation is initially located; the black line is $\mathcal{R}^c$, where the perturbation is absent. Together, $\mathcal{R}$ and $\mathcal{R}^c$ constitute the initial-data hypersurface Σ. The shaded red region is $J^+(\mathcal{R})$, where φ can propagate. The white region above Σ is $\mathcal{D}^+(\mathcal{R}^c)$, where $\varphi = 0$. The shades of red are a color map of φ (red large, white small). The hyperplanes at constants t_A and t_B are, respectively, the horizontal and oblique lines. Right panel: visualization of the sets $\mathcal{C}$ and $\tilde{\mathcal{C}}$ constructed in the proof of Theorem 1.

the principle of causality is respected. In this section, we focus our attention on a localized (possibly large) "perturbation," namely, a compactly supported deviation of the hydrodynamic fields from their equilibrium value.

Take an arbitrary spacelike Cauchy 3D surface Σ and decompose it into two regions $\mathcal{R}$ and $\mathcal{R}^c$, such that

$$\mathcal{R} \cup \mathcal{R}^c = \Sigma, \quad \mathcal{R} \cap \mathcal{R}^c = \varnothing, \quad \mathcal{R} \text{ is compact.} \tag{8}$$

Using Σ as the initial-data hypersurface, suppose that there is an initial (linear or nonlinear) displacement from equilibrium confined within $\mathcal{R}$. This is what we mean by "localized perturbation." Physically, such a perturbation can be any kind of nonequilibrium phenomenon, like a hot spot, a soliton, a vortex ring, a chemical imbalance, or even an "explosion" (in $\mathcal{R}$). We construct a non-negative scalar field φ, which measures how far the system is from equilibrium at each spacetime event, and vanishes wherever the perturbation is absent (hence, $\varphi = 0$ on $\mathcal{R}^c$). If the theory is well behaving, such a "perturbation-intensity field" (namely, φ) can always be constructed; see Appendix B (a rigorous mathematical definition of perturbation is provided in Appendix B 1). The following definition is natural [40–42]:

Definition 1. (subluminality). The perturbation is subluminal if $\varphi(p) = 0$ for any event $p \in \mathcal{D}^+(\mathcal{R}^c)$, the future Cauchy development of $\mathcal{R}^c$.

An equivalent definition of subluminality is that $\varphi \neq 0$ only on $J^+(\mathcal{R})$ (the causal future of $\mathcal{R}$); see Fig. 3, left panel. Now, if u_A^a is Alice's four-velocity, we can define Alice's time coordinate in a Lorentz-covariant fashion:

$$t_A = -x_a u_A^a. \tag{9}$$

Hence, interpreting t_A as a scalar field, we can define the sets

$$J_A^+(t) \coloneqq \{\text{events } p | t_A(p) \geq t\}. \tag{10}$$

Each set $J_A^+(t)$ is simply the causal future of the hyperplane $t_A = t$. Then, we can make a second definition:

Definition 2. (dissipation). A subluminal perturbation is dissipated in the reference frame of Alice if $\forall\, \varepsilon > 0$ there exists $t_\varepsilon \in \mathbb{R}$ such that $\varphi(p) < \varepsilon$ for any event $p \in J^+(\mathcal{R}) \cap J_A^+(t_\varepsilon)$.

This is a condition of uniform convergence of the perturbation to zero: After a certain time t_ε (in Alice's rest frame), the intensity of the perturbation falls below ε everywhere, and stays below ε for $t_A \geq t_\varepsilon$ (see shades of red in Fig. 3, left panel). Think of ε as the instrumental resolution: At t_ε, the system is back in equilibrium within resolution ε. Analogous definitions can be made for Bob: just replace A with B. We can finally present our theorem:

Theorem 1. (Lorentz invariance of dissipation). If a subluminal perturbation is dissipated in the reference frame of Alice, it is also dissipated in the reference frame of Bob.

Proof.—Let us assume that the subluminal perturbation is dissipated in Alice's frame. Then, taken an arbitrary $\varepsilon > 0$, we can find a time t_ε, future to $\mathcal{R}$, such that $\varphi < \varepsilon$ in $J^+(\mathcal{R}) \cap J_A^+(t_\varepsilon)$. Let $\mathcal{C}$ be the closure of $J^+(\mathcal{R}) \cap [J_A^+(t_\varepsilon)]^c$. Since $\mathcal{R}$ is bounded, $\mathcal{C}$ is compact (see Fig. 3, right panel). On the other hand, t_B is a continuous function; hence, also the image set $t_B(\mathcal{C}) \subset \mathbb{R}$ is compact. This implies that, fixed an arbitrary $\eta > 0$, the real number

$$\tilde{t}_\varepsilon \coloneqq \eta + \max[t_B(\mathcal{C})] \tag{11}$$

exists and is finite. Defined $\tilde{\mathcal{C}} \coloneqq J^+(\mathcal{R}) \cap J_B^+(\tilde{t}_\varepsilon)$, we have that $\mathcal{C} \cap \tilde{\mathcal{C}} = \varnothing$, because

$$\min[t_B(\tilde{\mathcal{C}})] = \tilde{t}_\varepsilon = \eta + \max[t_B(\mathcal{C})] > \max[t_B(\mathcal{C})]. \tag{12}$$

Considering that, by definition, $\tilde{\mathcal{C}} \subset J^+(\mathcal{R}) \subseteq \mathcal{C} \cup [J^+(\mathcal{R}) \cap J_A^+(t_\varepsilon)]$, it follows that

$$\tilde{\mathcal{C}} \subseteq J^+(\mathcal{R}) \cap J_A^+(t_\varepsilon). \tag{13}$$

However, if $\tilde{\mathcal{C}} = J^+(\mathcal{R}) \cap J_B^+(\tilde{t}_\varepsilon)$ is a subset of $J^+(\mathcal{R}) \cap J_A^+(t_\varepsilon)$, then $\varphi < \varepsilon$ in $J^+(\mathcal{R}) \cap J_B^+(\tilde{t}_\varepsilon)$. ■

The essence of the proof can be easily understood by looking at the color map in Fig. 3 (left panel): If the horizontal line $t_A = \text{const}$ is far enough in the future, the field φ becomes arbitrarily small in the shaded region above it. Then, we can always find an oblique line $t_B = \text{const}$ which slices $J^+(\mathcal{R})$ above the horizontal line, as in the figure. In this way, we are sure that φ is small also in Bob's frame for a given time t_B (and for later times).

Figure 3 (left panel) also shows why the condition of subluminality is needed: The lines $t_A = \text{const}$ and $t_B = \text{const}$ always intersect somewhere; hence, an infinite portion of the line $t_B = \text{const}$ lies in the past of $t_A = \text{const}$, where there is no bound on φ. Therefore, if $\varphi \to +\infty$ in the lower left corner of the figure (which is possible only if causality is violated), there is no limit on how large φ can get in Bob's frame. This is exactly what happens in the argument of Sec. III A. On the other hand, causality demands that $\varphi = 0$ outside $J^+(\mathcal{R})$, so that by pushing up the oblique line, we can make sure that $t_B = \text{const}$ is in the future of $t_A = \text{const}$ within the support of φ.

C. Lorentz invariance of linear instability

Theorem 1 deals with nonlinear perturbations, which are initially localized in space. However, in the linear approximation, it is usually convenient to study the evolution of sinusoidal plane waves, which have infinite support. Is there a straightforward analog of Theorem 1 for sinusoidal plane waves?

We work with linear perturbations to a homogeneous stationary state and call $\varphi := \{\delta\psi_i\}$ the array of perturbation fields $\delta\psi_i$. We take a global solution (i.e., a solution that is well defined across all Minkowski spacetime) of the form

$$\varphi = \text{“periodic field”} \times e^{\Gamma_B t_B}, \qquad (\Gamma_B \in \mathbb{R}), \tag{14}$$

where the periodic part is periodic both in space and in time. On hyperplanes $\{t_B = \text{const}\}$, we have $\varphi = \text{periodic field}$, which implies that the perturbation may be a plane wave (i.e., a Fourier mode) in Bob's frame. This is the type of solution that one considers while performing a linear stability analysis in Bob's frame [36,62]. Depending on the sign of Γ_B, the perturbation grows (if $\Gamma_B > 0$), decays (if $\Gamma_B < 0$), or has constant intensity (if $\Gamma_B = 0$) in Bob's frame. Working in Alice's frame, φ is no longer a Fourier mode (unless $\Gamma_B = 0$; see Appendix A 1), but it takes the form

$$\varphi = \text{periodic field} \times e^{\Gamma_B \gamma (t_A - v x_A)}. \tag{15}$$

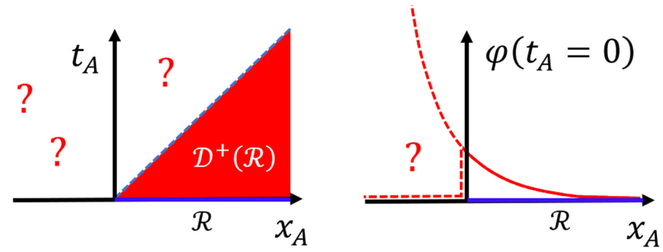

FIG. 4. Left panel: an observer inside the red region $\mathcal{D}^+(\mathcal{R})$ cannot know what the initial state of the system was for $x_A < 0$ (at $t_A = 0$). Right panel: therefore, both $\varphi(t_A = 0)$ and $\varphi^\star(t_A = 0) = \Theta(x_A)\varphi(t_A = 0)$ are initial states which are consistent with the data available to such observer; no experiment performed inside $\mathcal{D}^+(\mathcal{R})$ can tell φ and $\varphi^\star$ apart.

We can orient the x_A axis in a way that $v > 0$. Now, let us make two assumptions.

(i) The field equations are causal [40–42].
(ii) The perturbation grows in Bob's frame $\Gamma_B > 0$.

Our goal is to prove that the system is linearly unstable also in Alice's frame.

Consider an event $p \in \mathcal{D}^+(\mathcal{R}) = \{t_A \geq 0\} \cap \{x_A > t_A\}$, where $\mathcal{R}$ is the half hyperplane (see Fig. 4)

$$\mathcal{R} := \{t_A = 0\} \cap \{x_A > 0\}. \tag{16}$$

By causality, $\varphi(p)$ cannot depend on the initial state of the system outside $\mathcal{R}$. In particular, if we consider an alternative solution $\varphi^\star$, whose initial data (for $t_A = 0$) agree with φ on $\mathcal{R}$ and vanishes outside $\mathcal{R}$, i.e.,

$$\varphi^\star(t_A = 0) = \Theta(x_A)\varphi(t_A = 0)$$
$$(\Theta = \text{Heaviside step function}), \tag{17}$$

then we must have $\varphi^\star = \varphi$ on $\mathcal{D}^+(\mathcal{R})$. It follows that (for any $\varepsilon > 0$, $t_A \geq 0$)

$$\varphi^\star|_{x_A = t_A + \varepsilon} = \varphi|_{x_A = t_A + \varepsilon} \propto e^{\Gamma_B \gamma (1-v) t_A} \xrightarrow{t_A \to +\infty} \infty, \tag{18}$$

which means that both φ and $\varphi^\star$ have divergent amplitude at future lightlike infinity (see Fig. 5). Now, it is not so surprising that φ diverges somewhere in the future: In Alice's reference frame, one has $\varphi(t_A = 0) \propto \exp(-\Gamma_B \gamma v x_A)$, which is divergent at $x_A = -\infty$. Indeed, it is well known that if a perturbation has a divergent tail at $t_A = 0$, its later exponential growth cannot be taken as an indication of instability of the field equations [87]. On the other hand, $\varphi^\star$ has a much more “innocent” initial state [88]:

$$\varphi^\star(t_A = 0) = \text{periodic field} \times \Theta(x_A) e^{-\Gamma_B \gamma v x_A}. \tag{19}$$

It is evident that if such a perturbation diverges for later times, the system must be unstable in Alice's frame. We have therefore proven the following theorem:

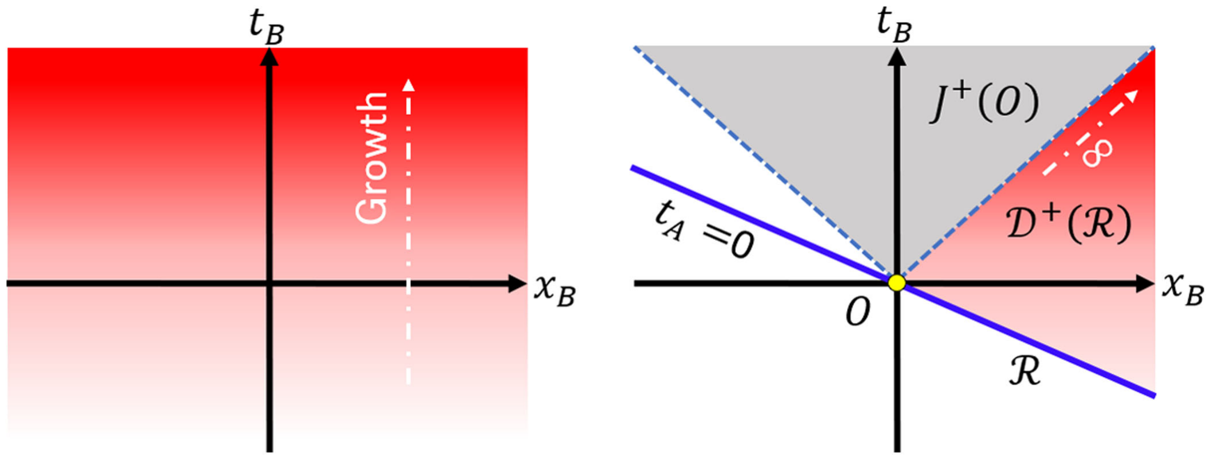

FIG. 5. Minkowski diagram of the two solutions φ (left panel) and $\varphi^\star$ (right panel) in Bob's coordinates. The shades of red are a color map of the perturbation (the oscillatory behavior of the periodic part is averaged out). In the gray area, we do not know the actual intensity of $\varphi^\star$. Left panel: φ is an unstable Fourier mode (i.e., a growing plane wave) in the B frame; it is well defined across the whole spacetime; its oscillation amplitude is constant along hyperplanes $t_B = \text{const}$ (horizontal lines) and grows exponentially for growing t_B ($\varphi \propto e^{\Gamma_B t_B}$). Right panel: $\varphi^\star$ is constructed on the half spacetime $\{t_A \geq 0\}$ by "gluing" initial data at $t_A = 0$. On the right (on $\mathcal{R}$), we take $\varphi^\star(t_A = 0) = \varphi(t_A = 0)$ so that (by causality) $\varphi^\star = \varphi$ on $\mathcal{D}^+(\mathcal{R})$. On the left, we set $\varphi^\star(t_A = 0) = 0$ (hence, $\varphi^\star = 0$ on the respective Cauchy development). In this way, $\varphi^\star$ has a well-defined Fourier transform on $\{t_A = 0\}$, but it diverges on $\mathcal{D}^+(\mathcal{R})$ (in the upper right corner), signaling an instability in Alice's frame.

Theorem 2. (Lorentz invariance of instability). If a causal (linear) theory presents a growing Fourier mode in one reference frame, then it is linearly unstable in all reference frames.

Equivalently, if a causal theory is stable in one reference frame, there cannot be any growing Fourier mode in the boosted frames (analog of Theorem 1 for plane waves). This result generalizes Theorem III of Bemfica *et al.* [18] to linear systems with arbitrary linear field equations. Theorem 2 is also a generalization of the "inverse argument" of Gavassino *et al.* [61] to theories that do not have an entropy current with strictly non-negative divergence, such as DNMR [20] and BDNK [58]. Note that for Theorem 2 to hold, the unperturbed state does not need to be the state of global thermodynamic equilibrium; instead, it may just be a homogeneous and stationary background state.

Finally, let us give a less rigorous but more intuitive explanation of Theorem 2. Assume that, working in Alice's frame, we can split a given solution of the field equations into the product

$$\varphi = (\text{intrinsic growth}) \times (\text{drift}) = e^{\Gamma_A t_A} \times \varphi_D(x_A - w t_A). \tag{20}$$

Stability means $\Gamma_A < 0$; causality requires $|w| \leq 1$. If we assume that $\varphi_D(x_A) = \text{periodic field} \times \exp(-\alpha x_A)$, with $\alpha > 0$, we obtain

$$\varphi \propto e^{-\alpha x_A + (\Gamma_A + \alpha w) t_A}. \tag{21}$$

Consistent with what we said before, we see that the fact that the perturbation grows in Alice's frame ($\Gamma_A + \alpha w > 0$) does not necessarily mean that the theory is unstable ($\Gamma_A > 0$) because a perturbation with an infinite tail (namely, $\varphi = \infty$ at $x_A = -\infty$) can mimic an effective growth by drifting its tail. However, since $|w| \leq 1$, such an effective growth cannot be too large in causal theories. Indeed, if we rewrite the perturbation (15) in the form (21), we find that

$$\Gamma_A = \Gamma_B \gamma (1 - v w) > 0 \quad \text{(by causality)}, \tag{22}$$

signaling instability in Alice's frame. The reader can see the Appendix of Gavassino *et al.* [38] for a similar argument.

IV. ACOUSTIC-CONE ARGUMENT

There is one "global argument," which unifies elegantly all the previous results and gives a clear physical intuition of the underlying mechanism relating acausality and instability.

We start from a well-known fact: The outer characteristics that pass through a spacetime point p bound the domain of influence of p [43,62]. This implies that if we perturb a system at p (e.g., by coupling the field equations with an external source), the induced disturbance will be confined within a conical-like region called the (future) acoustic cone [45,89,90]. In addition, if the unperturbed state is a state of global thermodynamic equilibrium, and if the theory is dissipative, we can assume that the perturbation will be more intense at the tip of the cone (i.e., closer to p), and it will become smaller as we move far away from p.

Let us first consider the case in which the theory is causal. Then, the acoustic cone is contained within (or overlaps) the light cone. Therefore, all observers experience the events in the following order: first p (external source), then the tip of the cone ("intense perturbation"), then the rest of the cone ("damped perturbation"). Hence, all observers will agree that the equilibrium state is stable against perturbations. We recover Theorem 1 (at least qualitatively). Furthermore, if we assume that the source at p excites all the Fourier modes, it is easy to recover Theorem 2.

Let us now move to the case in which the theory is acausal. In this case, a portion of the acoustic cone exits the light cone. Thus, there is an observer (Bob) who measures the perturbation before p has occurred. In Bob's frame, as t_B approaches $t_B(p)$ from below, the portion of the acoustic cone that intersects the hyperplane $\{t_B = \text{const}\}$ gets closer to the tip of the cone; see Fig. 6. This implies that

(i) for $t_B \ll t_B(p)$, the system is at equilibrium (the hyperplane $t_B = \text{const}$ is far from the tip of the cone).

(ii) for $t_B < t_B(p)$, the perturbation grows for increasing t_B.

(iii) at $t_B = t_B(p)$, the perturbation has a peak of intensity.

On the other hand, on the spacetime region $\{t_B < t_B(p)\}$, the perturbation is a solution of the field equations without sources because the only source is located at p. Therefore, we have just shown that there is a solution of the sourceless field equations, with initial data close to equilibrium [for $t_B \ll t_B(p)$], which departs from equilibrium at finite t_B [just before $t_B(p)$]. This behavior is a signature of instability in Bob's frame. We recover the argument of Sec. II A: If the source of a perturbation can be delayed, then the system can spontaneously depart from equilibrium in advance. But we also recover the argument of Sec. III A: Just identify the wave packet of Fig. 2 with the front of the perturbation induced by p (like a discontinuity, the front travels along the boundary of the acoustic cone [51]).

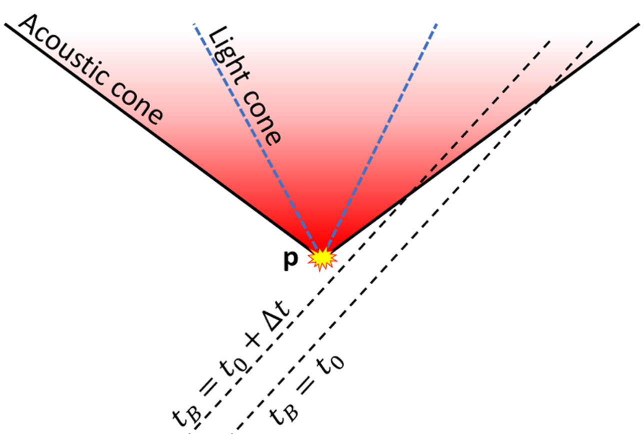

FIG. 6. Minkowski diagram of the "acoustic-cone argument." An external source at p (yellow star) generates a perturbation in the medium, which propagates within the outermost characteristic cone of the field equations (acoustic cone) and decays by dissipation as we move away from the source. If the field equations are acausal, the acoustic cone extends outside the light cone. Hence, there is an observer Bob in whose frame the perturbation exists before p has occurred. On the region $\{t_B < t_B(p)\}$, Bob observes a solution of the sourceless field equations, which is at equilibrium for $t_B \ll t_B(p)$ but grows spontaneously as t_B approaches $t_B(p)$ from below.

At this point, we need to make a clarification. Babichev *et al.* [45] suggested that, if the acoustic cone is larger than the light cone, then one should just use the acoustic cone in place of the light cone to define the causal structure of the spacetime and treat observers like Bob (Fig. 6) as "inappropriate" observers because they are not free to set the initial data at will. In this way, all paradoxes are avoided, and one has a new notion of causality. Their reasoning is valid, but we are working in different contexts. They are interested in what would happen in a universe in which there was some physical field which breaks the general-relativistic notion of causality at the fundamental level: For them, the limitations of Bob are real. On the other hand, here we are assuming that general-relativistic causality is fundamentally valid in our Universe (hence, Bob is physically capable of shaping the system), but we are using a field theory that contradicts such a principle. This is the actual origin of all paradoxes: not equations that break causality, but Cauchy problems that combine acausal theories with initial data on arbitrary spacelike surfaces [62].

A. Example: The boosted heat equation antidiffuses

Using the acoustic-cone argument outlined above, we are finally able to show that the instability of the heat equation in moving reference frames [62] is a consequence of its acausality. To this end, we consider the following thought experiment. A heat-conductive medium is at rest in Alice's frame. For $t_A < 0$, the temperature is everywhere zero. At $t_A = 0$, Alice injects a Dirac delta of energy in the location $x_A = 0$. For $t_A > 0$, the spike of energy diffuses across the medium, according to the heat equation. The temperature field is therefore given by [91]

$$T(t_A, x_A) = \frac{\Theta(t_A)}{\sqrt{4\pi D t_A}} \exp\left(-\frac{x_A^2}{4Dt_A}\right). \tag{23}$$

It can be easily verified (see Rauch [92], Sec. I.7, Problem 3) that this function is indeed a C^∞ solution of the heat equation for all values of t_A and x_A, except at the point $p = (0, 0)$, which is where the spike of energy is injected by Alice. Thus, when we boost to Bob's frame (treating T as a scalar field [93]),

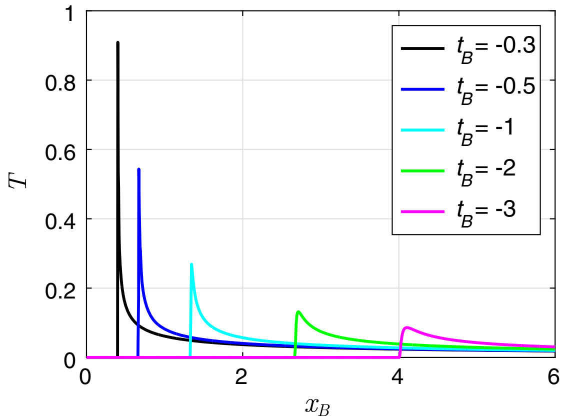

FIG. 7. Boosted Green's function of the heat equation for $t_B < 0$. We set $D = 30$ and $v = 3/4$. Each curve represents a snapshot of $T(t_B, x_B)$, for different choices of t_B. If someone knows the entire history of the system, the interpretation of this figure is quite straightforward: Alice injects a spike of energy at $t_B = x_B = 0$; because of acausality, a portion of such a spike propagates toward the past; as it travels backward in time, the spike diffuses and flattens. On the other hand, to Bob (who cannot predict the decisions of Alice), the situation looks very different. From his perspective, the material is initially in thermodynamic equilibrium (at $t_B = -\infty$). Then, a perturbation builds up spontaneously, developing a superluminal front on the characteristic line $x_B = -t_B/v$. As time goes ahead, the perturbation "antidiffuses," becoming more and more peaked. Eventually, when $t_B \to 0$, the peak diverges at $x_B = 0$. What we are observing is just an inversion of chronology (see Fig. 2).

$$T(t_B, x_B) = \frac{\Theta(t_B + v x_B)}{\sqrt{4\pi D\gamma(t_B + v x_B)}} \exp\left[-\frac{\gamma(x_B + v t_B)^2}{4D(t_B + v x_B)}\right], \tag{24}$$

and we restrict our attention to the spacetime region $\{t_B < 0\}$, we obtain a C^∞ solution of the boosted heat equation. In Fig. 7, we show some snapshots of such a solution.

As we can see, the qualitative behavior of $T(t_B, x_B)$ is consistent with our acoustic-cone argument. Before Alice injects the spike, the temperature is already nonzero in Bob's frame: Heat travels to the past. The characteristic line $x_B = -t_B/v$ (which is just the line $t_A = 0$ expressed in Bob's coordinates) defines the acoustic cone and plays the role of a superluminal wave front. There is a "temperature wave" on the right of such a front, which is initially infinitesimal (for $t_B \ll 0$) and grows with time, "antidiffusing," and becoming more and more peaked. In the end, T develops a singularity at $t_B = 0^-$. The very existence of a solution of this kind tells us that the boosted heat equation is antidissipative and unstable.

But there is more. Let us focus on the infinite strip $\{t_B, x_B\} \in [-1, 0) \times \mathbb{R}$. As we said, T is C^∞ on such a strip. In addition, the right tail of T decays faster than exponentially, while the left tail is identically zero. Therefore, we have constructed a solution of the boosted heat equation, whose initial data at $t_B = -1$ are regular (i.e., smooth and with well-defined Fourier transform), which nevertheless develops a singularity as $t_B \to 0$. It follows that the boosted heat equation must be ill-posed [62]. This fact is not surprising. The boost has inverted the chronology of the heat equation (see Sec. III A), converting it from diffusive to antidiffusive. Hence, the boosted heat equation should share some similarities with the "backward heat equation" $-\partial_t T = D\partial_x^2 T$, which is renowned for its ill-posedness.

V. SOME QUICK APPLICATIONS

As we mention in the Introduction, a relativistic theory should pass three tests to be considered reliable:

(i) causality.
(ii) stability in the background's rest frame.
(iii) stability in reference frames in which the background is moving.

Usually, one is content verifying these properties at least for linear deviations from equilibrium, although in principle conditions i–iii should be valid also in the nonlinear regime.

The main message of this paper is that, once properties i and ii have been tested, assessing property iii is superfluous. In fact, if causality is violated, we know from the argument of Sec. III A that the theory will be unstable (if dissipative). Furthermore, from the acoustic-cone argument of Sec. IV, we are also able to predict exactly in which reference frames the problems appear. If, on the other hand, i and ii are respected, then, by Theorems 1 and 2, iii follows automatically. Below, we list some direct applications of the present results, which span all areas of relativistic physics, including heavy-ion-collision simulations (point 1), accretion-disk simulations (point 2), alternative theories for dissipation (points 3–7), models for turbulent flow (point 8), Chern-Simons magnetohydrodynamics (point 9), and multiconstituent fluids (points 10–14).

(1) Plumberg *et al.* [94] have shown that viscous heavy-ion-collision simulations explore regimes of causality violation. This surely introduces uncertainty, but how much uncertainty? Each discrete time step in a simulation introduces an error, and may "activate" Fourier modes. Picture this error as a small source on the right-hand side of the field equations. As shown in Fig. 6, the effect of a source is dissipated away in those reference frames in which the acoustic cone points entirely toward the future. However, in the remaining frames, it triggers growing modes. Hence, a simulation is really nonreliable if and only if part of the acoustic cone "sinks" below the numerical time-step hypersurfaces. Plotting the acoustic cone will thus show the real entity of the problem (the formula for the acoustic cone can be deduced from the causality analysis of Bemfica *et al.* [95]).

(2) Fragile *et al.* [96] have performed relativistic viscous hydrodynamic simulations of accretion disks, adopting the Landau and Lifshitz [13] theory, which is acausal: The acoustic cone is the normal hyperplane to the fluid's velocity [62]. Thus, our reliability criterion (see point 1) is violated at any point where the flow velocity is not normal to the $3+1$ foliation: These simulations are probably nonreliable. However, the choice of approximating the viscous stress as constant (during the primitive solve) may have had the effect of erasing the second time derivatives, effectively collapsing the acoustic cone upon the foliation, removing the pathologies. This would explain why some of their simulations predict the existence of stable disks, which is surprising given the violence of the acausality-induced instabilities (see Sec. III A). We believe that this issue needs further investigation.

(3) Pu *et al.* [55] have shown that second-order viscous hydrodynamics is stable if and only if it is causal (in the linear regime). An analogous result has been found by Brito and Denicol [59] for third-order viscous hydrodynamics. We are in the position to predict that the same will also be true for higher-order viscous hydrodynamics.

(4) Andersson and Lopez-Monsalvo [57] have formulated a relativistic theory for heat conduction, proving that it satisfies conditions i and ii. Theorem 2 implies that also condition iii is satisfied: The theory is stable.

(5) Stricker and Öttinger [21] have formulated a relativistic viscous theory for liquids. In Ref. [21], they verify that, for some choice of parameters, condition ii is respected. However, we can see from Figs. 1–3 of Ref. [21] that for this same choice of parameters, the front velocity of some Fourier modes is superluminal. Since the signal velocity is not smaller than the front velocity [97], we can conclude that the liquid under consideration violates causality and is therefore unstable in some reference frames.

(6) Ván and Biró [22] have formulated a relativistic theory for viscosity and heat conduction, showing that it respects condition ii. However, upon inspection of the last column of their matrix $\mathbf{R}$ [Eq. (34)], we see that the field equations are not hyperbolic [51], suggesting the presence of causality violations and thus of instabilities. Indeed, if (in $\mathbf{R}$) we impose $\Gamma = \gamma\tilde{\Gamma}$ and $k = i\gamma v\tilde{\Gamma}$ (spatially homogeneous solution in a boosted frame [36]), we find that there is one growing solution for any $v \neq 0$.

(7) Ván and Biró [98] have formulated another theory for viscous hydrodynamics, similar to that discussed above. Unfortunately, it suffers exactly from the same problems as the previous one: The matrix $\mathbf{R}$ [Eq. (38)] models acausal perturbations, which become unstable when boosted.

(8) The Smagorinsky model [99] is a filtered theory for modeling turbulent flows in large eddy Newtonian simulations. Celora *et al.* [100] have shown that if the same approach is lifted to a relativistic setting, the resulting model is not "covariantly stable"; i.e., it satisfies condition ii but not condition iii. Applying Theorem 2, we can conclude that the relativistic Smagorinsky model is acausal.

(9) Kiamari *et al.* [101] have shown that Chern-Simons magnetohydrodynamics is causal but unstable in the rest frame. Using Theorem 2, we can conclude that the theory must be unstable in every reference frame.

(10) Many relativistic fluids can be modeled as reacting mixtures [102–104]. For a perfect-fluid reacting mixture, the rest-frame stability conditions coincide with the "textbook" conditions for thermodynamic stability [60], while the causality condition is simply the requirement that the sound speed at frozen chemical fractions should not exceed the speed of light [10]. Under these assumptions, by Theorem 2, a mixture is stable in all reference frames.

(11) Most models for radiation hydrodynamics assume that there is a matter fluid with stress-energy tensor M^{ab} and a radiation fluid with stress-energy tensor R^{ab}, which interact dissipatively though the equation $\nabla_a M^{ab} = -\nabla_a R^{ab} = G^b$, where G^b is a hydrodynamic force [34,35,105]. Since G^b usually does not depend on the gradients, its presence does not modify the characteristic determinant of the system. Therefore, the causality properties of the two fluids are unaffected by the coupling: If the dynamics of the matter fluid is acausal, the total radiation-hydrodynamic theory will also be acausal. On the other hand, radiation hydrodynamics is dissipative by construction [31,34]. Therefore, invoking the argument of Sec. III A, we can conclude that all acausal fluids become unstable when coupled with radiation through G^b.

(12) The argument above can be easily generalized: Assume that an arbitrary number of fluids and classical fields interact dissipatively through some equations $\nabla_a T^{ab}_n = G^b_n$ (n is an index counting the fluids), where G^b_n does not depend on the gradients. Then, if any of these fluids is acausal (and its dissipative coupling with the other fluids is not zero), the resulting composite system is unstable.

(13) Carter and Khalatnikov [106] have formulated a relativistic theory for superfluid mixtures. The simplest way of implementing dissipation in their theory is by coupling the currents through hydrodynamic forces which do not contain gradients [63] (analogous to the case above). It follows that dissipative superfluid mixtures (and, more in general, "multifluids") are stable only if their nondissipative analog

is causal. The only exception is when the dissipative coupling is mediated by quantum vortices [107,108], in which case the drag force depends nonlinearly on the gradients, changing completely the causal structure of the system.

(14) Superfluid neutron stars exhibit a phenomenon called "entrainment," according to which the superfluid momentum of the paired neutrons is not collinear to the flow of neutrons [24]. If we imagine to remove this effect, the acoustic cone becomes that of Carter's regular theory for heat conduction [109], which can be acausal, for certain equations of state [78]. Hence, the existence of the entrainment may be necessary to guarantee the stability of the equilibrium. The thermodynamic origin of this fact is studied in another work [79].

VI. CONCLUSIONS

In this paper, we identify the physical mechanism that connects causality, stability, and dissipation. Our reasoning can be summarized as follows. First, we abstract from the general notion of "dissipation" its key feature, namely, the existence of a decaying-over-time scalar field (which measures "how large" a perturbation is at a point). Next, we interpret the word "stability" as the statement that all possible observers agree on the fact that such a field is nonincreasing with respect to their proper time. Finally, we set up a simple argument: Suppose that a perturbation moves superluminally (i.e., outside the light cone) and decays over time from the point of view of one observer. Because the perturbation is superluminal, it links causally disconnected spacetime points which can, via a Lorentz transformation, be chronologically inverted, making the decaying quantity appear increasing from the point of view of another observer. In a nutshell, the lack of causality always allows one to transform dissipation into antidissipation (i.e., dissipation backward in time). This also explains why acausal theories always turn out to be thermodynamically unstable [61].

As a concrete example, we study how the retarded Green's function of the heat equation transforms under Lorentz boosts. We find that, due to the relativity of simultaneity, one of its Gaussian tails must always "sink" to the past (no matter how small the boost velocity), so that the boosted Green's function presents an advanced part. This acausal precursor undergoes an inversion of chronology: It antidiffuses instead of diffusing (see Fig. 7). As a consequence, thermodynamics now is time reversed: Spikes tend to pinch (instead of flattening), energy tends to concentrate (instead of spreading), and the medium wants to move away from equilibrium (rather than toward it). That is why the boosted heat equation is unstable [36], antidissipative [63], and ill-posed [62].

With a similar reasoning, we rigorously prove that, instead, if a causal theory is stable in one reference frame, it is stable in all reference frames. The reason is that Lorentz transformations can never invert the chronological order of causally connected events: A decaying subluminal perturbation cannot be Lorentz transformed into a growing one. In other words, causality guarantees that the "thermodynamic arrow of time" points toward the future in all reference frames, not only in the rest frame.

Our analysis reveals that the causality-stability assessment is much easier than we thought, because the boosted-frame stability analysis (which is notoriously the most difficult part) is superfluous. Causality alone takes care of ensuring the Lorentz invariance of a stability assessment, which can just be performed in a preferred reference frame. This result is a more general formulation of Theorem III of Bemfica *et al.* [18] and of the inverse argument of Gavassino *et al.* [61]. The main advantage of our Theorems 1 and 2 is that they do not make any assumption about the structure of the field equations, besides causality.

We also formulate a general criterion based on the notion of an acoustic cone, which allows one to predict exactly in which reference frames an acausal theory becomes problematic. This criterion can be used to understand whether the reliability of state-of-the-art heavy-ion-collision simulations [94] is really compromised by the causality violations of Israel-Stewart-type theories.

This paper clarifies several fundamental aspects of relativistic hydrodynamics and thermodynamics, providing a definitive answer to some old open questions.

(1) What is the "physical interpretation" of the instabilities that we observe in relativistic hydrodynamics? They are just dissipative processes under time reversal. Without causality, there is no absolute notion of chronology, because the "cause" and the "effect" may be exchanged via a Lorentz boost. As a result, the thermodynamic arrow of time may point toward the past for some observers. When this happens, systems evolve away from equilibrium rather than toward it. That is why these instabilities are present in some reference frames and not in others.

(2) Is it possible to make these instabilities small enough to be irrelevant? No. If the beginning and the end of a process can be chronologically reordered via a boost, then there is an intermediate reference frame in which they are simultaneous. In such a frame, the whole process occurs instantaneously. Therefore, one cannot hope that the instabilities will grow "slowly" (for a given acausal theory), because there is always some reference frame in which the growth rate is infinite.

(3) Why does this problem appear only when we turn on dissipation? Because nondissipative theories are

invariant under time reversal (strictly speaking, they are invariant under the action of CPT [110]). Hence, in the absence of dissipation, an inversion of chronology does not produce any observable effect on the laws of thermodynamics.

(4) Is it possible to observe a similar phenomenon in Newtonian physics? No. In Newtonian physics, time (and in particular, chronology) is absolute. As a consequence, the thermodynamic arrow of time is Galilean invariant, and all observers agree on whether a system is stable or not.

Theorems 1 and 2 are also interesting from the point of view of the foundations of relativistic thermodynamics. In fact, the essence of these theorems may be summarized as follows: If a system exhibits a tendency to evolve toward thermodynamic equilibrium in one frame of reference, it exhibits the same tendency in all frames, provided that the principle of causality holds. This suggests that, once thermodynamics is valid in one reference frame, it should "look the same" in all reference frames. This is perfectly in line with our recent proof [82] of van Kampen's argument [111] for the existence of a relativistically covariant theory of thermodynamics. There, causality and stability are implicitly assumed when the concept of "kick" is introduced.

ACKNOWLEDGMENTS

This work is supported by the Polish National Science Centre Grant No. OPUS 2019/33/B/ST9/00942. The author thanks M. Antonelli, B. Haskell, and F. Bemfica for reading the manuscript and providing useful comments. I am particularly grateful to M. Disconzi, whose observations allow me to significantly improve the mathematical rigor of the discussion. My gratitude also goes to the editorial team and the referees of PRX: Their recommendations and criticisms play a crucial role in giving this paper its final form.

APPENDIX A: THE RELATIVISTIC STABILITY ASSESSMENT

Let us compare the Galilean boost with the Lorentz boost (we ignore the variables y and z):

$$\text{Galileo}\begin{cases} t_A = t_B, \\ x_A = x_B + v t_B, \end{cases} \qquad \text{Lorentz}\begin{cases} t_A = \gamma(t_B + v x_B), \\ x_A = \gamma(x_B + v t_B). \end{cases} \tag{A1}$$

In addition to the Lorentz factor $\gamma = (1-v^2)^{-1/2}$, there is an additional term in the Lorentz boost that catches the eye: the position-dependent shift "vx_B" in the relativistic transformation of time. This term is responsible for a counterintuitive phenomenon called "relativity of simultaneity," according to which two events that are simultaneous for one observer ($\Delta t_B = 0$) may not be simultaneous for another observer ($\Delta t_A = \gamma v \Delta x_B \neq 0$). It is this effect that makes the relativistic stability assessment more complicated than its Newtonian counterpart. Let us see why.

1. Boosted Fourier modes are no longer Fourier modes

A physical system is in thermodynamic equilibrium. We perturb it a bit. We expect that after an initial transient, the system will relax back to equilibrium. If instead the perturbation grows with time, we say that the theory is "unstable."

In practice, given a system of partial differential equations, how do we assess the stability of the equilibrium? The standard approach is the same both in Newtonian physics and in relativity, and works as follows. Let us say, for clarity, that we are interested in tracking the evolution of the local temperature $T(t_A, x_A)$ interpreted as a scalar field [93]. For small perturbations, we can work in the linear approximation and expand a generic solution of the field equations as a superposition of sinusoidal plane-wave solutions (Fourier modes). For each of these solutions, the perturbation to the local temperature takes the form below:

$$\delta T(t_A, x_A) = e^{\Gamma_A t_A} \sin(k_A x_A - \omega_A t_A + \phi), \tag{A2}$$

where Γ_A, k_A, ω_A, ϕ are real numbers, which do not depend on the spacetime location. The numbers k_A and ϕ called, respectively, "wave number" and "phase" are treated as free parameters, whereas Γ_A and ω_A called, respectively, "growth rate" and "frequency" are constrained by the equations of motion and depend on k_A. It is evident that if Γ_A is always nonpositive (for all values of k_A), the system is stable, otherwise, it is unstable.

The only difference between a Newtonian stability analysis and a relativistic stability analysis lies in what happens when we change the frame of reference. Our intuition suggests that once we verify that the system is stable in one reference frame, it should be stable in all reference frames. And, indeed, this is true in Newtonian physics. In fact, when we change reference frame (in a Newtonian world), Eq. (A2) is transformed into

$$\delta T(t_B, x_B) = e^{\Gamma_B t_B} \sin(k_B x_B - \omega_B t_B + \phi), \tag{A3}$$

with $\Gamma_B = \Gamma_A$, $k_B = k_A$, and $\omega_B = \omega_A - v k_A$. As we can see, the Galilean boost always maps sinusoidal plane waves into sinusoidal plane waves, with the same wave number and growth rate. Hence, if $\Gamma_A(k_A)$ cannot be positive, neither can $\Gamma_B(k_B)$. However, things change dramatically in relativity. In fact, when we make a Lorentz boost, relativity of simultaneity mixes space with time, and the exponential in Eq. (A2) becomes

$$e^{\Gamma_A t_A} = e^{\Gamma_A \gamma t_B} e^{\Gamma_A \gamma v x_B}. \tag{A4}$$

Because of the extra factor $e^{\Gamma_A \gamma v x_B}$, the wave is no longer sinusoidal in the boosted frame (see Fig. 8) unless $\Gamma_A = 0$. This is telling us that solutions of the form (A2) are

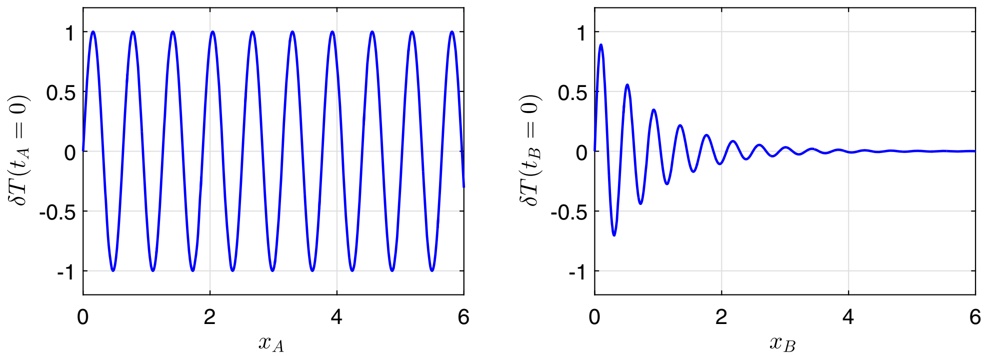

FIG. 8. The same plane wave viewed by two observers in motion with respect to each other. The full spacetime dependence of the temperature perturbation is assumed to be $\delta T(t_A, x_A) = e^{-t_A} \sin(10 x_A)$. In the "(A)" frame, we have a conventional Fourier mode (left panel), whose amplitude decays in time. In the "(B)" frame, there is an exponential modulation also in space (right panel). This is a direct consequence of the relativity of simultaneity: If something decays only in time in one frame, it may decay both in time and space in another frame. We choose the boost velocity $v = 3/4$, which corresponds to the Lorentz factor $\gamma \approx 1.5$.

intrinsically different from the solutions of the form (A3). One is not the boosted version of the other. We cannot even express Eq. (A2) as a superposition of solutions like Eq. (A3) because the factor $e^{\Gamma_A \gamma v x_B}$ has a divergent tail for $x_B \to \infty$ (plus or minus, depending on the sign of $v\Gamma_A$), so that the plane wave (A2) does not have a well-defined Fourier transform in the B frame. This fact can lead to a surprising phenomenon: Sometimes, a system is stable in one reference frame but unstable in another one.

2. The case of the heat equation

The most striking example of how relativity of simultaneity can destabilize a system is the case of the heat equation:

$$\frac{\partial T}{\partial t_A} = D \frac{\partial^2 T}{\partial x_A^2}. \tag{A5}$$

In the A frame, this equation is clearly stable. In fact, if we plug Eq. (A2) into Eq. (A5), we obtain $\Gamma_A = -D k_A^2 \leq 0$. No Fourier mode can grow. However, quite surprisingly, there are unstable Fourier modes in all other frames of reference. For example, consider a solution of the form $\delta T(t_B) = e^{\Gamma_B t_B}$. In the B frame, this is a sinusoidal plane wave, with $k_B = 0$. Physically, it models a configuration with no gradients in space for observer B. Intuitively, we then expect that the only possible solution will be $\Gamma_B = 0$ (no gradients $\Rightarrow$ no heat flux $\Rightarrow$ no temperature changes). However, this is not the case. Because of relativity of simultaneity, δT acquires an exponential profile in the A coordinates:

$$\delta T(t_A, x_A) = e^{\Gamma_B t_B} = e^{\Gamma_B \gamma (t_A - v x_A)} = e^{\Gamma_B \gamma t_A} e^{-\Gamma_B \gamma v x_A}. \tag{A6}$$

As a consequence, when we plug Eq. (A6) into Eq. (A5), we obtain two possible solutions. One is $\Gamma_B = 0$ and the other is

$$\Gamma_B = \frac{1}{D \gamma v^2} > 0. \tag{A7}$$

As we can see, in the B frame, the temperature is allowed to grow uniformly (with no bound), even in the absence of spatial gradients. Note that this is not in contradiction with the Fick law ("fluxes" $\propto$ "spatial gradients"), because in the rest frame of the medium (the A frame) there are gradients. In Sec. IV A. we finally explain why the boosted heat equation must necessarily be unstable.

APPENDIX B: THE PERTURBATION-INTENSITY FIELD

The essence of Theorem 1 is the following: If a causal perturbation with compact support converges to zero uniformly in Alice's frame, it converges to zero uniformly also in Bob's frame. To prove it, we rely only on the existence of a scalar field φ that measures "how large" a perturbation is at a point. The simplest way of constructing φ is the following. Suppose that Alice and Bob are interested in measuring a finite set of relevant scalar observables $\mathcal{O}_n$ (e.g., temperature T, pressure P, chemical potential μ, electromagnetic field strength $F_{ab}F^{ab}$, etc.) at each spacetime event. Then, they may define φ as

$$\varphi = \sum_n [\mathcal{O}_n - \mathcal{O}_n^{\mathrm{eq}}]^2, \tag{B1}$$

where $\mathcal{O}_n^{\mathrm{eq}}$ is the equilibrium value of $\mathcal{O}_n$. Viewed in this light, the theorem tells us that if the differences $\mathcal{O}_n - \mathcal{O}_n^{\mathrm{eq}}$ go below experimental resolution uniformly in Alice's frame, then the same happens in Bob's frame, provided that $\mathcal{O}_n = \mathcal{O}_n^{\mathrm{eq}}$ on $\mathcal{D}^+(\mathcal{R}^c)$.

On the other hand, one would also like to interpret the theorem in a more "mathematical" sense. For a given deterministic field theory with some set of field equations, what are the underlying assumptions that make Theorem 1 applicable? We address this (rather technical) problem in the remaining part of this appendix.

1. What is a perturbation?

We consider the following Cauchy problem:

$$\begin{cases} \mathcal{F}_h(\psi_i, \nabla_a \psi_i, \nabla_a \nabla_b \psi_i, \ldots) = 0 & \text{on } \overset{\circ}{\mathcal{D}}{}^{+}(\Sigma), \\ \psi_i = f_i^{(0)}, n^a \nabla_a \psi_i = f_i^{(1)}, \ldots & \text{on } \Sigma, \end{cases} \tag{B2}$$

where ψ_i are the fields of the theory, Σ is the spacelike Cauchy 3D surface introduced in the main text, n^a is the unit normal to Σ, $\{f_i^{(n)}\}_{i,n}$ is a set of functions on Σ (they constitute the initial data), and $\mathcal{F}_h$ are some tensor-valued functions, which are smooth in all the arguments. We also assume that Σ is smooth, and we restrict our attention to smooth initial data. The following two assumptions are standard [41] but not so easy to guarantee in general [112]:

(i) The Cauchy problem (B2) is globally well posed; i.e.. the solution exists, is unique, and depends continuously on the initial data [across all $\mathcal{D}^+(\Sigma)$].

(ii) The field equations are causal; i.e., if the initial data for $\psi_i^\star$ agree with those of ψ_i on a subset $\mathcal{S}$ of Σ, then $\psi_i^\star = \psi_i$ on $\mathcal{D}^+(\mathcal{S})$.

Now we can define rigorously what we mean by a perturbation. Any state of global thermodynamic equilibrium is modeled in a deterministic field theory as a specific solution of the field equations, with certain properties (e.g., $\nabla_a s^a = 0$ and $\nabla_a \beta_b + \nabla_b \beta_a = 0$ [14]). We simply call ψ_i a solution of this kind, which plays the role of the background equilibrium state. A localized perturbation (of the type considered in Sec. III B) is another solution $\psi_i^\star$, whose initial data agree with those of ψ_i on $\mathcal{R}^c$, but differ on $\mathcal{R}$ (there is no need for $\psi_i^\star$ to be "close to ψ_i" inside $\mathcal{R}$). Then, by causality, we know that

$$\psi_i^\star = \psi_i \quad \text{on } \mathcal{D}^+(\mathcal{R}^c). \tag{B3}$$

The final step consists of constructing a scalar field φ which quantities how far $\psi_i^\star$ is from ψ_i at a point. There are infinitely many ways of constructing such a field, but the simplest one works as follows: Taken a preferred tetrad $e_A = e_A^a \partial_a$ (with $\nabla e_A = 0$) and its dual $e^A = e_a^A dx^a$, introduce the operation

$$\|A\|_e^2 \coloneqq \sum_{A_1,\ldots,A_l,B_1,\ldots,B_k}^{\text{from 0 to 3}} |A(e^{A_1},\ldots,e^{A_l},e_{B_1},\ldots,e_{B_k})|^2 \tag{B4}$$

for a generic complex-valued (l,k)-tensor A. Then, define φ as

$$\varphi = \sum_i \|\psi_i^\star - \psi_i\|_e^2. \tag{B5}$$

It is evident that given a spacetime point p, $\varphi(p) = 0$ if and only if $\psi_i^\star(p) = \psi_i(p)$. Hence, from Eq. (B3), we see that $\varphi = 0$ on $\mathcal{D}^+(\mathcal{R}^c)$, proving that Definition 1 follows directly from causality. Furthermore, global well posedness guarantees that φ exists everywhere in $\mathcal{D}^+(\Sigma)$, which is another central assumption of the theorem.

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[81] The frame dependence of the maximum entropy state is not in contradiction with the Lorentz invariance of the entropy [82] because the total entropy is Lorentz invariant only at equilibrium [83]. Indeed, it is easy to see from Fig. 2 (considering that $\nabla_a s^a \neq 0$ along the red arrows) that any attempt to use the Gauss theorem to prove that $S_A = S_B$ is doomed to fail.

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[88] For example, $\varphi^\star(t_A = 0)$ has a well-defined Fourier transform. The reader should not be concerned about the discontinuity at $x_A = 0$ because the step function can be

replaced by any smooth function $\tilde{\Theta}$ such that $\tilde{\Theta}(x_A) = \Theta(x_A)$ for $x_A \in (-\infty, -1) \cup (0, +\infty)$ without affecting the result.

[89] By "acoustic cone" we actually mean the outermost cone: the fastest characteristic. Here, we are using the evocative word "acoustic" to mean that observers inside it can "feel" the disturbance. But such a disturbance does not need to be sound in a strict sense. It may also be a shear or an Alfvén wave. Also, note that the acoustic cone is an actual 3D cone (like the light cone) only if the field equations are hyperbolic and the medium is isotropic in some frame. For anisotropic media, the shape of the acoustic cone may be distorted. Furthermore, if the field equations are parabolic, the acoustic cone degenerates to a 3D hyperplane, but the argument still applies.

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[112] For example, we know that the Israel-Stewart theory is not globally well posed [113]: A singularity may appear in finite time, even for smooth initial data.

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