A Postquantum Theory of Classical Gravity?

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The effort to discover a quantum theory of gravity is motivated by the need to reconcile the incompatibility between quantum theory and general relativity. Here, we present an alternative approach by constructing a consistent theory of classical gravity coupled to quantum field theory. The dynamics is linear in the density matrix, completely positive, and trace preserving, and reduces to Einstein's theory of general relativity in the classical limit. Consequently, the dynamics does not suffer from the pathologies of the semiclassical theory based on expectation values. The assumption that general relativity is classical necessarily modifies the dynamical laws of quantum mechanics; the theory must be fundamentally stochastic in both the metric degrees of freedom and in the quantum matter fields. This breakdown in predictability allows it to evade several no-go theorems purporting to forbid classical quantum interactions. The measurement postulate of quantum mechanics is not needed; the interaction of the quantum degrees of freedom with classical space-time necessarily causes decoherence in the quantum system. We first derive the general form of classical quantum dynamics and consider realizations which have as its limit deterministic classical Hamiltonian evolution. The formalism is then applied to quantum field theory interacting with the classical space-time metric. One can view the classical quantum theory as fundamental or as an effective theory useful for computing the backreaction of quantum fields on geometry. We discuss a number of open questions from the perspective of both viewpoints.

I. MUST WE QUANTIZE GRAVITY?

The two pillars of modern theoretical physics are general relativity, which holds that gravity is the bending of space-time by matter, and quantum field theory, which describes the matter that lives in that space-time. Yet, they are fundamentally inconsistent. Einstein's equation for gravity

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \tag{1}$$

has a left-hand side which encodes the space-time degrees of freedom via the Einstein tensor and is treated classically, while on the right-hand side sits the energy-momentum tensor encoding the matter degrees of freedom which must become an operator $\hat{T}^{\mu\nu}$ according to quantum theory (we henceforth denote quantum operators by placing a ^ on them). The widespread belief is that this inconsistency should be remedied by finding a quantum theory of gravity so that the left-hand side of Eq. (1) is also an operator. Yet,

although we have candidates such as string theory, which is in its mid-50's [1], and loop quantum gravity turning just over 40 [2–4], a convincing theory of quantum gravity remains elusive.

Another possibility is that gravity should still be treated classically, but that Einstein's equation should be modified. There is some sense to this. Space-time, though dynamical, can be understood as describing the background on which quantum fields of matter interact. Gravity, through the equivalence principle, is unique in this regard—no other fields can be described as a universal geometry in which quantum fields live. Having this classical background structure appears crucial to our current understanding of quantum mechanics. Quantum field theory allows us to predict the result of future measurements on a spacelike surface, based on past initial data. The identification of a Cauchy surface on which to give these initial data, not to mention the identification of a one parameter family of future hypersurfaces, is natural when the background structure is classical, while it is unclear whether it is possible to quantize this causal structure in a way which is independent of these background choices. While this difficulty may be merely technical, background-independent approaches such as loop quantum gravity, string field theory [5], and others [6,7] have yet to demonstrate that this is possible.

And while the ultraviolet catastrophe required the quantization of electromagnetism, the analogous issue in gravity is less well understood [8] and muted by the fact that gravitational systems in equilibrium occur in exotic situations [9], or would emit gravitational radiation over very long timescales [10].

Nonetheless, there is no strong case that gravity is exceptional, and the overwhelming consensus has been that we must quantize it along with all the other fields. In part, this is due to a number of no-go theorems and arguments over the years [11–21] purporting to require the quantization of the gravitational field. A recent no-go theorem [21] holds not only for coupling classical mechanics with quantum theory, but also with modifications to quantum mechanics, so-called postquantum theories which alter the quantum state space or dynamics [22]. But given the difficulties faced in constructing such a theory, it seems important to revisit this consensus. The question of why space-time might be fundamentally classical in contrast to other forces will be considered elsewhere [24], while here we merely consider the necessity of quantizing gravity. Indeed, not only do we find that it is logically possible to have quantum fields interacting with classical space-time, we construct such a theory. The no-go theorems on the need to quantize gravity implicitly or explicitly assume that the coupling of quantum fields to space-time is not stochastic. This loophole can be exploited to construct a theory of classical space-time which can be acted upon by quantum fields.

Although the construction of a consistent fundamental theory is the principal motivation of the present work, two other motivations remain even if one believes that the fundamental theory of gravity is quantum. The first is to better understand the backreaction that quantum fields have on space-time. In many cases, such as in the evaporation of large black holes, or in inflationary cosmology, we are interested in the limit in which space-time can be treated classically, while the matter itself, such as the black-hole radiation, or vacuum fluctuations, must be treated quantumly. We would like to compute the backreaction that quantum matter produces on space-time, yet we have no consistent theory which enables us to do this. The current approach to describe gravity classically is to take the expectation value of the right-hand side of Eq. (1), leading to the semiclassical Einstein equation [25–27]

$$G^{\mu\nu} = \frac{8\pi G}{c^4} \langle \hat{T}^{\mu\nu} \rangle. \tag{2}$$

Although this equation is often used to understand the backreaction on space-time, it is valid only when fluctuations are small, while to understand quantum effects, we are precisely interested in the case where quantum fluctuations are significant. Taking the semiclassical Einstein equation seriously leads to pathological behavior. If a heavy object is in a statistical mixture of being in two locations (as depicted in Fig. 1), a test mass will fall toward the point between the two locations, and the object itself will be attracted to the place where it might have been. This has been ruled out by Page and Geilker through the performance of a gedanken experiment in which a lead mass was placed in a position that depended on the reading of a Geiger counter situated near a radioactive source [28]. It will come as no surprise to the reader, that they verified that the dynamics depicted in the right panel of Fig. 1 is what occurs.

This problematic behavior of the semiclassical Einstein equation has generally been viewed as a reason why we must quantize gravity [31–34]. However, one sees the same apparent paradox in a fully quantum or fully classical theory of two systems. Take for example the interaction Hamiltonian $\hat{H}_{MG} = \hat{H}_M \otimes \hat{H}_G$, and the limit where the M and G system are close to a product state with density matrix $\hat{\rho}_{MG} \approx \hat{\sigma}_M \otimes \hat{\sigma}_G$. Then, for as long as the two systems remain approximately uncorrelated, Heisenberg's equation of motion for the M system, after tracing out the G system gives

$$\frac{\partial \hat{\sigma}_M}{\partial t} \approx -i[\hat{\sigma}_M, \hat{H}_M] \otimes \langle \hat{H}_G \rangle, \tag{3}$$

where we use units where $\hbar = c = 16\pi G = 1$. The state of the system M also appears to be coupled to an expectation value, even though we know that this is not what happens. By tracing out the G system, we have lost view of the fact that the two systems will get correlated. Once the systems become correlated, the reduced dynamics is no longer even linear in the reduced state $\hat{\sigma}_M$ since $\text{tr}_G \hat{H}_G \hat{\rho}_{MG} \neq \langle \hat{H}_G \rangle \hat{\sigma}_M$. While this in no way suggests that the semiclassical Einstein equation is correct even on average, its pathologies have led to a widely held objection to theories which treat gravity classically, while it is clear from the above example that the pathologies of the semiclassical Einstein equations have nothing to do with treating one system classically, and everything to do with taking expectation values, and thus losing sight of correlations. The same apparent pathologies occur if both systems are classical, but we allow for probabilistic mixtures of states. The theory presented here does not suffer from these issues; the equations of motion are linear in the density matrix, just like quantum mechanics and the classical Liouville equation. In order to understand the interaction of two systems, whether they are both quantum, both classical, or are classical quantum, we need to account for the correlations between them, and this is more easily done by considering the evolution of a joint density matrix or probability density.

This leads us to the second motivation for the present work: Even if one does not regard the theory as fundamental, it provides a sandbox, or toy model, for understanding quantum gravity. Indeed, a natural first step to quantizing gravity is to begin with the Liouville equation
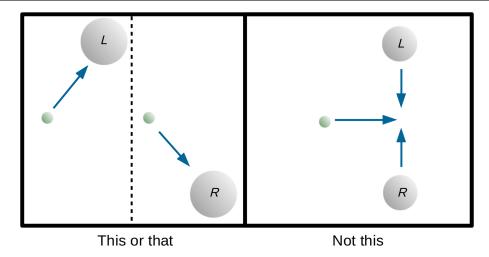
FIG. 1. The prediction of the semiclassical Einstein equation is depicted in the right panel, compared with what we expect to occur based on the statistical interpretation of the density matrix in the left panel. If a planet is in a statistical mixture of being in two possible locations ("*L*" and "*R*"), then we expect the gravitational field to become correlated with it. A test particle therefore falls toward one of the planets. If one treats the semiclassical Einstein equations as fundamental, the test particle falls down the middle, which is indeed the average trajectory of its path in the left-hand panel. The planets are also attracted toward the place they might have been. The systems in the linear theory considered here exhibit the behavior depicted in the left-hand panel. The case where we attempt to put the planet in a superposition of the two locations is discussed further in Refs. [29,30]. If the gravitational field is suitably different in the two cases, in the sense of how distinguishable the field configurations are when noise is added to them, then there will be a limit on how coherent the superposition can be [30]. In the case where there is coherent superposition, the initial trajectory of the test mass will be insufficiently correlated with the planet's position to enable a determination of where the planet is. However, after the decoherence time, it may be possible to determine the planet's location from the trajectory of the test mass, since its trajectory will become correlated to the planet's location [29]. This is also what one expects from a fully quantum theory, with the stochasticity of the test mass being due to vacuum fluctuations of the metric.

for general relativity and a probability density $\rho(t)$ over its phase space. After all, the probability density and the Liouville equation have a lot in common with the quantum density matrix and the Heisenberg equations of motion. As we see with the semiclassical Einstein equation, some of the conceptual difficulties faced in understanding quantum gravity are also found in classical probabilistic theories and are resolved by considering instead the Liouville equation. The classical limit of the theory we present here should thus be close to the Liouville equation for classical Einstein gravity introduced as Eq. (103). Many issues which arise in quantum gravity, especially in the canonical quantization framework, will turn out to have a similar but more tractable form here. In particular, the difficulty encountered in understanding the role of diffeomorphism invariance when one does not have a single 4-geometry or single time slicing, as well as the technical difficulty encountered in calculating the constraint algebra, also appear in the theory considered here. And even in some cases, in the fully classical limit of the theory.

Let us now turn to another widely held belief on why a classical theory of gravity is incompatible with superpositions and the uncertainty relation. This argument is usually attributed to Feynman [12,35] and Aharonov and Rohrlich [36] in relation to the gravitational field (see also Refs. [14,37–40]). As depicted in Fig. 2, if the gravitational field produced by a particle is classical, the field can in

principle be measured to sufficiently high precision to determine the particle's position without disturbance. This would reveal which slit it went through, and would either prevent particles from being in superposition or would violate the uncertainty principle if the interference experiment of Fig. 2 is carried out. However, if the classical field responds to the quantum system stochastically [43], then measuring the classical degrees of freedom will not necessarily determine the particle's quantum state to arbitrary precision. This could be because the gravitational field reacts in an indeterministic way to the presence of matter, or because there is only a probability that the gravitational field reacts to the position of the particle during the time of the interference experiment [45]. In either case, the gravitational degrees of freedom contain only partial information about the location of the particle. We thus see that previous arguments for requiring the quantization of the space-time metric implicitly assume that theory is deterministic, and are not a barrier to the theory considered here.

The fact that coupling classical gravity to quantum theory necessarily requires stochasticity [49] is particularly compelling in light of the black-hole information problem [51–53], and its sharper version, the Almheiri-Marolf-Polchinski-Sully (AMPS) "paradox" [54,55]. Since deterministic theories appear to require a breakdown of gravity at low energies [56], serious consideration to nondeterministic
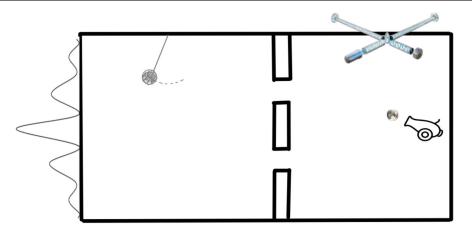
FIG. 2. Consider a variation of the gedanken experiment proposed by Feynman [12,35] and Aharonov [36] in which a massive particle travels through two slits as a plane wave. Quantum theory predicts an interference pattern on the screen behind the slits which enables a determination of the particle's momentum. The local Newtonian potential it creates can be measured everywhere. At late times, the state of the electromagnetic field far from the slits will be slightly different depending on which slit the particle goes through, but these two states need not be orthogonal. The quantum nature of the electromagnetic field prevents an experimenter from distinguishing these two nonorthogonal states and while the electromagnetic interaction might lead to some decoherence of the particle, if the interaction is small enough the interference pattern remains. On the other hand, if the gravitational field is classical and the interaction deterministic, then one could imagine measuring the gravitational field close to the particle and during its flight, without disturbing the field to unambiguously determine which path the particle went through. One could therefore never get interference fringes (or the uncertainty principle would be violated). Two classical states cannot be different but nonorthogonal. Actually measuring the gravitational field is not required since classical systems are undisturbed; even if we cannot actually measure the field to the required accuracy [48], it is impossible to write down a pure quantum state of the particle which is correlated with a classical gravitational field determined by the particle's position. Since the theory presented here modifies quantum theory so that the interaction is stochastic, the gravitational field need not become unambiguously correlated with the particle's path and the interference pattern is restored. Probability densities over classical states can be different and nonorthogonal.

theories should be given [59]. However, theories with information destruction face imposing obstacles. Banks, Peskin, and Susskind (BPS) argued that they lead either to violations of locality or anomalous heating [60,61], and Coleman argued that they merely result in false decoherence corresponding to unknown coupling constants [62]. There have been a few attempts to work around the issue raised by BPS. Unruh and Wald proposed that if the information destruction was at sufficiently high energy, then violations in momentum and energy conservation would also be at high energy, and hence, not observable in the lab [63]. Models which are local, yet violate cluster decomposition [64] were proposed in Ref. [65], and ones which have only mild violations of energy conservation at the expense of some nonlocality were proposed in Ref. [66]. However, none of these attempts have been theories of gravity, and once the backreaction to space-time is considered and the constraints of general relativity taken into account, the situation is different enough to warrant reconsidering the issue. The class of theories presented here may allow us to resolve the black-hole information paradox in favor of information loss. We return to these points in the Discussion section.

We find that the issue of anomalous heating takes on a different form in this theory but continues to raise questions. These may be resolved simply by regularizing the theory, but it is also possible that we need to introduce new physics at a distance scale larger than the Planck scale, as needs to happen in spontaneous collapse models [67–70]. Further research is needed because the theory we present contains a number of unspecified couplings and choices. These can be shown to be greatly reduced if one demands that the theory should be continuous in phase space [71]. However, this does not completely fix the theory, and there remains some additional freedom which we expect to be constrained by demanding the theory be regularizable, and by experimental constraints. Whether there is sufficient freedom in the couplings to satisfy these demands, as well as ones coming from symmetry principles such as diffeomorphism invariance, remains open.

Having argued that nothing forbids the coupling of quantum systems to classical ones, let us turn to attempts to construct such a theory. An early attempt to couple toy models for classical gravity to quantum mechanics was made in Ref. [72] using the Aleksandrov-Gerasimenko (AG) bracket [73,74]

$$\frac{\partial \hat{\rho}(z;t)}{\partial t} = -i[\hat{H}(z), \hat{\rho}(z;t)] + \frac{1}{2}(\{\hat{H}(z), \hat{\rho}(z;t)\} - \{\hat{\rho}(z;t), \hat{H}(z)\}), \tag{4}$$

where $\hat{H}$ is a quantum Hamiltonian which can depend on phase-space points z, and $\{\cdot, \cdot\}$ are Poisson brackets.

The object $\hat{\rho}(z; t)$, about which we say more in Sec. II, is the classical quantum (CQ) state at time t—a positive matrix in Hilbert space, and a functional of the classical degrees of freedom z normalized such that

$$\int dz \mathrm{tr}\hat{\rho}(z; t) = 1. \tag{5}$$

However, the dynamics the AG bracket generated leads to negative probabilities [72,75]. Unless we are prepared to modify the Born rule, or equivalently, our interpretation of the density matrix, any dynamics must be trace preserving (TP) to conserve probability, and be completely positive (*CP*), for probabilities to remain positive. So, although the dynamics of Eq. (4) makes a frequent appearance in attempts to couple quantum and classical degrees of freedom [76–78] (cf. Ref. [79]), and while it may give insight into some regimes, it cannot serve as a fundamental theory. Other attempts, such as using the Schrödinger-Newton equation [80,81] (cf. Refs. [82,83]), suffer from the problem that the equations are nonlinear in the density matrix, and thus will lead to superluminal signaling [84–86] and a breakdown of the statistical mechanical interpretation of the density matrix [87]. And yet, experimentalists working in the field of quantum control routinely couple quantum and classical degrees of freedom together, usually by invoking the mysterious measurement postulate of quantum theory. This presents another path to treat gravity classically, where one imagines the system of interest is fundamentally quantum, but that observables are restricted or a measurement is performed on them [75,88–90]. However, for gravity, this would require first quantizing the theory, which might well be an impossible task. Another approach is to apply a measurement to the quantum system instead, and then couple the output to the classical degrees of freedom such as the Newtonian potential [91–94]. Both approaches, at least initially, invoke the measurement postulate of quantum mechanics, or an *ad hoc* field which produces a stochastic collapse of the wave function [95–100], which then sources the gravitational field.

Here lies the essence of the problem: If nature is fundamentally quantum, our notion of classicality hinges on the measurement postulate. Thus, the emphasis in this work is to invert the logic: Assume that since space-time describes causal structure and relationships between the matter degrees of freedom, it is *a priori* and fundamentally classical. Then, examine the implications of this. One outcome will be that the quantum degrees of freedom inherit classicality from space-time. By treating the space-time metric as a classical degree of freedom, we obtain for free not only a theory of quantum matter and gravity, but also a theory which produces the gravitationally induced collapse of the wave function conjectured by Karolyhazy [101], Diósi [102], and Penrose [103,104]. Collapse arises as a simple consequence of treating space-time classically.

Such an approach can be found in Ref. [105] where a variation of Eq. (2) is studied, but it is argued that the interaction of a quantum system with a classical measuring device necessarily leads to decoherence and consistency of the measurement postulate. Likewise, Tilloy and Diósi [92,93] point out that although they introduce a spontaneous collapse model to source the Newtonian potential, this can be thought of as formal, and it is the consistency of hybrid dynamics which requires decoherence of the quantum matter. Poulin [66] also note that the classical coupling leads to decoherence of the quantum system.

The theory considered here, however, cannot be reduced to a linear spontaneous collapse model, because the dynamics is not Markovian or even linear in the density matrix when the gravitational degrees of freedom are traced out (although it is linear and Markovian when we include the metric degrees of freedom). It also leads not merely to decoherence, but to a genuine objective collapse of the wave function. In particular, there are realizations of the discrete theory considered here, where stochastic jumps in phase space can be finite and correlate to Lindblad operators. As a result, changes in the quantum state correspond to changes in the classical system. Consequently, the quantum state remains pure and uniquely determined by the trajectory of the classical system [106]. Purity of the quantum system conditional on the classical trajectory has since been shown to also hold in the continuous realizations under natural conditions [29].

Before proposing a theory which couples classical general relativity with quantum field theory, we first construct a general framework for classical quantum dynamics. In order to do so, we derive general linear dynamics which maps the set of classical quantum states to itself. More precisely, we wish to find the most general linear dynamics which is completely positive, trace preserving, and preserves the division of degrees of freedom into classical and quantum. Two examples of such dynamics have previously been proposed. In Refs. [46,107], a master equation with diagonal coupling to Lindblad operators $\hat{L}_\alpha$ was considered of the form

$$\frac{\partial \hat{\rho}(z; t)}{\partial t} = -i[\hat{H}(z), \hat{\rho}(z; t)] + \sum_{\alpha, z'} W^\alpha(z|z')\hat{L}_\alpha \hat{\rho}(z'; t)\hat{L}_\alpha^\dagger$$
$$- \frac{1}{2}W^\alpha(z)\{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{\rho}(z; t)\}_+, \tag{6}$$

where z are discrete classical degrees of freedom, and $\{\cdot\}_+$ is the anticommutator. When $W^\alpha(z|z') = W^\alpha(z)\delta(z - z')$, this reduces to the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) or Lindblad equation [108,109] controlled by an external degree of freedom z. Equation (6) has been shown by Poulin to be the most general master equation in the case of discrete classical variables and bounded Lindblad operators by embedding the classical system in Hilbert space [110]. However, for trajectories which are continuous in the

classical degrees of freedom, it will turn out that nondiagonal couplings are required. A master equation due to the early work of Diósi [47] corresponds to the constant force case $\hat{H}(q) = \int dx \, q(x)\hat{L}(x)$ or sum of such terms [113], $\hat{H}(q) = \sum_i \int dx \, q_i(x)\hat{L}_i(x)$

$$
\begin{aligned}
\frac{\partial \hat{\rho}(z;t)}{\partial t} &= \{H_0(z), \hat{\rho}(z;t)\} - i[\hat{H}(q), \hat{\rho}(z;t)] \\
&+ \frac{1}{2}(\{\hat{H}(q), \hat{\rho}(z;t)\} - \{\hat{\rho}(z;t), \hat{H}(q)\}) \\
&+ \frac{1}{2}\int dx dx' \{q_i(x'), \{q_j(x), D_2^{ij}(x,x')\hat{\rho}(z;t)\}\} \\
&+ \frac{1}{2}\int dx dx' D_{0,ij}(x,x') \\
&\times \left[\{\hat{H}(q), p^i(x)\}, [\hat{\rho}(z;t), \{\hat{H}(q), p^j(x')\}] \right],
\end{aligned} \quad (7)
$$

which adds diffusion and a Lindbladian to the Aleksandrov-Gerasimenko bracket of Eq. (4), and is completely positive provided the matrix inequality $D_2 D_0 \succeq \frac{1}{4}$ is satisfied. For arbitrary Hamiltonians and dynamics which is continuous in phase space, one should turn to the most general continuous master equation since derived in [71]. Unravellings for arbitrary Hamiltonians has since been taken up in [29]. In this work, the classical quantum interaction Hamiltonian $\hat{H}(q)$ is chosen to be minimally coupled, meaning that it is a functional of phase-space fields $q_i(x)$ and not on the conjugate momenta $p_i(x)$, while $H_0(z)$ is any purely classical Hamiltonian. The constant force case considered in Ref. [47] corresponds to an interaction Hamiltonian which is linear in a single classical degree of freedom $q(x)$, $\hat{H}(q) = \int dx q(x)\hat{L}(x)$. The constant force case has been studied in the context of measurement theory [111], in the classical limit of the collisionless Boltzmann equation [112], and in a model for Newtonian gravity [113]. The discrete dynamics has appeared in discussions around black-hole evaporation [66].

In Sec. III, we show that Eq. (14) is the most general form of time-local [114] classical quantum dynamics. It includes the discreet dynamics of Eq. (6) and the continuous Hamiltonian dynamics of Eq. (7). Our derivation uses a fully classical quantum formalism and gives necessary conditions for the dynamics to be completely positive. These conditions are also sufficient in the case of discrete dynamics, and in Ref. [71] with Sparaciari, Šoda and Weller-Davies, we derive the necessary and sufficient conditions when the dynamics is continuous in phase space. This continuous master equation generalizes Ref. [47] to the case of arbitrary interaction Hamiltonians and multiple Lindblad operators and classical degrees of freedom. Since the continuous master equation requires off-diagonal Lindblad couplings, this can be shown to imply that the

dynamics of Eq. (6) has discrete, finite-sized jumps in phase space [71].

With the general form of dynamics in hand, the next step is to restrict ourselves to dynamics which is generated by a Hamiltonian. This allows us to apply the framework to the Hamiltonian formalism of general relativity. This follows from considering a perturbative expansion of the more general dynamics presented as Eq. (17). This expansion can be thought of as a CQ version of the Kramers-Moyal expansion [115,116] familiar in classical stochastic dynamics. We show in Sec. IV that continuous and deterministic classical evolution emerges in some limit. Away from this limit, the quantum system can retain some coherence, but complete positivity requires the classical system to undergo diffusion or discrete, finite-sized jumps in phase space. A discrete realization of this dynamics makes use of a finite and noncommuting directional divergence, Eqs. (73)–(75). This dynamics corresponds to a discrete version of Liouville dynamics on the classical system and the action of a dynamical semigroup on the quantum system, thus providing a natural generalization of both classical and quantum evolution. A parameter τ governs how continuous and classical the total system is, with the system's trajectory in the classical phase space going from finite-sized hops to continuous deterministic dynamics as $\tau \to 0$. In this limit, the quantum system decoheres and becomes classical. For larger τ (or $\hbar$), the decoherence times are longer, while the jump distance in the classical phase space increases resulting in greater diffusion in the system's trajectory. The dynamics of Eq. (17) also includes the fully continuous evolution of Eq. (7).

In Sec. IV B, we present a simple example of the discrete dynamics, namely, a quantum spin system with a classical position and momentum interacting with a potential. We see that if the particle is initially in a superposition of spin states, then it eventually collapses to one of them through its interaction with the classical degrees of freedom. In essence, the classical potential acts like a Stern-Gerlach device, so that after some time, the particle's classical degrees of freedom uniquely determine the value of the spin. Unlike standard decoherence, the collapse of the quantum state happens suddenly, when the system undergoes a classical jump in its momentum. At that point, monitoring the momentum unambiguously reveals the particle's spin. The quantum state of the system remains pure, conditioned on the state of the classical system. There is thus no need for the mysterious measurement postulate of quantum theory, a postulate which has meant that the interpretation of quantum theory has been open to dispute, and whose problematic nature has received renewed attention [117].

In Sec. IV C, we then extend the framework to include a hybrid version of quantum field theory. In Sec. V, we apply this hybrid formalism to the case of general relativity. We work in the Arnowitt-Deser-Misner (ADM) formalism [118],

treating the metric as classical, while the matter living on space time is quantum fields. In the classical limit, the equations of motion reduce to Einstein equations, in the sense of a CQ version of Ehrenfest's theorem, that after taking the expectation value on the quantum fields, the equations of motion are close to the semiclassical version of the Liouville equation [Eq. (21)] applied to general relativity [Eq. (103)]. There are nonetheless some deviations from general relativity in the classical limit, because there are additional contributions relating to diffusion and decoherence, with some of the experimental bounds on these terms since given in Ref. [50].

The equations we derive allow one to calculate the backreaction of quantum fields on classical space-time. This allows us to go beyond the semiclassical Einstein equation, and we expect it to prove useful in attempts to understand black-hole evaporation or the effect of vacuum fluctuations in cosmology. The interaction of the quantum system with classical space-time causes the wave function to collapse at a rate determined by the system's energy-momentum tensor. The stochastic nature of the interaction allows for the possibility that information is destroyed in black holes, although the quantum state conditioned on the metric degrees of freedom can remain pure. The theory suggests a number of possible experiments which could falsify or verify it, and we discuss this in Sec. VI. We now begin in Sec. II by reviewing the formalism of classical quantum dynamics and then discuss how the evolution law we derive relates to both classical stochastic dynamics and quantum maps. The form of the master equation, Eq. (14), is then derived in Sec. III. Since the initial posting of this manuscript in 2018 [119], a number of advancements have been made. We continue to highlight some of these developments while maintaining the integrity of the original draft.

II. CONSISTENT CLASSICAL QUANTUM DYNAMICS

The state of a quantum system is represented in a Hilbert space $\mathcal{H}_S$ by a density matrix—a positive trace one matrix $\hat{\sigma}(t)$ at time t. A discrete classical system is represented by a discrete random variable z, and the system has a probability $p(z;t)$ of being at z. In the continuous case, we consider a classical system to live in phase space, although we could just as well consider any other classical sample space. Phase space is a $2n$-dimensional manifold $\mathcal{M}$ of points $z = (q, p)^T$, and the state of the classical system can be represented by a probability density—a non-negative distribution $\rho(z;t)$ defined over z at time t normalized such that $\int dz \rho(z;t) = 1$.

Here, as in Refs. [46,47,66,73,74,107,111,112], we consider a hybrid classical quantum system which lives on both the Hilbert space and the classical phase space, an illustrative example being the classical quantum qubit of Eq. (77). The state of a hybrid system $\hat{\rho}(t)$ at time t is defined over an interval around z by a positive matrix $\hat{\rho}(z;t)$ in $\mathcal{H}_S$ [120],

such that the normalization condition Eq. (5) holds. In quantum theory, such a system is sometimes said to be in a cq state (or CQ state), and has a density matrix of the form

$$\hat{\rho}_{\text{cq}} = \int dz \rho(z;t) |z\rangle\langle z| \otimes \hat{\sigma}(z;t) \qquad (8)$$

with $\hat{\sigma}(z;t)$ a normalized density matrix and $|z\rangle$ an orthonormal basis. Since $\hat{q}$ and $\hat{p}$ do not commute in quantum theory, if we want to encode phase-space degrees of freedom in a quantum system, we could consider two separate subsystems, one which encodes q and one which encodes p. Here, we do not consider such an embedding, but rather describe the classical quantum state as

$$\hat{\rho}(z;t) = \rho(z;t)\hat{\sigma}(z;t) \qquad (9)$$

with $\rho(z;t) := \text{tr}\hat{\rho}(z;t)$ a phase-space density and $\int \text{tr}\hat{\rho}(z;t)dz \leq 1$ when integrated over any interval. Conditioned on the classical system being in state z, the quantum system is in the state $\hat{\sigma}(z;t)$, the normalized state of the quantum system. In finite dimension, we can also write the CQ state as a matrix whose diagonal entries are phase-space densities, as with the CQ qubit of Eq. (77). When we consider gravity, each point z is a classical field (the 3-metric, and 3-momentum) over $\mathbb{R}^3$.

We now ask, what is the most general dynamical map which take any cq state at time $t = 0$ to another cq state at a later time t? This requires us to adapt Kraus's representation theorem [121] to the classical quantum case. Table I compares the classical-quantum map to the Kraus map for quantum systems and the stochastic matrix for classical systems. Table II compares the corresponding master equations which we now review. Let us recall that for quantum systems, a linear map $\mathcal{E}(\hat{\sigma})$ which is completely positive trace preserving at every time t (we say it is infinitesimally divisible [122]) is always generated by the GKSL or Lindblad equation [108,109]

$$\frac{\partial \hat{\sigma}(t)}{\partial t} = -\frac{i}{\hbar}[\hat{H}(t), \hat{\sigma}(t)]$$
$$- \lambda^{\alpha\beta}(t)\left[\hat{L}_\alpha \hat{\sigma}(t)\hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger \hat{L}_\alpha, \hat{\sigma}(t)\}_+\right] \qquad (10)$$

with $\hat{H}(t)$ the Hamiltonian, the set $\{\hat{L}_\alpha\}$ a traceless basis of operators orthogonal in the Hilbert-Schmidt norm, and $\lambda^{\alpha\beta}(t)$ are elements of a positive-semidefinite (PSD) matrix, which we write as $\lambda \succeq 0$. Here, and throughout, we use Einstein summation notation when there are raised and lowered indices. One can always diagonalize λ, in which case $\hat{L}_\alpha \hat{\sigma} \hat{L}_\alpha^\dagger$ is sometimes called the jump term since one could interpret it as giving the rate at which the system jumps from being in another quantum state into $\hat{\sigma}$, while $\frac{1}{2}\{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{\sigma}\}_+$ is sometimes called the no-event term and acts to preserve the trace. If the map is non-Markovian

TABLE I. Summary of Secs. II–III of the finite-time map acting on states for stochastic classical, quantum, and classical quantum dynamics. We use Einstein summation notation, so an index which appears both raised and lowered should be summed over. In the classical case, the probability density $\rho(z;t)$ over phase space z is given in terms of the earlier probability density $\rho(z;0)$ and the transition probabilities $P(z|z';t)$ which define a classical channel. In the quantum case, the necessary and sufficient condition for complete positivity is that $\lambda(t)$ as a matrix in $\alpha\beta$ is positive semidefinite [which we write as $\lambda(t) \succeq 0$]. $\{\hat{L}_\alpha\}$ are a basis of operators. The classical quantum positivity and norm conditions combine those of the purely classical and purely quantum dynamics, and generalizes the Kraus decomposition theorem to the hybrid case in Eq. (34).

	Map	Positivity condition	Norm condition
Classical	$\rho(z;t) = \int dz' P(z|z';t)\rho(z',0)$	$P(z|z';t) \geq 0, \ \forall z, z'$	$\int dz P(z|z',t) = 1$
Quantum	$\hat{\sigma}(t) = \lambda^{\alpha\beta}(t)\hat{L}_\alpha \hat{\sigma}(0)\hat{L}_\beta^\dagger$	$\lambda(t) \succeq 0$	$\lambda^{\alpha\beta}(t)\hat{L}_\beta^\dagger \hat{L}_\alpha = \mathbb{I}$
Classical quantum	$\hat{\rho}(z;t) = \int dz' \Lambda^{\alpha\beta}(z|z';t)\hat{L}_\alpha \hat{\rho}(z';0)\hat{L}_\beta^\dagger$	$\Lambda(z|z';t) \succeq 0, \ \forall z, z'$	$\int dz \Lambda^{\alpha\beta}(z|z';t)\hat{L}_\beta^\dagger \hat{L}_\alpha = \mathbb{I}$

TABLE II. Summary of Secs. II and III of the master equations governing classical, quantum, and classical quantum dynamics, along with the positivity condition in the case of continuous dynamics, or discrete jumps in the classical degrees of freedom. In the classical case, continuous dynamics corresponds to the Fokker-Planck equation specified by a drift term $D_1(z)$ and a positive-semidefinite diffusion matrix $D_2(z) \succeq 0$ with elements $D_{2,ij}(z)$. One can always choose $\hat{L}_0 = \mathbb{1}$, and the remaining $\hat{L}_\alpha$ to be traceless [71]. The CQ master equation is a natural generalization of the classical rate equation (or Fokker-Planck equation in the continuous case) and the Lindblad equation, and is here presented as Eq. (14) (in this table, we use the convention of including the $\lambda^{\alpha\beta}$ in the $W^{\alpha\beta}$ terms). In general, the coupling constants can have an explicit time dependence which we drop for ease of presentation.

Classical	
Master equation	$\partial_t \rho(z;t) = \int dz' W(z|z')\rho(z';t) - \int dz' W(z'|z)\rho(z;t)$
Positivity (jumps)	$W(z|z') \geq 0, \ \forall z \neq z'$
Positivity (continuous)	$W(z|z') = -D_{1,i}(z')\partial_{z_i'}\delta(z-z') + D_{2,ij}(z')\partial_{z_i'}\partial_{z_j'}\delta(z-z'), \ D_2 \succeq 0$

Quantum	
Master equation	$\partial_t \hat{\sigma} = -i[H, \hat{\sigma}] + h^{\alpha\beta} L_\alpha \hat{\sigma} L_\beta^\dagger - \frac{1}{2}\{h^{\alpha\beta} L_\beta^\dagger L_\alpha, \hat{\sigma}\}_+$
Positivity condition	$h^{\alpha\beta}$ a PSD matrix in α, β $(h \succeq 0)$

Classical quantum	
Master equation	$\partial_t \hat{\rho}(z;t) = \int dz' W^{\alpha\beta}(z|z') L_\alpha \hat{\rho}(z';t) L_\beta^\dagger - \frac{1}{2}\int dz' W^{\alpha\beta}(z'|z)\{L_\beta^\dagger L_\alpha, \hat{\rho}(z;t)\}_+$
Positivity (jumps)	$W^{\alpha\beta}$ PSD in $\alpha\beta$ $\forall z \neq z'$, $W^{\alpha\beta}$ PSD in $\alpha\beta$ for $z = z'$
Positivity (continuous)	Equation (49)

(or more precisely, not infinitesimally divisible), but has an inverse, then it can still be represented by Eq. (10), with the positive-semidefinite condition on the $\lambda^{\alpha\beta}(t)$ lifted, provided the finite-time map is still CP [123]. In finite-dimensional Hilbert spaces, CP maps have an inverse, but this condition can fail in infinite dimension [124,125], in which case, it may not be possible to write the master equation in time-local form.

For a classical stochastic process, on the other hand, an initial state at $t = 0$, $\rho(z;0)$ evolves into

$$\rho(z;t) = \int dz P(z|z';t)\rho(z';0) \qquad (11)$$

with $P(z|z';t) \geq 0$ and $\int dz P(z|z';t) = 1$. This simply ensures that any initial probability density $\rho(z;0)$ remains positive and normalized. This is equivalent to $P(z|z';t)$ being the conditional probability that the state makes a transition to z at time t given that it was initially at z'. For a time-local process, the most general dynamics is given by the rate equation

$$\frac{d\rho(z;t)}{dt} = \int dz' W(z|z';t)\rho(z';t) - W(z;t)\rho(z;t), \qquad (12)$$

where $W(z|z';t)$ are rates for the system to transition from state z' to state z (this can be thought of as the jump term). To conserve probability, we require

$$W(z;t) = \int dz' W(z'|z;t) \qquad (13)$$

being the total rate for the system to transition away from state z to one of the states z'. $W(z;t)\rho(z)$ is analogous to the no-event term in the GKSL equation.

The dynamics can be discontinuous in z, containing discrete finite-sized jumps, in which case it is necessary and sufficient that $W(z|z';t) \geq 0$, $\forall z \neq z'$ for the dynamics to be positive. When $W(z|z';t) = -D_{1,i}(z')\partial_{z'_i}\delta(z-z') + D_{2,ij}(z')\partial_{z'_i}\partial_{z'_j}\delta(z-z')$, the dynamics is continuous in z, and positive if and only if $D_{2,ij}$ is a positive-semidefinite matrix in ij corresponding to diffusion. D_{1_i} is called the drift term and includes Hamiltonian evolution as a special case. $W(z;t) = 0$ by integration by parts in Eq. (13). The continuous dynamics is sometimes called the Fokker-Planck equation or forward Kolmogorov equation, while the general form of the dynamics is sometimes referred to as the differential Chapman-Kolmogorov equation [126].

Here we present the most general classical quantum dynamical map which is linear and completely positive. In Sec. III, we find the generators of this map, deriving the most general continuous-time master equation for the dynamics. As in the proof of the generality of the GKSL equation, the proof presented here is valid for a separable Hilbert space, and countable set of bounded Lindblad operators $\hat{L}_\alpha$, or alternatively, when the map has an inverse at infinitesimal short times. In the case of unbounded operators [127], one can cast the equation in the form we derive, but it is possible the dynamics are not unique.

Under these assumptions, we find that the most general master equation can be put into the form

$$
\begin{aligned}
\frac{\partial\hat{\rho}(z;t)}{\partial t} = {}& -i[\hat{H}(z;t),\hat{\rho}(z;t)] + \lambda^{\alpha\beta}(z;t)\bigg(\hat{L}_{\boldsymbol{\alpha}}\hat{\rho}(z;t)\hat{L}_{\boldsymbol{\beta}}^\dagger \\
& -\frac{1}{2}\{\hat{L}_{\boldsymbol{\beta}}^\dagger\hat{L}_{\boldsymbol{\alpha}},\hat{\rho}(z;t)\}_+\bigg) \\
& + \int dz' W^{\alpha\beta}(z|z';t)\hat{L}_\alpha\hat{\rho}(z';t)\hat{L}_\beta^\dagger \\
& -\frac{1}{2}W^{\alpha\beta}(z;t)\{\hat{L}_\beta^\dagger\hat{L}_\alpha,\hat{\rho}(z;t)\}_+.
\end{aligned}
\tag{14}
$$

The condition that the master equation preserve probability requires

$$
W^{\alpha\beta}(z;t) = \int dz' W^{\alpha\beta}(z'|z;t)
\tag{15}
$$

for each α, β. One can verify this is the case by taking the trace of Eq. (15) and integrating over phase space, and then requiring the total probability to be conserved in time

$$
\frac{\partial}{\partial t}\int dz \mathrm{tr}\hat{\rho}(z;t) = 0.
\tag{16}
$$

This norm-preserving condition is analogous to the purely classical case of Eq. (13) and the pure-Lindbladian case.

Our derivation is general enough to allow for the coupling terms to be time dependent, which would allow for the inclusion of various non-Markovian effects, but following our general derivation, we drop the time dependence from $\lambda^{\alpha\beta}(z;t)$ and $W^{\alpha\beta}(z|z';t)$ (one refers to such dynamics as being autonomous). For the dynamics of Eq. (14) to be infinitely divisible (i.e., CP as a generator at every time t), it is necessary that $W^{\alpha\beta}(z|z';t)$ be a positive-semidefinite matrix in α,β $\forall z \neq z'$, and the matrix with elements $\lambda^{\alpha\beta} + W^{\alpha\beta}(z|z;t)$ must be PSD. These are sufficient conditions for discrete, jumping dynamics while the conditions on $W^{\alpha\beta}(z|z';t)$ in the continuous case are derived in Ref. [71] and previewed here as Eq. (49). For continuous quantum systems, the sum over $\alpha\beta$ is replaced by an integral, an example being the case where we take Lindblad operators to be local field operators written in terms of the coordinate x. $\hat{L}_\alpha$ can be any operator, but we can always rewrite the above equation in terms of a basis of Lindblad operators orthogonal in Hilbert-Schmidt norm, such that the $\hat{L}_{\boldsymbol{\alpha}}$ (distinguished by having α in bold font) are traceless, and $\hat{L}_0 = \mathbb{1}$.

The terms proportional to $W^{00}(z|z')$ and $W^{00}(z)$ are thus purely classical and account for both stochastic and deterministic evolution of the classical degrees of freedom. In the case of continuous deterministic dynamics, this term is just the Poisson bracket $\{h^{00}(z),\hat{\rho}(z;t)\}$ with $h^{00}(z)$ a classical Hamiltonian. In Eq. (14), we separate out the Lindbladian term with couplings $\lambda^{\alpha\beta}(z)$ to be consistent with conventions of the classical literature, but it is often convenient to include them in $W^{\alpha\beta}(z|z';t)$. The $W^{\alpha\beta}(z|z')$ can be interpreted as the jump term in both phase space or Hilbert space or both since it can induce transitions in both the classical quantum system. $\lambda^{\alpha\beta}(z)$ causes transitions in Hilbert space only, and $W^{00}(z|z')$ transitions only in phase space. The anticommutator terms are the collective no-event terms.

In the case that the quantum and classical systems do not get correlated in z (for example, when one system acts as a large heat bath weakly coupled to the other), the dynamics derived here reduces to a semiclassical version of the stochastic master Eq. (12) when we trace out the quantum system, while if we integrate out z we get an averaged version of GKSL dynamics. However, more generally, tracing out the quantum or classical system results in dynamics that are non-Markovian, since there can be correlations between the classical and quantum systems, and the reduced dynamics are not even linear on the quantum state. When we integrate over z, the presence of correlations between the classical system and the quantum one translate into different effective coupling strengths $W(z)$ for different quantum states [128,129].

In Sec. IV, we introduce a class of theories whose dynamics is close to Hamiltonian dynamics. This master equation can be expanded as

$$\frac{\partial \hat{\rho}(z;t)}{\partial t} = -i[\hat{H}(z), \hat{\rho}(z;t)]$$

$$+ \frac{W^{\alpha\beta}}{\tau}\left[\hat{L}_\alpha \hat{\rho}(z;t)\hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger \hat{L}_\alpha, \hat{\rho}(z;t)\}_+\right]$$

$$- \hat{L}_\alpha X_h^{\alpha\beta}(z) \cdot \nabla \hat{\rho}(z;t)\hat{L}_\beta^\dagger + \tau \hat{L}_\alpha \nabla \cdot [\gamma_h^{\alpha\beta}(z)]\hat{\rho}(z;t)\hat{L}_\beta^\dagger$$

$$+ \tau \hat{L}_\alpha \nabla \cdot D^{\alpha\beta}(z)\hat{\rho}(z;t) \cdot \overleftarrow{\nabla}\hat{L}_\beta^\dagger + \cdots, \quad (17)$$

where $\nabla = (\partial_q, \partial_p)^T$ is the divergence in phase space $z = q,\ p$, and $X_h^{\alpha\beta}$ a matrix of vector fields which, as we detail, is generated by a Hamiltonian

$$\hat{H}(z) = h^{\alpha\beta}(z)\hat{L}_\beta^\dagger \hat{L}_\alpha \quad (18)$$

over quantum and classical degrees of freedom via the symplectic matrix

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \quad (19)$$

The couplings

$$X_h^{\alpha\beta}(z) = \Omega \nabla h^{\alpha\beta}(z) \quad (20)$$

are the terms which correspond to Hamiltonian evolution in the classical limit. $\gamma_h^{\alpha\beta}(z)$ is taken to correspond to a friction term and is a vector in the $2n$-dimensional phase space at each z, with components $\gamma_{h,i}^{\alpha\beta}(z)$, while the $D^{\alpha\beta}(z)$ corresponding to diffusion terms define a matrix both in $\alpha\beta$ and in phase space with elements $D_{ij}^{\alpha\beta}(z)$ at each z. One typically considers $D^{\alpha\beta}(z)$ to depend only on $\bar{q}$ and the diffusion to be in $\bar{p}$. Here, we include the pure-Lindbladian term proportional to $\lambda^{\alpha\beta}(z)$ in the $W^{\alpha\beta}(z)$ term.

Dropping terms of higher order in τ and taking the expectation value on both sides by performing the trace over the quantum degrees of freedom gives a semiclassical version of the Liouville equation

$$\frac{\partial \rho(z;t)}{\partial t} = \text{tr}\{\hat{H}(z), \hat{\rho}(z;t)\} + \cdots. \quad (21)$$

If we retain the term proportional to τ, an appropriate choice of $\gamma_h^{\alpha\beta}(z)$ and $D^{\alpha\beta}(z)$ corresponds to a hybrid version of the Fokker-Planck equation. Truncating the expansion in Eq. (21) can result in negative probabilities, but the full evolution equation is completely positive and provides a natural generalization of both classical Hamiltonian dynamics and the GKSL equation.

We also see that taking the trace over the quantum system does not result in the pathological behavior found in

the semiclassical Einstein equation (2) as depicted on the right side of Fig. 1. Instead, the dynamics of Eq. (21) is linear in the total density matrix and similar to the situation described in our discussion of Eq. (3) where we have linear evolution of two coupled systems which account for the correlation. For example, the correlated state of two components labeled L and R,

$$\hat{\rho}(z) = \frac{1}{2}\rho_L(z)\hat{\sigma}_L + \frac{1}{2}\rho_R(z)\hat{\sigma}_R, \quad (22)$$

would evolve according to Eq. (21) as

$$\frac{\partial \rho(z)}{\partial t} = \frac{1}{2}\{\text{tr}\hat{H}(z)\hat{\sigma}_L, \rho_L(z)\} + \frac{1}{2}\{\text{tr}\hat{H}(z)\hat{\sigma}_R, \rho_R(z)\}\cdots \quad (23)$$

with the component $\rho_L(z)$ evolving differently into $\rho_R(z)$ as in the left side of Fig. 1. One can also think of $\partial/\partial p$ in Eq. (21) acting on each matrix element of $\hat{\rho}(z;t)$, while the velocities $\partial\hat{H}(z)/\partial q$ are a matrix, and thus the backreaction force depends on which state the quantum system is in and correlates the quantum system with the classical one.

III. THE CLASSICAL QUANTUM MAP AND MASTER EQUATION

We first derive the classical quantum map which takes a cq state $\hat{\rho}(z;0)$ to another cq state $\hat{\rho}(z;t)$ and then derive the continuous-time master equation. Recall that we restrict ourselves to the case of trace-class operations. We also restrict ourselves to the case where the map has an inverse at infinitesimal times (which need not be completely positive), so that we may write the master equation in time-local form [123]. Recall also Kraus's theorem, that the most general map on quantum systems $\mathcal{E}(\hat{\sigma})$ that is CP and TP can be written so that

$$\hat{\sigma}(t) = \mathcal{E}(\hat{\sigma}(0)) = \sum_\mu K_\mu \hat{\sigma}(0) K_\mu^\dagger \quad (24)$$

with the operators K_μ satisfying

$$K_\mu^\dagger K_\mu = \mathbb{1}, \quad (25)$$

where CP means that $\mathbb{1} \otimes \mathcal{E}$ acting on $\mathcal{H}_S \otimes \mathcal{H}_S$ takes positive operators to positive operators, and together with the trace-preserving condition, ensures that density matrices are mapped to density matrices, even if the map acts on part of a system.

Let us now consider the case of a map which acts on cq states. Because the map has to be linear in $\hat{\rho}(z;t)$, we can write the total map Λ_t acting on $\hat{\rho}(0)$ in terms of quantum maps $\Lambda_{z|z';t}$ such that

$$\hat{\rho}(z;t) = \int dz' \Lambda_{z|z';t}\hat{\rho}(z';0). \quad (26)$$

Since the initial state could be $\hat{\rho}(z';0) = \delta(z' - z_0)\hat{\sigma}(z')$, i.e., a normalized quantum state, with vanishing support a finite distance away from some initial point z_0, the map $\Lambda_{z|z';t}$ must be completely positive for all z, z'. The initial state which could be peaked around any z' must be mapped to another positive (but not necessarily normalized) quantum state around z. In the case of discrete z, the maps need not preserve the trace; they must merely be completely positive, trace nonincreasing, since if we act on the normalized state around some point z', a final state around z cannot have a trace larger than 1, as this would correspond to a probability $\int dz\rho(z;t)$ larger than 1. Thus, one easily sees that a necessary and sufficient condition for a cq map is that each $\Lambda_{z|z';t}$ must be a completely positive, trace nonincreasing map, with the total map

$$\Lambda_{z';t} := \int dz\Lambda_{z|z';t} \tag{27}$$

being trace preserving. This is the analog of the classical condition $\int dzP(z|z') = 1$. In the case where z is continuous, the above statements hold, except that because $\text{tr}\hat{\rho}(z;t)$ is a distribution, we should interpret statements about the nonincreasing of its trace as referring to its trace smeared over an indication function $\int dz\text{tr}\hat{\rho}(z;t)\xi(z)$.

We can then write the map in terms of sets of Kraus operators $K_\mu(z|z';t)$,

$$\hat{\rho}(z;t) = \int dz'\sum_\mu K_\mu(z|z';t)\hat{\rho}(z';0)K_\mu(z|z';t)^\dagger, \tag{28}$$

where $\sum_\mu K_\mu(z|z';t)^\dagger K_\mu(z|z';t) \leq \mathbb{1}$. This follows from the fact that the action of the entire map must preserve the trace; thus, summing over μ and taking the trace of both sides of Eq. (28) on an arbitrary initial state $\hat{\rho}(z';0)$, requires

$$\sum_\mu \int dzK_\mu(z|z';t)^\dagger K_\mu(z|z';t) = \mathbb{1}. \tag{29}$$

Equation (28), together with the normalization constraint of Eq. (29), gives the most general map on hybrid classical quantum states. It is the hybrid generalization of both the classical probability map of Eq. (11), and in some sense, an extension of the Kraus representation theorem Eq. (24). However, it can also be considered a special case of a Kraus decomposition, as we could restrict ourselves to quantum states that have the form of Eq. (8) and write its action as

$$\hat{\rho}_{\text{cq}}(t) = \int dzdz'\sum_\mu |z\rangle\langle z'| \otimes K_\mu(z|z';t)\hat{\rho}_{\text{cq}}(0)|z'\rangle\langle z|$$
$$\otimes K_\mu(z|z';t)^\dagger. \tag{30}$$

where the set of Kraus operators is $\{|z\rangle\langle z'| \otimes K_\mu(z|z';t)\}$.

We now want to derive the generator $\mathcal{L}$ of this map

$$\Lambda_t = \exp\left(\int_0^t ds\mathcal{L}_s\right) \tag{31}$$

using

$$\mathcal{L}_t\hat{\rho}(z;t) = \lim_{\delta t \to 0} \frac{\hat{\rho}(z;t+\delta t) - \hat{\rho}(z;t)}{\delta t}. \tag{32}$$

This is possible, if at short times, the map has an inverse (which need not be CP), since then it can always be written in time-local form [123] (the proof contained in Refs. [124,125] also applies to maps on CQ states). I.e.,

$$\dot{\hat{\rho}}(z;t) = \mathcal{K}(t)\hat{\rho}(z';t). \tag{33}$$

To this end, we write the Kraus operators in terms of an orthonormal basis of operators $\hat{L}_\alpha$ to obtain

$$\hat{\rho}(z;t) = \sum_{\alpha,\beta=0} \int dz'\Lambda^{\alpha\beta}(z|z';t)\hat{L}_\alpha\hat{\rho}(z';0)\hat{L}_\beta^\dagger, \tag{34}$$

where complete positivity for all z, z' requires the $\Lambda^{\alpha\beta}(z|z';t)$ be a positive-semidefinite Hermitian matrix in α, β for all z, z' (see Ref. [130] and also Appendix A). The trace-preserving condition Eq. (29) requires

$$\sum_{\alpha\beta} \int dz\Lambda^{\alpha\beta}(z|z';t)\hat{L}_\beta^\dagger\hat{L}_\alpha = \mathbb{1}. \tag{35}$$

Since $\{\hat{L}_\alpha\}$ is an arbitrary orthogonal basis of the operator space, we can always take $\hat{L}_0 = \mathbb{1}$, and the orthogonality condition $\text{tr}\hat{L}_\beta^\dagger\hat{L}_\alpha = 0$ for $\alpha \neq \beta$ means that the other $\hat{L}_\alpha$ are traceless, and we denote them with bold font $\boldsymbol{\alpha}$. We are interested in changes of the state over infinitesimally short times δt. Since the dynamics is time local [Eq. (33)], we can expand the $\Lambda^{\alpha\beta}$ as

$$\Lambda^{00}(z|z';t+\delta t) = \Lambda^{00}(z|z';t) - \delta(z-z')\gamma^{00}(z';t)\delta t$$
$$+ W^{00}(z|z';t)\delta t + O(\delta t^2), \tag{36}$$

$$\Lambda^{\alpha\beta}(z|z';t+\delta t) = \Lambda^{\alpha\beta}(z|z';t) + \delta(z-z')\lambda^{\alpha\beta}(z';t)\delta t$$
$$+ W^{\alpha\beta}(z|z';t)\delta t + O(\delta t^2), \quad \alpha\beta \neq 00. \tag{37}$$

This is just a Taylor expansion where we divide up the terms proportional to δt into two contributions for later convenience and for comparison with existing convention. If the map acts on an initial state $\hat{\rho}(z;t)$, then $\Lambda^{00}(z|z';t) = \mathbb{1}\delta(z-z')$, since when $\delta t \to 0$ the system has to remain at the same point in phase space and the quantum system does not change. The other $\Lambda^{\alpha\beta}(z|z';t) = 0$, since these terms are

associated with Lindblad operators $\hat{L}_\alpha$ acting on the state and nothing happens to the state in the limit $\delta t \to 0$. This leaves

$$\Lambda^{00}(z|z';t+\delta t) = \delta(z-z')(1-\delta t\gamma^{00}(z';t))$$
$$+ W^{00}(z|z';t)\delta t + O(\delta t^2), \quad (38)$$

$$\Lambda^{\alpha\beta}(z|z';t+\delta t) = \delta(z-z')\lambda^{\alpha\beta}(z';t)\delta t$$
$$+ W^{\alpha\beta}(z|z';t)\delta t + O(\delta t^2), \quad \alpha\beta \neq 00. \quad (39)$$

The first term proportional to δt in Eq. (39) can be thought of as the probability per time δt the system does not move in phase space while the quantum system has $\hat{L}_\alpha\hat{\rho}(z;t)\hat{L}_\beta^\dagger$ applied to it, while the $W^{\alpha\beta}(z|z';t)$ term in both equations gives the rate that the system goes from z' to z and has $\hat{L}_\alpha\hat{\rho}(z;t)\hat{L}_\beta^\dagger$ applied to it. We adopt the convention that the sign in front of γ^{00} is negative, anticipating that it will contribute negatively; it is the no-event term, and the probability that the system is unchanged can decrease only with time.

The expansion for the Λ^{00} component is similar to the one used to derive Eq. (12) for continuous classical stochastic processes. Taking the limit $dt \to 0$ and keeping only the first-order terms usually require that the set of operators $\hat{L}_\alpha$ be bounded in order for such terms to be small, but here we replace that typical assumption with the assumption that the map is invertible at short times [leading to Eq. (33)]. To find the generator of the dynamics, we substitute Eqs. (38) and (39) into Eq. (34) to get

$$\hat{\rho}(z;t+\delta t) = \int dz'\delta(z-z')((1-\delta t\gamma^{00}(z';t))\hat{\rho}(z';t)$$
$$+ \delta t\lambda^{\alpha 0}(z;t)\hat{L}_\alpha\hat{\rho}(z';t) + \delta t\lambda^{0\beta}(z;t)\hat{\rho}(z';t)\hat{L}_\beta^\dagger$$
$$+ \delta t\lambda^{\alpha\beta}(z;t)\hat{L}_\alpha\hat{\rho}(z';t)\hat{L}_\beta^\dagger)$$
$$+ \sum_{\alpha\beta}\int dz'\delta t W^{\alpha\beta}(z|z';t)\hat{L}_\alpha(x)\hat{\rho}(z';t)\hat{L}_\beta^\dagger,$$
$$(40)$$

and subtracting $\hat{\rho}(z;t)$, dividing by δt, and taking the limit $\delta t \to 0$ gives

$$\frac{\partial\hat{\rho}(z;t)}{\partial t} = -\gamma^{00}(z;t)\hat{\rho}(z;t) + \lambda^{\alpha 0}(z;t)\hat{L}_\alpha\hat{\rho}(z;t)$$
$$+ \lambda^{0\beta}(z;t)\hat{\rho}(z;t)\hat{L}_\beta^\dagger + \lambda^{\alpha\beta}(z;t)\hat{L}_\alpha\hat{\rho}(z;t)\hat{L}_\beta^\dagger$$
$$+ \sum_{\alpha\beta}\int dz' W^{\alpha\beta}(z|z';t)\hat{L}_\alpha\hat{\rho}(z';t)\hat{L}_\beta^\dagger. \quad (41)$$

As in the derivation of the GKSL equation, we now define the two Hermitian operators

$$\hat{H}(z;t) := \frac{1}{2i}(\lambda^{0\beta}\hat{L}_\beta^\dagger - \lambda^{\alpha 0}\hat{L}_\alpha),$$

$$\hat{N}(z;t) := -\gamma^{00}(z;t) + \frac{1}{2}(\lambda^{0\beta}\hat{L}_\beta^\dagger + \lambda^{\alpha 0}\hat{L}_\alpha), \quad (42)$$

so that we can write Eq. (41) as

$$\frac{\partial\hat{\rho}(z;t)}{\partial t} = -i[\hat{H}(z;t),\hat{\rho}(z;t)] + \lambda^{\alpha\beta}(z;t)\hat{\rho}(z;t)\hat{L}_\alpha\hat{\rho}(z;t)\hat{L}_\beta^\dagger$$
$$+ \int dz' W^{\alpha\beta}(z|z';t)\hat{L}_\alpha\hat{\rho}(z';t)\hat{L}_\beta^\dagger$$
$$+ \{\hat{N}(z;t),\hat{\rho}(z;t)\}_+. \quad (43)$$

Having subtracted $\hat{\rho}(z;t)$ from Eq. (35) the trace-preservation condition Eq. (27) becomes Eq. (16), i.e.,

$$\int dz\,\mathrm{tr}\frac{\partial\hat{\rho}(z;t)}{\partial t} = 0$$

for any input state $\hat{\rho}(z';t)$, which requires

$$\hat{N}(z';t) = \frac{1}{2}\left(\sum_{\alpha\beta}\lambda^{\alpha\beta}(z';t)\hat{L}_\beta^\dagger\hat{L}_\alpha\right.$$
$$\left. + \int dz\sum_{\alpha\beta}W^{\alpha\beta}(z|z';t)\hat{L}_\beta^\dagger\hat{L}_\alpha\right). \quad (44)$$

Defining the outgoing flux $W^{\alpha\beta}(z;t)$ by Eq. (15) gives the desired form of the master equation (14).

The first term of Eq. (14) is the free evolution of the quantum system at each point in phase space, the second term corresponding to $\lambda^{\alpha\beta}(z;t)$ is pure Lindbladian, with a rate determined by the classical degrees of freedom. The remaining term corresponding to $W^{\alpha\beta}$ gives the interaction term between the classical and quantum degrees of freedom. It is worth noting that the term corresponding to $\alpha\beta = 00$,

$$\nabla^{00}\hat{\rho}(z;t) := \int dz' W^{00}(z|z')\hat{\rho}(z';t) - W^{00}(z)\hat{\rho}(z;t), \quad (45)$$

is the purely classical evolution of the system and is identical to Eq. (12). It can be stochastic or it can correspond to deterministic dynamics generated by a classical Hamiltonian $H(z)$, in which case, we can write it in terms of the Poisson bracket, or equivalently, as

$$\nabla^{00}\hat{\rho}(z;t) = -X_h^{00}\cdot\nabla\hat{\rho}(z;t), \quad (46)$$

where X_h^{00} is the Hamiltonian vector field

$$X_h^{00} = \Omega\nabla H. \quad (47)$$

Recalling Eqs. (20) and (18)

$$X_h^{00} = \left(\frac{\partial h^{00}}{\partial p}, -\frac{\partial h^{00}}{\partial q}\right)^T, \quad (48)$$

∇^{00} could also include a diffusion term as in the Fokker-Planck equation, but it will be natural to treat it in the same

way as the other $W^{\alpha\beta}$, in which case, it will be a stochastic version of Hamiltonian dynamics.

Let us now return to Eqs. (38) and (39) and the condition that complete positivity places on $\Lambda^{\alpha\beta}(z|z';t)$. In general, all that is required is that the total map from some initial state at $t=0$ to some final time is completely positive. In this case, the generators at intermediate times t need not generate completely positive dynamics if one were to replace the current state of the system $\hat{\rho}(z;t)$ with some other state as long as they generate CP dynamics on the state which had been evolving since $t=0$. In this way, one can consider non-Markovian dynamics. However, in the case where we demand that the generator is infinitesimally divisible, i.e., generate CP dynamics at all times on any state, then the positivity conditions for $\Lambda^{\alpha\beta}(z|z';t)$ to be completely positive can be determined from the right-hand side of Eqs. (38) and (39). For $z \neq z'$ [we can smear

Eqs. (38) and (39) over an indicator function with support on or away from $z=z'$ as is done in [71]], $\Lambda^{\alpha\beta}(z|z';t+\delta t)$ is equal to $W^{\alpha\beta}(z|z';t)\delta t$, and so it is necessary that $W^{\alpha\beta}(z|z';t)$ be a positive-semidefinite matrix in $\alpha\beta$. When $z=z'$, we just have the usual GKSL dynamics, and so we require $W^{\alpha\beta}(z|z';t) + \lambda^{\alpha\beta}(z,t)$ be positive semi-definite. These condition are also sufficient in the case of discrete dynamics, because these two cases completely specify the dynamics, and the only condition it has to satisfy is complete positivity and norm preserving. The necessary and sufficient conditions on $W^{\alpha\beta}(z|z';t)$ for continuous evolution are derived in Ref. [71]. They are best given in terms of the moment expansion of $W^{\alpha\beta}(z|z';t)$, which is not derived until Sec. IV, but for completeness we conclude this section with it, expressed as a matrix in α, β:

$$W(z|z';t) = \begin{bmatrix} \sum_{n=0}^{2} \frac{M_n^{00}(z',t)}{n!} \nabla_{z'}^{\otimes n} \delta(z-z'), & \sum_{n=0}^{1} \frac{M_n^{0\beta}(z',t)}{n!} \nabla_{z'}^{\otimes n} \delta(z-z') \\ \sum_{n=0}^{1} \frac{M_n^{\alpha 0}(z',t)}{n!} \nabla_{z'}^{\otimes n} \delta(z-z'), & M_0^{\alpha\beta}(z',t)\delta(z-z') \end{bmatrix} \tag{49}$$

with

$$M_2 \succeq M_1 M_0^{-1} M_1^{\dagger}, \qquad (\mathbb{I} - M_0 M_0^{-1})M_1 = 0. \tag{50}$$

Here, M_0^{-1} is the generalized inverse of the PSD matrix with elements $M_0^{\alpha\beta}$, M_1 is a matrix in both α, i where i indexes the components of z. It has entries $M_{1,i}^{0\alpha}$. M_2^{00} is a PSD matrix in i, j with entries $M_{2,ij}^{00}$. This generalizes the constant force dynamics of Ref. [47], and gives Eq. (7), allowing for any Hamiltonian $\hat{H}(z;t)$ to generate classical quantum dynamics, as well as more general non-Hamiltonian backreaction, such as those which include friction.

In the remainder of the article, we restrict our attention to the autonomous case where the coupling constants have no explicit time dependence.

IV. NONCOMMUTING HAMILTONIAN DYNAMICS

We would like stochastic hybrid dynamics, which has continuous Hamiltonian evolution in the classical limit. For the continuous dynamics of Eq. (49), this is given as Eq. (7). In this section, we explore the discrete master equation corresponding to Eq. (17) whose first few terms in its expansion correspond to classical quantum versions of the Liouville equation at first order and Fokker-Planck equation at second next order. We begin by taking z to be phase-space variables q, p, and later present the case where z represents canonically conjugate pairs of field degrees of freedom over x, since the generalization is straightforward,

while the notation is more cumbersome. The dynamics is generated by a Hamiltonian operator, which can also depend on phase-space variables z and a decomposition of the Hamiltonian in terms of Lindblad operators $\hat{L}_\alpha$ given by Eq. (18), which we recall as

$$\hat{H}(z) = h^{\alpha\beta}(z)\hat{L}_\beta^{\dagger}\hat{L}_\alpha.$$

When we get to the field theory case, we see that Lorentz invariance, and later, diffeomorphism invariance, place very tight constraints on this decomposition. The $h^{00}(z)$ term is purely classical, since it couples only via $\mathbb{1}$ to the quantum degrees of freedom. There is some ambiguity in what is included in the $\hat{L}_\alpha$ and what in $h^{\alpha\beta}(z)$, but it will turn out that this ambiguity is inconsequential when applying this formalism to general relativity. The matrix $h^{\alpha\beta}$ is Hermitian, and it is sometimes convenient to take it to be positive as is the case for the coupling term of general relativity. We define a generator of dynamics X_h which has both phase-space components and components α, β in Hilbert space determined by the Lindblad operators, i.e.,

$$X_h^{\alpha\beta} = \left(\frac{\partial h^{\alpha\beta}}{\partial p}, -\frac{\partial h^{\alpha\beta}}{\partial q}\right)^T. \tag{51}$$

Just as the Hamiltonian vector field determines the rate of flow along different directions in phase space in the case of purely deterministic classical evolution, here, $X_h^{\alpha\beta}$, additionally sets the rate of quantum jumps along different Lindblad operators.

We have already expanded Λ_t in terms of infinitesimally small times δt, and we now also expand it in terms of the

phase-space vector $\Delta := z - z'$, which we can then take arbitrarily small if we choose. To this end, let us write $\Lambda^{\alpha\beta}(z|z';t) = \Lambda^{\alpha\beta}(z|z - \Delta;t)$ and perform both an expansion in δt and a Kramers-Moyal expansion [115,116] in Δ. We can either do this directly on Eqs. (38) and (39), but it is slightly less cumbersome to perform it on $W^{\alpha\beta}(z|z - \Delta;t)$,

$$W^{\alpha\beta}(z|z - \Delta;t)\hat{\rho}(z - \Delta;t)$$
$$= W^{\alpha\beta}(z + \Delta - \Delta|z - \Delta;t)\hat{\rho}(z - \Delta;t)$$
$$= \sum_{n=0}^{\infty} \frac{(-\Delta \cdot \nabla)^n}{n!} [W^{\alpha\beta}(z + \Delta|z;t)\hat{\rho}(z;t)]. \quad (52)$$

We now integrate this expression over Δ to write

$$\int d\Delta W^{\alpha\beta}(z|z - \Delta;t)\hat{\rho}(z - \Delta;t)$$
$$= \sum_{n=0}^{\infty} (-\nabla)^{\otimes n} \times \left[\frac{M_n^{\alpha\beta}(z;t)}{n!} \hat{\rho}(z;t) \right] \quad (53)$$

in terms of the moments

$$M_n^{\alpha\beta}(z;t) = \int d\Delta (\Delta)^{\otimes n} W^{\alpha\beta}(z + \Delta|z;t). \quad (54)$$

Here, we write $M_n^{\alpha\beta}(z;t)$ as an n-fold tensor power, since $\Delta^{\otimes n} \cdot \nabla^{\otimes n} = (\Delta \cdot \nabla)^n$ with $\Delta \cdot \nabla$ the directional divergence in the direction Δ. This has the advantage that the notation is compact, but the disadvantage that it distinguishes the moment corresponding to $\partial_q \partial_p$ to that of $\partial_p \partial_q$ (for example). The zeroth moments

$$M_0^{\alpha\beta} = \int d\Delta W^{\alpha\beta}(z + \Delta|z;t) \quad (55)$$

are just $W^{\alpha\beta}(z;t)$ of Eq. (15). The first moments are the vectors

$$M_1^{\alpha\beta} = \int d\Delta \Delta W^{\alpha\beta}(z + \Delta|z;t). \quad (56)$$

For a Hamiltonian $\hat{H}$ and decomposition $h^{\alpha\beta}$, we see that there is a natural choice of $W^{\alpha\beta}(z + \Delta|z;t)$ so that the vector of first moments has a contribution corresponding to Hamiltonian flow

$$X_h^{\alpha\beta}(z) := \Omega \nabla h^{\alpha\beta}(z), \quad (57)$$

where we indicate that $W^{\alpha\beta}(z + \Delta|z)$ is generated from the field $h^{\alpha\beta}$, with the subscript h. This we do in Sec. IV A. We can also consider a contribution corresponding to friction $\gamma_h^{\alpha\beta}$ so that

$$\int d\Delta \Delta W_h^{\alpha\beta}(z + \Delta|z) := X_h^{\alpha\beta}(z) - \gamma_h^{\alpha\beta}(z). \quad (58)$$

Although there could be some arbitrariness in this division, a standard friction term is of the form $\gamma_h^{\alpha\beta}(z) = \tau\gamma\partial h^{\alpha\beta}/\partial p$ with γ a constant and τ included for later convenience.

Next, there is the matrix of diffusion terms

$$D^{\alpha\beta}(z) := \frac{1}{2} \int d\Delta \Delta \otimes \Delta W_h^{\alpha\beta}(z + \Delta|z). \quad (59)$$

Making the Markovian approximation so that we may drop the explicit dependence on time, and putting this together gives

$$\int d\Delta W^{\alpha\beta}(z|z - \Delta)\hat{\rho}(z - \Delta;t)$$
$$= W_h(z)\hat{\rho}(z;t) - X_h^{\alpha\beta} \cdot \nabla\hat{\rho}(z;t) + \nabla \cdot \gamma_h^{\alpha\beta}\hat{\rho}(z;t)$$
$$+ \nabla \cdot D^{\alpha\beta}(z)\hat{\rho}(z;t) \cdot \overleftarrow{\nabla} + \cdots, \quad (60)$$

where we use the fact that $\nabla \cdot X_h^{\alpha\beta} = 0$, and the notation $\overleftarrow{\nabla}$ is to indicate that ∇ acts on the second Δ in the tensor product of $D^{\alpha\beta}(z)$, while ∇ acts on the first. One typically refers to the terms which contain $\nabla\hat{\rho}(z;t)$ as the drift, since they determine the rate at which the expectation values of observables change. The terms corresponding to second derivatives lead to diffusion in phase space. Note that if $D^{\alpha\beta}(z)$ depends on both the q and p, then it can also include contributions to the drift, sometimes referred to as anomolous drift. Here we take $D^{\alpha\beta}(z)$ to depend only on the q so that there are no anomalous drift terms coming from this and higher-order terms.

The Pawula theorem [131] says that either the Kramers-Moyal expansion truncates at the second moment or requires an infinite sum of terms. In the latter case, our truncated master equation will not be completely positive, and should be regarded as an approximation of the dynamics of Eq. (14). We call a particular expansion of $\hat{H}$ in terms of Lindblad operators, and the choice of $W^{\alpha\beta}(z + \Delta|z)$ or $\Lambda^{\alpha\beta}(z + \Delta|z)$ a realization, and although we present two simple realizations in Sec. IV A, the main requirement we demand is that the moment expansion include $X_h^{\alpha\beta}(z)$, the Hamiltonian vector field for $h^{\alpha\beta}(z)$ in its first moment, Eq. (58). This ensures that we are able to reproduce the deterministic classical equations of motion in the appropriate limit. When we consider applying our results to field theory and later general relativity, Lorentz covariance and parameter invariance implies further constraints on $W^{\alpha\beta}(z|z')$.

Substituting the Kramers-Moyal expansion of $W^{\alpha\beta}(z|z')$ into Eq. (14) gives

$$\frac{\partial\hat{\rho}(z;t)}{\partial t} = -i[\hat{H}(z), \hat{\rho}(z;t)] + (\lambda^{\alpha\beta}(z)$$
$$+ M_0^{\alpha\beta}(z)) \left[\hat{L}_\alpha \hat{\rho}(z;t) \hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger \hat{L}_\alpha, \hat{\rho}(z;t)\}_+ \right]$$
$$- \hat{L}_\alpha X_h^{\alpha\beta}(z) \cdot \nabla\hat{\rho}(z;t)\hat{L}_\beta^\dagger + \hat{L}_\alpha \nabla \cdot [\gamma_h^{\alpha\beta}(z)\hat{\rho}(z;t)]\hat{L}_\beta^\dagger$$
$$+ \hat{L}_\alpha \nabla \cdot D^{\alpha\beta}(z)\hat{\rho}(z;t) \cdot \overleftarrow{\nabla}\hat{L}_\beta^\dagger + \cdots.$$

When we consider realizations of $W^{\alpha\beta}$, we typically find that each order in the expansion carries with it higher powers of some constant τ. If we identify $(1/\tau)W^{\alpha\beta}(z) := \lambda^{\alpha\beta}(z) + M_0^{\alpha\beta}$ and $D^{\alpha\beta} \to \tau D^{\alpha\beta}$, this is Eq. (17) highlighted in the Sec. I.

An important category of realizations are ones in which $M_n^{\alpha\beta}(z)$ for $n \geq 2$ and $\gamma_h^{\alpha\beta}(z)$ depend only on the conjugate coordinates q, and the diffusion is only in the conjugate momenta p, i.e.,

$$
\begin{aligned}
\frac{\partial\hat{\rho}(z;t)}{\partial t} = {}& -i[\hat{H}(z), \hat{\rho}(z;t)] \\
& + \frac{W^{\alpha\beta}(z)}{\tau}\left[\hat{L}_\alpha\hat{\rho}(z;t)\hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger\hat{L}_\alpha, \hat{\rho}(z;t)\}_+\right] \\
& - \hat{L}_\alpha X_h^{\alpha\beta}(z) \cdot \nabla\hat{\rho}(z;t)\hat{L}_\beta^\dagger \\
& + \tau\hat{L}_\alpha\frac{\partial}{\partial p}\cdot[\gamma_h^{\alpha\beta}(z)\hat{\rho}(z;t)]\hat{L}_\beta^\dagger \\
& + \tau\hat{L}_\alpha\frac{\partial}{\partial p}\cdot D^{\alpha\beta}(z)\hat{\rho}(z;t)\cdot\frac{\overleftarrow{\partial}}{\partial p}\hat{L}_\beta^\dagger + \cdots.
\end{aligned}
$$ (61)

This means that the q still obeys the equation of motion

$$
\dot{q} = \frac{\partial h^{00}}{\partial p}
$$ (62)

and allows one to define p deterministically in terms of q and $\dot{q}$. This can be verified using the Heisenberg representation for all moments of q (see Appendix B). Equation (61) is thus a classical quantum version of the Wiener process or Fokker-Planck equation, and we refer to such a master equation as Brownian. For such a master equation, it is the conjugate momentum which is influenced by the quantum system. In the more general case, p is defined only in terms of expectation values.

It is worth noting that in our expansion, we have a purely Lindbladian term which is itself trace preserving, and a total divergence term, with the quantity $X_h^{\alpha\beta} - \gamma_h^{\alpha\beta} - D^{\alpha\beta} \cdot \overleftarrow{\nabla}$ acting as a current. Thus, if we integrate over phase space, this term also conserves probability, contributing only a boundary term. Generally though, this term is not positive by itself, and one typically has to include the full expansion to ensure this. Looking at each of the terms in Eq. (17), we see that the first term is just the deterministic

quantum evolution. One can have a purely deterministic classical evolution term as well. Since $\hat{L}_0 = \mathbb{1}$, the X_h^{00} term is generated by the pure classical term h^{00} in the expansion of $\hat{H}(z)$ in Eq. (18),

$$
-X_h^{00}(z) \cdot \nabla\psi = \{h^{00}(z), \hat{\rho}(z;t)\}.
$$ (63)

The other deterministic term, at least on the classical system and for a particular $\alpha\beta$, is $\hat{L}_\alpha X_h^{\alpha\beta}(z) \cdot \nabla\hat{\rho}(z;t)\hat{L}_\beta^\dagger$, which, if we take the trace over the quantum system, gives the required dynamics we are looking for, Eq. (21).

The other nondeterministic parts include the pure-Lindblad term, and if we trace out the quantum system it contributes nothing to the dynamics of the classical system, except indirectly, as it evolves the quantum system which backreacts on the classical system. On the quantum system, it can generate decoherence, and if the Lindblad operators are diagonal in raising and lowering operators, it raises and lowers the state of the quantum system. The final two terms, if we trace out the quantum degrees of freedom are the friction and diffusion terms on phase space reminiscent of the Fokker-Planck equation, although here, there is again some non-Markovianity as the quantum system can serve as a memory for the classical system.

The Lindbladian term can determine the rate at which the wave function decoheres, and if this rate is small then the diffusion term tends to be large. This is to be expected since as discussed in Sec. I, a purely deterministic evolution on phase space would decohere the wave function instantly, and so a slow rate of decoherence requires either a very noisy stochastic evolution or a very slow jump rate.

It may be worth noting that we can now write a formal solution of Eq. (17). One way to do this is to double the Hilbert space [132–134] to write the density matrix $\hat{\rho}(z;t)$ in its eigenbasis

$$
\hat{\rho}(z;t) = \sum_i p_i(z)|i\rangle\langle i|
$$ (64)

as a pure entangled state in the Schmidt basis

$$
|\hat{\rho}(z;t)\rangle_{AB} = \sum_i \sqrt{p_i(z)}\,|i\rangle_A \otimes |i\rangle_B.
$$ (65)

The CQ master equation of Eq. (61) can then be written as

$$
\begin{aligned}
\frac{\partial|\hat{\rho}(z;t)\rangle_{AB}}{\partial t} = {}& \left[-i\hat{H}(z) \otimes \mathbb{1} + i\mathbb{1} \otimes \hat{H}^T(z) + [\lambda^{\alpha\beta}(z) + W^{\alpha\beta}(z)]\left(\hat{L}_\alpha \otimes \hat{L}_\beta^* - \frac{1}{2}\hat{L}_\beta^\dagger\hat{L}_\alpha \otimes \mathbb{1} - \frac{1}{2}\mathbb{1} \otimes \hat{L}_\alpha^T\hat{L}_\beta^*\right)\right. \\
& \left. - \hat{L}_\alpha \otimes \hat{L}_\beta^* X_h^{\alpha\beta} \cdot \nabla + \hat{L}_\alpha \otimes \hat{L}_\beta^* \nabla \cdot \gamma_h^{\alpha\beta}(z) + \hat{L}_\alpha \otimes \hat{L}_\beta^* \nabla^{\otimes 2} \cdot D^{\alpha\beta}(z) + \cdots\right]|\hat{\rho}(z;t)\rangle_{AB},
\end{aligned}
$$ (66)

where we use the fact that applying an operator to one subsystem of the entangled state is equivalent to applying the transpose to the other subsystem [135]. If we designate the operator in square brackets as $\mathcal{L}$, then the formal solution to the rate equation is

$$|\hat{\rho}(z;t)\rangle_{AB} = e^{\mathcal{L}t}|\hat{\rho}(z;0)\rangle. \tag{67}$$

The generator on the doubled system is in some sense a Hamiltonian, albeit one with an imaginary part.

A. Realizations of $W^{\alpha\beta}(z,z-\Delta)$

Here, we present some simple and constructive examples for $W^{\alpha\beta}(z, z - \Delta)$ which have the properties we require, namely, positive-semidefinite matrices whose first moment contain a term which generates Hamiltonian flow via Eq. (58). A simple realization valid for small τ is

$$W^{\alpha\beta}(z, z - \Delta) \approx \frac{1}{\tau}\delta^{(2N)}[\Delta - \tau X_h^{\alpha\beta}(z)]. \tag{68}$$

This gives the Liouville term as the first moment as required, and $1/\tau$ as the zeroth moment governing the collapse rate. We could take τ to depend on α, β, or include a distribution $f^{\alpha\beta}$ over τ. The realization of Eq. (68) can be considered an approximation of

$$\int d\Delta W^{\alpha\beta}(z, z - \Delta)\hat{\rho}(z - \Delta; t) = \frac{1}{\tau}e^{\tau\{h^{\alpha\beta},\cdot\}}\hat{\rho}(z; t) \tag{69}$$

with the right-hand side being the exponential of the Poisson bracket evolution $\{h^{\alpha\beta}(q, p), \cdot\} := \sum_k[(\partial h^{\alpha\beta}/\partial q_k)(\partial/\partial p_k) - (\partial h^{\alpha\beta}/\partial p_k)(\partial/\partial q_k)]$ which generates a finite shift in phase space along the flow generated by h (dropping the $\alpha\beta$ index for simplicity). This is the classical analog of the evolution operator induced by the quantum Hamiltonian $\hat{H}$, $e^{-i\hat{H}t}$ since

$$e^{\tau\{h,\cdot\}}\hat{\rho}(q, p; 0) = \hat{\rho}\left(q - \int_0^\tau dt \frac{\partial h}{\partial p}, p + \int_0^\tau dt \frac{\partial h}{\partial q}; 0\right)$$

$$\approx \hat{\rho}\left(q - \tau\frac{\partial h}{\partial p}, p + \tau\frac{\partial h}{\partial q}; 0\right). \tag{70}$$

Then $e^{\tau\{h,\cdot\}} - 1$ acts like a finite-time divergence along the direction of Hamiltonian flow generated by h.

This leads to a natural noncommuting generalization of this directional divergence which for a CQ Hamiltonian with a single Lindblad operator $\hat{H} = h\hat{L}^\dagger\hat{L}$, and for small τ, is

$$\boldsymbol{\nabla}\hat{\rho}(q, p) := \frac{1}{\tau}e^{\tau\{h,\cdot\}}\hat{L}^\dagger\hat{\rho}\,\hat{L} - \frac{1}{2\tau}\{\hat{L}^\dagger\hat{L}, \hat{\rho}\}_+. \tag{71}$$

In the case of many Lindblad operators, let us define basis vectors $e^{\alpha\beta}$ and define

$$\boldsymbol{\nabla}_{\alpha\beta}\hat{\rho}(z;t)e^{\alpha\beta} := \frac{1}{\tau}\left[\int d\Delta W_h^{\alpha\beta}(z|z - \Delta)\hat{L}_\alpha\hat{\rho}(z - \Delta; t)\hat{L}_\beta^\dagger \right.$$
$$\left. - W_h^{\alpha\beta}(z)\frac{1}{2}\{\hat{L}_\beta^\dagger\hat{L}_\alpha, \hat{\rho}(z;t)\}_+\right] \tag{72}$$

leading to

$$\boldsymbol{\nabla}_{\alpha\beta}\hat{\rho}(z;t)e^{\alpha\beta} = \frac{\hat{L}_\alpha e^{\tau\{h^{\alpha\beta},\cdot\}}\hat{\rho}(z;t)\hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger\hat{L}_\alpha, \hat{\rho}(z;t)\}_+}{\tau}, \tag{73}$$

which for small τ looks very much like a finite-directional divergence (both along the direction $X_h^{\alpha\beta}$ and along the Lindblad operators).

This definition depends on the decomposition of $\hat{H}$ into Lindblad operators, but one can first diagonalize $h^{\alpha\beta}(z)$, in which case, we write the left-hand side of Eq. (73) as $\boldsymbol{\nabla}_{\hat{H}}\hat{\rho}(z;t)$. In cases where

$$\boldsymbol{\nabla}_{\alpha\beta}\hat{\rho}(z;t)e^{\alpha\beta} \approx \frac{1}{\tau}\left[\hat{L}_\alpha\hat{\rho}(z - \tau X_h^{\alpha\beta})\hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger\hat{L}_\alpha, \hat{\rho}(z;t)\}_+\right], \tag{74}$$

it is sometimes the case that $X_h^{\alpha\beta}$ is non-negative (or nonpositive), in which case this discrete, noncommuting directional derivative takes the simpler form of a noncommuting finite difference

$$X_h^{\alpha\beta}\boldsymbol{\nabla}_{\alpha\beta}\hat{\rho}(z;t) \approx \frac{X_h^{\alpha\beta}}{\tau}\left[\hat{L}_\alpha\hat{\rho}(z - \tau)\hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger\hat{L}_\alpha, \hat{\rho}(z;t)\}_+\right]. \tag{75}$$

Care must be taken to ensure it is norm preserving, which is the case if, for example, $X_h^{\alpha\beta}$ depends only on q and the jump is in p. Although Eq. (73) looks like a discrete CQ Poisson bracket, $\boldsymbol{\nabla}_{\hat{H}}\hat{\rho}(z;t)$ is only linear in $\hat{\rho}(z;t)$ and not $\hat{H}$. On the other hand, the right-hand side of Eq. (4), which uses the Aleksandrov-Gerasimenko bracket, is bilinear and so it does have some properties more akin to a novel bracket although its action is not CP. The same holds true if we replace the Aleksandrov-Gerasimenko bracket in Eq. (4) with the zeroth and first moment of $\boldsymbol{\nabla}_{\hat{H}}\hat{\rho}(z;t)$. In some cases, we can have a bilinear map which is also CP as in Eq. (75).

The noncommuting divergence is trace preserving and completely positive, and allows us to write the master equation as

$$\frac{\partial \hat{\rho}(z;t)}{\partial t} = -\frac{i}{\hbar}[\hat{H}(z), \hat{\rho}(z;t)] + \mathbf{\nabla}_{\alpha\beta}\hat{\rho}(z;t)e^{\alpha\beta}$$

$$= -\frac{i}{\hbar}[\hat{H}(z), \hat{\rho}(z;t)]$$

$$+ \frac{1}{\tau}\left[\hat{L}_\alpha\hat{\rho}(z;t)\hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger\hat{L}_\alpha, \hat{\rho}(z;t)\}_+\right]$$

$$- X_h^{\alpha\beta} \cdot \hat{L}_\alpha\nabla\hat{\rho}(z;t)\hat{L}_\beta^\dagger + \cdots. \tag{76}$$

If we make the model more deterministic ($\tau \to 0$), the strength of the pure-Lindbladian term increases to compensate, while if we make the Lindbladian term small, the diffusion terms at second order become significant. Here, we have temporarily switched units to show $\hbar$, since it strongly suggests that we set $\tau = \hbar$. Thus, as $\hbar \to 0$, the decoherence renders the system completely classical, the diffusion terms and those of higher order go to zero, and we are left with purely continuous classical equations of motion. For gravity, the realizations are more tightly constrained due to diffeomorphism invariance, but similar realizations can be used to obtain dynamics which recover general relativity in the classical limit.

B. Example: Qubit coupled to a classical potential

Let us now review a simple system solved in Ref. [106] with Sparaciari, Šoda and Weller-Davies, which contains some features we will encounter with gravity [136]. While the model bears some similarity to the Stern-Gerlach experiment (cf. also Ref. [75]), the behavior of the spin can differ depending on how one decomposes the Hamiltonian in terms of Lindblad operators. Our system consists of a classical particle which couples to the spin of a two-level system or the energy of a quantum harmonic oscillator. The classical degrees of freedom are the position

$q = x$ and momentum p of the particle, with free evolution given by $h^{00} = p^2/2$. We could take the purely classical evolution from $W^{00}(z|z';t)$ to be deterministic or stochastic, and we consider the former here and discuss the latter in Ref. [106]. We take the total cq Hamiltonian to be given by $\hat{H}(z) = p^2/2m + B(q)\hat{H}^{(m)}$ with $\hat{H}^{(m)} = \omega\frac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$ for the oscillator and $\hat{H}^{(m)} = \left(\begin{smallmatrix} \omega & 0 \\ 0 & -\omega \end{smallmatrix}\right)$ for the qubit.

To model a Stern-Gerlach experiment, we consider a two-level system with states $|\uparrow\rangle, |\downarrow\rangle$ and a cq-density matrix which depends on the classical degrees of freedom.

$$\hat{\rho}(q,p) = \begin{pmatrix} u_\uparrow(q,p) & \alpha(q,p) \\ \alpha^*(q,p) & u_\downarrow(q,p) \end{pmatrix}. \tag{77}$$

For simplicity, we consider a linear potential $B(q) = Bq$ and a decomposition of the Hamiltonian, Eq. (18), in terms of Lindblad operators $\hat{L}_\uparrow = \sqrt{\omega}|\uparrow\rangle\langle\uparrow|$, $\hat{L}_\downarrow = \sqrt{\omega}|\downarrow\rangle\langle\downarrow|$ with $h^{\uparrow\uparrow}(z) = -h^{\downarrow\downarrow}(z) = Bq$ and the others zero. An example which builds toward understanding the field theoretic case is the harmonic oscillator with Lindblad operators $\hat{a}$ and $\hat{a}^\dagger$. This is more interesting in the sense that the pure-Lindbladian term raises and lowers the state of the quantum system. In the qubit system, this corresponds to choosing Lindblad operators $\hat{L}_\uparrow = \sqrt{\omega}|\downarrow\rangle\langle\uparrow|$, $\hat{L}_\downarrow = \sqrt{\omega}|\uparrow\rangle\langle\downarrow|$ with diagonal couplings $W^{\alpha\beta}(z|z-\Delta) = \delta_{\alpha\beta}W^{\alpha\beta}(z|z-\Delta)$, or even a single Lindblad operator $\hat{L} = \left(\begin{smallmatrix} 0 & \sqrt{\omega} \\ \sqrt{\omega} & 0 \end{smallmatrix}\right)$. Here, we just present the more trivial first case decomposition; the other cases, including nontrivial $B(q)$, are studied both analytically and numerically using an unraveling approach in Ref. [106]. We take a simple realization of the discrete dynamics, that of Eq. (68) for small τ,

$$\frac{\partial\hat{\rho}(q,p)}{\partial t} = -i[\hat{H}^{(m)}(q,p), \hat{\rho}(q,p)] + \{h^{00}, \hat{\rho}(q,p)\} + \sum_{\alpha,\beta}\frac{1}{\tau}\left[\hat{L}_\alpha\hat{\rho}\left(q - \tau\frac{\partial h^{\alpha\beta}}{\partial p}, p + \tau\frac{\partial h^{\alpha\beta}}{\partial q}\right)\hat{L}_\beta^\dagger - \frac{1}{2}\{\hat{L}_\beta^\dagger\hat{L}_\alpha, \hat{\rho}(q,p)\}_+\right]$$

$$= -iBq\left[\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}, \hat{\rho}(q,p)\right] + \left\{\frac{p^2}{2m}, \hat{\rho}(q,p)\right\} + \frac{\omega}{\tau}|\uparrow\rangle\langle\uparrow|\hat{\rho}(q, p + \tau B)|\uparrow\rangle\langle\uparrow|$$

$$+ \frac{\omega}{\tau}|\downarrow\rangle\langle\downarrow|\hat{\rho}(x, p - \tau B)|\downarrow\rangle\langle\downarrow| - \frac{\omega}{2\tau}\{\mathbb{1}, \hat{\rho}(q,p)\}_+. \tag{78}$$

In terms of the matrix elements of the cq state, Eq. (78), the master equation decouples into two equations for the diagonal components and one for the off-diagonal coherence term

$$\frac{\partial u_\uparrow(q,p)}{\partial t} = -\frac{p}{m}\frac{\partial u_\uparrow(q,p)}{\partial q} + \frac{\omega}{\tau}[u_\uparrow(q, p + \tau B) - u_\uparrow(q,p)],$$

$$\frac{\partial u_\downarrow(q,p)}{\partial t} = -\frac{p}{m}\frac{\partial u_\downarrow(q,p)}{\partial q} + \frac{\omega}{\tau}[u_\downarrow(q, p - \tau B) - u_\downarrow(q,p)], \tag{79}$$

$$\frac{\partial\alpha(q,p)}{\partial t} = -2iBq\omega\alpha(q,p) - \frac{p}{m}\frac{\partial\alpha(q,p)}{\partial q} - \frac{\omega}{\tau}\alpha(q,p). \tag{80}$$

Equation (80) has a solution

$$\alpha(q,p) = \rho\left(q - \frac{p}{m}t\right)e^{-2iB\omega t(q - \frac{pt}{2m}) - \frac{\omega t}{\tau}}, \tag{81}$$

where $\rho(q - pt)$ is any normalized function of its argument. We see that the coupling of the qubit results in decoherence, since the off-diagonal elements decay exponentially fast.

More than that, the wave function collapses into a definite state of being in the up or down state. This we see through the equation for the diagonal terms. Comparing Eq. (79) to Eq. (12), we see that at a rate of ω/τ the system undergoes jumps in momentum by $\pm\tau B$ depending on its state. Since we can monitor the classical degrees of freedom without disturbing the system, measuring the change in momentum will uniquely determine the value of the spin. In this sense, the interaction leads to more than just decoherence, but an objective collapse of the wave function [69,100]. While the basis to which the quantum state collapses can appear arbitrary in spontaneous collapse models, here it is determined by the form of the CQ interaction which will be tightly constrained by the requirement of reproducing Einstein gravity in the classical limit. More generally, if the backreaction force $\nabla h^{\alpha\beta}$ is unique when diagonalized, then monitoring the classical system uniquely determines which Lindblad operators were applied to the state. Thus, an initially pure quantum state remains pure conditioned on the classical degrees of freedom. In the continuous realizations of the theory, this has since been shown to be generically true in Ref. [29] with Layton and Weller-Davies, provided an inequality we call the decoherence-diffusion trade-off is saturated.

The solution (a Poisson distribution) for the diagonal terms of $\hat{\rho}$ in this and a more general setting is given in Ref. [106], but the evolution can be understood in broad terms by inspection. The particle is making a momentum jump of $\pm\tau B$ at a rate of ω/τ, so on average it has an acceleration of $\pm\omega B$ exactly as we expect for a Stern-Gerlach device. We can imagine that at each dt, there is a probability of the particle undergoing a jump in momentum, so by the law of large numbers, we expect the particle to have made on average $\omega t/\tau$ jumps, normally distributed with a variance of $\frac{\omega}{\tau}[1 - (\omega/\tau)]$. If we Taylor expand Eq. (79)

$$\frac{\partial u(q, p)}{\partial t} = -\frac{p}{m}\frac{\partial u(q, p)}{\partial x} \pm \omega B \frac{\partial u(q, p)}{\partial p}$$
$$+ \frac{\omega B^2 \tau}{2}\frac{\partial^2 u(q, p)}{\partial p^2} + \cdots \qquad (82)$$

(where u is $u_\uparrow$ or $u_\downarrow$ for $\pm$, respectively), then to first order we have a particle undergoing acceleration $\pm\omega B$ depending on its spin, as expected, with a diffusion term at higher order. However, care must be taken, as if we truncate the Taylor expansion, the density matrix can become negative. Equation (78) is completely positive however, and can be understood as follows: An initial pure state in superposition will initially have a constant momentum, but will then undergo jumps in momentum of finite size τB, and at a rate

ω/τ. This causes the state to collapse to the up or down state, at a rate ω/τ.

If we trace out the qubit and look at the average value of the particle's trajectory, then for a uniform superposition of up and down spin, it will have zero average acceleration, but if we look at the full state, the particle undergoes acceleration (albeit stochastically) in a direction which depends on the value of the spin. We thus see that the evolution of the particle corresponds to the left panel of Fig. 1, not the right one.

C. Postquantum field theory

In order to consider applying our master equation to field theories, we must first ensure that it can be made Lorentz covariant. Although it was initially thought that such dynamics are impossible [137], GKSL equations which are covariant under the proper orthochronous Lorentz group were introduced in Ref. [138]. This was further taken up in Refs. [65,66] (see also Ref. [139]). A Lorentz-invariant Schwinger-Keldish action for open quantum scalar field theory was used to show that the GKSL equation for scalar field theories is renormalizable [140]. The formalism used there is manifestly Lorentz invariant. There are however subtleties that have since been clarified with Weller-Davies in Ref. [141]. The full master equation of Ref. [140] is not completely positive, but one can find Lorentz-invariant master equations which are CP, Lorentz invariant, and renormalizable, as well as master equations which are CP, renormalizable, and Lorentz covariant, but not Lorentz invariant. The Lindbladian corresponding to the discrete realization presented in Sec. V is covariant but not invariant, while the Lindbladian of the continuous realization is invariant but not covariant. This apparent paradox is resolved by showing that transformations of space-time are not generated by a unitary, but rather, by the semigroup inherited from the Lindbladian, and thus, there are different notions of invariance and covariance [141]. Here, we focus on covariance conditions, and address the subtleties in more detail in Refs. [141,142].

In order to make the dynamics here Lorentz covariant, one can begin by choosing Lindblad operators and couplings, such that the right-hand side of Eq. (14) transforms like $\partial/\partial t$, so that both sides of the equation transform in the same way under a Lorentz boost. This has been referred to as a minimal Lorentz-covariance requirement [137]. We first construct a field theory version of Eq. (14) and then discuss how to make it covariant under the proper orthochronous Lorentz group.

To obtain the field-theoretic version of the master equation, we take the Lindblad operators $\hat{L}_\alpha$ to also depend on the spatial coordinate x and in particular, take them to be local field operators $\hat{L}_\alpha(x)$. We can think of them as carrying a double index α, x. This requires the couplings W to carry not just an index $\alpha\beta$ but also x, y. Then, instead of a sum or integral over just α, β, we also have an integral

over x, y, and write the couplings as $W^{\alpha\beta}(z|z'; x, y)$ with the phase-space degrees of freedom z being local fields at each point x.

Cluster separability [64] demands that the coupling constants $W^{\alpha\beta}(z|z'; x, y)$ be delta functions in x, y [60], and that the coupling to phase-space variables z be to the local classical field $z(x)$. A violation of cluster separability can result in violations of Lorentz invariance, with correlations or entanglement being created over spacelike separations. However, the creation of correlations need not violate causality or allow for superluminal signaling [65]. Thus, while it is desirable to have $W^{\alpha\beta}(z|z'; x, y)$ be delta functions, for greater generality we allow the coupling constants $W^{\alpha\beta}(z|z'; x, y)$ to have support at finite $|x - y|$.

Should these couplings not fall off fast enough with increasing $|x - y|$, the theory is likely to run afoul of experiment. For the cq state at z, we continue to write it as $\hat{\rho}(z; t)$ with the understanding that it lives in Fock space, and is an operator-valued distribution functional of the classical fields $z(x)$ over the entire manifold of points x. If we consider the points x to be on a lattice, with the continuum obtained in the limit that the lattice spacing goes to zero, then we can think of the measure $\mathcal{D}z$ over phase space as $\Pi_x dq(x) dp(x)$, and in the continuum limit, the master equation is given by a functional integral.

If we take the Lindblad operators to be local field operators $\hat{L}_\alpha(x)$, then the field-theoretic version of Eqs. (14) and (17) is

$$
\begin{aligned}
\frac{\partial \hat{\rho}(z; t)}{\partial t} = &-i[\hat{H}(z), \hat{\rho}(z; t)] + \int dx d^3y \, \mathcal{D}\Delta W_h^{\alpha\beta}(z|z - \Delta; x, y) \hat{L}_\alpha(x) \hat{\rho}(z - \Delta; t) \hat{L}_\beta^\dagger(y) \\
&- \frac{1}{2} W_h^{\alpha\beta}(z; x, y) \{\hat{L}_\beta^\dagger(y) \hat{L}_\alpha(x), \hat{\rho}(z; t)\}_+
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
= &-i[\hat{H}(z), \hat{\rho}(z; t)] + \int dx d^3y \, W_h^{\alpha\beta}(z; x, y) \left[\hat{L}_\alpha(x) \hat{\rho}(z; t) \hat{L}_\beta^\dagger(y) - \frac{1}{2} \{\hat{L}_\beta^\dagger(y) \hat{L}_\alpha(x), \hat{\rho}(z; t)\}_+ \right] \\
&- \int dx \left[X_h^{\alpha\beta}(z; x) \hat{L}_\alpha(x) \cdot \nabla \hat{\rho}(z; t) \hat{L}_\beta^\dagger(x) - \nabla \cdot D_h^{\alpha\beta}(z; x, y) \cdot \hat{L}_\alpha(x) \hat{\rho}(z; t) \hat{L}_\beta^\dagger(y) \overleftarrow{\nabla} \right. \\
&\left. - \int dx \nabla \cdot [\gamma_h^{\alpha\beta}(z; x, y) \hat{L}_\alpha(x) \hat{\rho}(z; t) \hat{L}_\beta^\dagger(y)] \right] + \cdots.
\end{aligned}
\tag{84}
$$

One can also consider master equations which contain multiple jumps at higher order, i.e.,

$$
\int \mathcal{D}\Delta dx_1 dx_2 \ldots W^{\alpha_1\beta_1, \alpha_2\beta_2 \cdots}(z|z - \Delta; x_1, x_2, \ldots) \cdots \hat{L}_{\alpha_2}(x_2) \hat{L}_{\alpha_1}(x_1) \hat{\rho}(z - \Delta; t) \hat{L}_{\beta_1}^\dagger(x_1) \hat{L}_{\beta_2}^\dagger(x_2) \cdots,
$$

but we do not do so here.

In flat space, one can consider as an example the scalar field discussed later in Sec. V, and take the pure-Lindbladian coupling term $W^{\alpha\beta}(z; x, y)$ to be close to a delta function and proportional to $1/\tau$. If we wish to reproduce Hamiltonian dynamics in the classical limit, then we generally take the $X_h^{\alpha\beta}$ term to transform just as the Poisson bracket would for a Lorentz-invariant classical field theory, since the only difference between it and $\{\hat{H}(z), \hat{\rho}(z; t)\}$ is operator ordering.

As an example, we could decompose the Hamiltonian of the field theory in terms of the Lindblad operators used in Refs. [66,138,140]

$$
\hat{H}(x) = \int dx h^{\alpha\beta}(z; x) \hat{L}_\beta^\dagger(x) \hat{L}_\alpha(x)
\tag{85}
$$

with $h^{\alpha\beta}(z; x)$ coupling the quantum field to a classical field $z(x)$, which in the case of gravity, we can take to be the 3-metric and its conjugate 3-momentum. The couplings

$$
X_h^{\alpha\beta}(z; x) = \frac{\delta h^{\alpha\beta}(z; x)}{\delta z(x)}
\tag{86}
$$

then reproduce the Hamiltonian backreaction of the quantum fields on the classical ones. If one wants to be explicit about $W^{\alpha\beta}(z|z'; x, y)$, one can use the realization presented in Sec. V.

In Ref. [71], we show that depending on the decomposition $h^{\alpha\beta}(z; x)$, there are discrete dynamics or continuous dynamics. In particular, it is shown that the Kramers-Moyal expansion in Eq. (84) never terminates and leads to finite-sized jumps in phase space unless the only nonzero $D_h^{\alpha\beta}(z; x, y)$ is $D_h^{00}(x, y)$, and the only nonzero $X_h^{\alpha\beta}(z; x)$ is $X_h^{\alpha 0}(z; x)$ and $X_h^{0\alpha}(z; x)$, and the higher-order terms all vanish. In other words, the continuous master equation is obtained by restricting the decomposition of Eq. (85) to include only terms

$$\hat{H}(x) = \int dx h^{0\alpha}(z;x)\mathbb{1}\hat{L}_\alpha(x), \qquad (87)$$

while otherwise, one has discrete dynamics.

The positivity conditions in the field-theoretic case are analogous to the finite-dimensional conditions of Eq. (50). Recalling that for $2n$ phase-space degrees of freedom per point, $D_h^{00}(z;x,y)$ is a $2n \times 2n$ matrix over x, y and denoting $W_h(z;x,y)$ as the matrix with elements $W_h^{\alpha\beta}(z;x,y)$, and $X_h(z;x)$ as the vector with elements $X_h^{0\alpha}(z;x)$, the continuous master equation is completely positive, provided that the square matrix

$$\begin{bmatrix} D_h^{00}(x,y) & \delta(x,y)X_h(z;y) \\ X_h^\dagger(z;x)\delta(x,y) & W_h(z;x,y) \end{bmatrix} \succeq 0, \qquad (88)$$

which for invertible $W_h(z;x,y)$ is equivalent to the condition that $D_h^{00}(x,y)$ be positive semidefinite along with its Schur complement

$$D_h^{00}(x,y) - X_h^\dagger(z;x)W_h^{-1}(z;x,y)X_h(z;y) \succeq 0, \qquad (89)$$

where $W_h^{-1}(z;x,y)$ is the generalized inverse of $W_h(z;x,y)$. The positivity condition for singular $W_h(z;x,y)$ is a straightforward extension of Eq. (89); see, for example, Ref. [143]. We have since shown that the condition given by Eq. (89) holds for the zeroth, first, and second moments of any consistent classical quantum coupling, and provides an experimentally testable signature of classical gravity [50],

Turning back to Lorentz covariance, one might worry that in the master equation of Eq. (84), there is a dependence on the choice ∂_t,

$$\partial_t \hat{\rho}(z;t) = \mathcal{L}_t \hat{\rho}(z;t). \qquad (90)$$

As discussed in Ref. [138], we can consider any four-vector a from the future cone $\mathcal{F}^+$, i.e., $a \cdot a \geq 0$, and $a_0 \geq 0$, and consider the dynamics $\partial_a \hat{\rho}(z;t) = \mathcal{L}_a$. Indeed, consider M an element of the proper orthochronous Lorentz group. Let $\pi(a, M)$ be a representation of the Poincaré group and let the evolution of the state $\hat{\rho}(z;t)$ be given by a completely positive trace-preserving Markovian map $\mathcal{E}_a$. The Lorentz-covariance condition as defined in Ref. [138] (see also Refs. [66,144]) is then

$$\pi(a, M)\mathcal{E}_b\pi(a, M)^{-1} = \mathcal{E}_{Mb} \qquad (91)$$

for any $b \in \mathcal{F}^+$. In other words, if we perform a boost from the lab frame, evolve the state, then boost back into the lab, one gets the same final state as evolving the initial state in the lab for the boost time. The open quantum scalar field theory we consider in the next section satisfies this property when the background space-time is flat, as can be seen from the fact that in that limit, it is equivalent to the Lindbladian

considered in Ref. [138]. The relationship between this covariance condition and Lorentz invariance is discussed in Ref. [141].

V. POSTQUANTUM GENERAL RELATIVITY

We now have a procedure to take a Hamiltonian, which couples classical and quantum degrees of freedom, and use it to generate consistent hybrid dynamics. Let us now do that for general relativity coupled to quantum matter via the energy-momentum tensor. We do so in the ADM formulation of general relativity [118,145], whose classical formulation we now recall. One starts with a slicing up of space-time into spacelike hypersurfaces labeled by the parameter t, and with coordinates x. The hypersurfaces are endowed with a 3-metric g_{ab} with determinant g at each point x. We often drop the explicit x dependence from field degrees of freedom, since all quantities should be assumed to be locally defined, unless they are a functional, in which case, this is indicated with square brackets. We put in the explicit x dependence when we have two integrals over space, or if there could be any ambiguity. The components of the 3-metric are the canonical coordinates, and π^{ab} their conjugate momenta

$$\pi^{ab} = \sqrt{g}(K^{ab} - g^{ab}K) \qquad (92)$$

with K^{ab} the extrinsic curvature and K its trace.

In general relativity, one is free to choose the timelike direction by arbitrarily specifying a lapse N and shift vector N^a at each point x. The evolution of the hypersurfaces is then governed by the classical ADM Hamiltonian

$$H_{\text{ADM}}[N, \vec{N}] := \int dx(N\mathcal{H}^{(\text{tot})} + N^a\mathcal{H}_a^{(\text{tot})})$$
$$+ \oint ds \times \mathcal{H}^{(\text{boundary})}, \qquad (93)$$

and we now define the various terms appearing in it. $\mathcal{H}^{(\text{tot})}$ is called the total super-Hamiltonian and is typically written in terms of a pure gravity part, and a term which contains the matter fields

$$\mathcal{H}^{(\text{tot})} = \mathcal{H}^{(\text{gr})} + \mathcal{H}^{(m)}. \qquad (94)$$

The first term is the purely gravitational part of the super-Hamiltonian given by

$$\mathcal{H}^{(\text{gr})} := G_{abcd}\pi^{ab}\pi^{cd} - g^{1/2}R \qquad (95)$$

with R the intrinsic curvature on the 3-manifold analogous to a potential term, while the first term is analogous to a kinetic term, with G_{abcd} the deWitt metric,

$$G_{abcd} = \frac{1}{2}g^{-1/2}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}). \qquad (96)$$

$\mathcal{H}^{(m)}$ is the matter Hamiltonian density, and we take it to also include the cosmological constant term $\sqrt{g}g_{ab}\Lambda_{cc}$.

The total supermomentum $\mathcal{H}_a^{(\mathrm{tot})}$ is given by

$$\mathcal{H}_a^{(\mathrm{tot})} = \mathcal{H}_a^{(\mathrm{gr})} + \mathcal{H}_a^{(m)} \tag{97}$$

with $\mathcal{H}_a^{(\mathrm{gr})} := -2g_{ac}\nabla_b\pi^{cb}$ the purely gravitational contribution, and ∇_b the covariant derivative with respect to the Levi-Civita connection for the 3-metric. $\mathcal{H}_a^{(m)}$ is the matter contribution. The final term in Eq. (93) is the boundary term, and we ignore it in the present paper, although the formalism can be extended to apply to its inclusion. Finally, it proves convenient to define the total gravitational Hamiltonian $H^{(\mathrm{gr})}[N, \vec{N}]$ and the total matter Hamiltonian $H^{(m)}[N, \vec{N}]$ as functionals so that $H_{\mathrm{ADM}}[N, \vec{N}] = H^{(\mathrm{gr})}[N, \vec{N}] + H^{(m)}[N, \vec{N}]$ and

$$H^{(\mathrm{gr})}[N, \vec{N}] := \int dx (N\mathcal{H}^{(\mathrm{gr})} + N^a\mathcal{H}_a^{(\mathrm{gr})}),$$
$$H^{(m)}[N, \vec{N}] := \int dx (N\mathcal{H}^{(m)} + N^a\mathcal{H}_a^{(m)}). \tag{98}$$

With the Hamiltonian now defined, let us turn to the equations of motion. The dynamics is given by Hamilton's equations

$$\frac{\partial g_{ab}}{\partial t} = \frac{\delta H_{\mathrm{ADM}}[N, \vec{N}]}{\delta\pi^{ab}}, \qquad \frac{\partial\pi^{ab}}{\partial t} = -\frac{\delta H_{\mathrm{ADM}}[N, \vec{N}]}{\delta g_{ab}}, \tag{99}$$

where δ indicates the Fréchet derivative. In order for these dynamics to be invariant under the choice of the lapse N and shift N^a, we need to impose primary constraints on their conjugate momenta

$$P_N \approx 0, \qquad P_{N_a} \approx 0, \tag{100}$$

where $\approx$ denotes that the constraints are weakly zero in the sense that they vanish only on part of the phase space (the constraint surface). To ensure that these primary constraints are conserved in time, i.e., that

$$\dot{P}_N = 0, \qquad \dot{P}_{N_a} = 0, \tag{101}$$

we require secondary constraints

$$\mathcal{H}^{(\mathrm{tot})} \approx 0, \qquad \mathcal{H}_a^{(\mathrm{tot})} \approx 0. \tag{102}$$

The constraints, once initially satisfied, are preserved by the dynamical equations of motion.

One can write a Liouville equation for classical general relativity

$$\frac{\partial\rho(g, \pi; t)}{\partial t} = \{H_{\mathrm{ADM}}[N, \vec{N}], \rho\}$$
$$= \{H^{(\mathrm{gr})}[N, \vec{N}], \rho\}_g + \{H^{(m)}[N, \vec{N}], \rho\}_m$$
$$+ \{H^{(m)}[N, \vec{N}], \rho\}_g, \tag{103}$$

where we divided up the Poisson bracket into one with respect to the pure gravitational degrees of freedom

$$\{H^{(m)}[N, \vec{N}], \rho\}_g$$
$$:= \int dx \left(\frac{\delta H^{(m)}[N, \vec{N}]}{\delta g_{ab}}\frac{\delta\rho}{\delta\pi^{ab}} - \frac{\delta H^{(m)}[N, \vec{N}]}{\delta\pi^{ab}}\frac{\delta\rho}{\delta g_{ab}}\right), \tag{104}$$

and one with respect to the matter degrees of freedom, which for a scalar field with conjugate variables ϕ, π_ϕ at each point x would be

$$\{H^{(m)}[N, \vec{N}], \rho\}_m$$
$$= \int dx \left(\frac{\delta H^{(m)}[N, \vec{N}]}{\delta\phi}\frac{\delta\rho}{\delta\pi_\phi} - \frac{\delta H^{(m)}[N, \vec{N}]}{\delta\pi_\phi}\frac{\delta\rho}{\delta\phi}\right). \tag{105}$$

The Poisson bracket with respect to the lapse and shift and their conjugate momenta is zero when the constraints are satisfied and is usually not included in the definition of the Poisson bracket. When we discuss the classical quantum theory, it may be useful to also include them in the definition of the Poisson bracket.

When only matter is quantum, the matter Poisson bracket of Eq. (105) just becomes the commutator $-i[\hat{H}^{(m)}[N, \vec{N}], \hat{\rho}]$ which is unproblematic, and just represents the evolution of the quantum fields on a spatial manifold endowed with an evolving classical metric. This is just quantum field theory in curved space. However, the Poisson bracket of Eq. (104) is the backreaction of the quantum fields on space-time, and this is the dynamics we now seek to describe in terms of CQ dynamics.

We first replace $T^{\mu\nu}$ with the stress tensor of quantum field theory, $\hat{\mathcal{H}}^{(m)} = \sqrt{g}\hat{T}^{00}$. We can now write

$$\hat{\mathcal{H}}^{(\mathrm{tot})} = \mathcal{H}^{(\mathrm{gr})} + \hat{\mathcal{H}}^{(m)}, \tag{106}$$

as long as we recall that this object is hybrid and cannot be used to derive classical or quantum evolution in the traditional way. For the discrete master equation, we expand $\hat{\mathcal{H}}^{(\mathrm{tot})}$ in terms of local field operators $\hat{L}_\alpha(x)$, so that dropping the explicit x dependence

$$\hat{\mathcal{H}}^{(\mathrm{tot})} = \mathcal{H}^{(\mathrm{gr})}\mathbb{1} + h^{\alpha\beta}(g, \pi)\hat{L}_\beta^\dagger\hat{L}_\alpha \tag{107}$$

with $\hat{L}_0(x) = \mathbb{1}$ multiplying the purely classical part of the super-Hamiltonian. For convenience, we explicitly separate out the pure gravity contribution $\mathcal{H}^{(\mathrm{gr})}$, and one can then

take $h^{00} = 0$. Although in our derivation of the master equation, we take the $\hat{L}_\alpha$ to be a basis and $\hat{L}_\alpha$ to be traceless, one can show as with the Lindblad equation, that any set of Lindblad operators can be put into this standard form [71], and so we henceforth do not assume that the other $\hat{L}_\alpha$ are traceless. We often consider matter which is minimally coupled, meaning $\hat{\mathcal{H}}^{(m)}$ does not depend on π^{ij} so that we can write $h^{\alpha\beta}(g)$. For the continuous master equation, which we consider elsewhere [146], we use Lindblad operators $\hat{L}_{ij} = \delta\hat{\mathcal{H}}^{(m)}[N,\vec{N}]/\delta g_{ij}, \hat{\mathcal{H}}^{(m)}[N,\vec{N}]$, and $L_0 = \mathbb{1}$.

The supermomentum can be treated similarly, where the stress-energy term in Eq. (97) is replaced by its quantum counterpart

$$\mathcal{H}_a^{(m)} \to \hat{\mathcal{H}}_a^{(m)} \tag{108}$$

leading to

$$\hat{\mathcal{H}}_a^{(\mathrm{tot})} = \mathcal{H}_a^{(\mathrm{gr})}\mathbb{1} + p_a^{\alpha\beta}(g)\hat{L}_\beta^\dagger\hat{L}_\alpha \tag{109}$$

for the discrete master equation. Although the matrix with elements $N^a p_a^{\alpha\beta}$ is not generally a positive-semidefinite matrix, the combination $N h^{\alpha\beta} + N^a p_a^{\alpha\beta}$ is.

To make things concrete, it is useful to have in mind a scalar field, especially since the general case can be treated similarly and Lindbladian evolution of ϕ^4 theory has been shown to be renormalizable in this case [140], albeit with the caveats mentioned in Sec. IV C. We have [147]

$$\hat{\mathcal{H}}^{(m)} = \frac{1}{2}(g^{-1/2}\pi_\phi^2 + \sqrt{g}g^{ab}\nabla_a\phi\nabla_b\phi + \sqrt{g}m^2\phi(x)^2$$
$$- 4\sqrt{g}\Lambda_{cc}), \tag{110}$$

$$\hat{\mathcal{H}}_a^{(m)} = \pi_\phi\nabla_a\phi. \tag{111}$$

For the discrete evolution, we can now decompose the Hamiltonian as in Eq. (107), with the metric degrees of freedom included in $h^{\alpha\beta}(g,\pi)$.,

$$h^{\pi\pi} := g^{-1/2}, \qquad h^{\phi\phi} := \sqrt{g},$$
$$h^{ab} := \sqrt{g}g^{ab}, \qquad h^{\Lambda\Lambda} = \sqrt{g}, \tag{112}$$

with the rest zero, leaving the local Lindblad operators as

$$\hat{L}_\pi = \frac{1}{\sqrt{2}}\pi_\phi, \qquad \hat{L}_\phi = \frac{m}{\sqrt{2}}\phi,$$
$$\hat{L}_a = \frac{1}{\sqrt{2}}\partial_a\phi, \qquad \hat{L}_\Lambda = \sqrt{2\Lambda}|\Lambda\rangle\langle\Lambda|. \tag{113}$$

Note that the $h^{\alpha\beta}(g,\pi)$ do not include the N prefactor, since we use it to apply the constraint equation; however, when we look at the dynamical equations of motion, we need to

include it. We choose a simple form for $\hat{L}_\Lambda$, but, *a priori*, other choices are possible, for example, the shift operator. Their particular dependence is not important in terms of the broad outline of this discussion, rather, what is important is that we can decompose $\hat{\mathcal{H}}^{(m)}$ in terms of local field operators. The expansion of the supermomentum in terms of local operators could be treated similarly, with

$$\hat{L}_\pi = \pi_\phi, \qquad \hat{L}_a = \partial_a\phi, \qquad \hat{L}_\mathbb{1} = \mathbb{1}. \tag{114}$$

Note that $\hat{\mathcal{H}}_a^{(m)}$ has no metric dependence and thus does not contribute to the backreaction.

Although we cannot generally define Fock states locally on a Cauchy slice, if we know, for example, that the dynamics will lead to a space-time which asymptotically has a timelike Killing vector, then we can choose a vacuum state and set of creation and annihilation operators $\hat{a}_{\vec{p}}(g), \hat{a}_{\vec{p}}^\dagger(g)$, which depend on the metric to expand the matter Hamiltonian as

$$H^{(m)}[N,\vec{N}] = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}(\hat{a}_{\vec{p}}^\dagger(g)\hat{a}_{\vec{p}}(g) + \hat{a}_{\vec{p}}(g)\hat{a}_{\vec{p}}^\dagger(g)),$$
$$\tag{115}$$

where we drop the cosmological constant term for simplicity. In a fixed background, for example, the Lindblad equation using the Lindblad operators of Eq. (113) are equivalent to those of Eq. (115). In this way, one sees the covariance of the master equation; however, for general dynamical space-times, an expansion of local operators in terms of creation and annihilation operators is not possible, and we use local Lindblad operators such as those of Eq. (113).

The dynamical equations of motion are generated by the CQ-ADM Hamiltonian

$$\hat{H}_{\mathrm{ADM}}[N,\vec{N}] = \int dx(N\hat{\mathcal{H}}^{(\mathrm{tot})} + N^a\hat{\mathcal{H}}_a^{(\mathrm{tot})}) \tag{116}$$

with $\hat{\mathcal{H}}^{(\mathrm{tot})}$ and $\hat{\mathcal{H}}_a^{(\mathrm{tot})}$ given by Eqs. (107) and (109). If we wish the dynamics to be invariant under the choice of lapse and shift, this would place strong constraints on it. It seems to require that the stochastic part of the master equation be generated by $NW_h^{\alpha\beta}(z|z-\Delta; x, y)$, i.e., that the model be linear in the sense that $W_{Nh}^{\alpha\beta}(z|z-\Delta; x, y)$ can be replaced by $NW_h^{\alpha\beta}(z|z-\Delta; x, y)$ and $W_{N_a p^a}^{\alpha\beta}(z|z-\Delta; x, y)$ by $N_a W_{p^a}^{\alpha\beta}(z|z-\Delta; x, y)$. This is analogous to replacing $\{N\mathcal{H}^{(\mathrm{tot})}, \rho\}$ by $N\{\mathcal{H}^{(\mathrm{tot})}, \rho\}$ in the classical case, since they are equal when the constraint $\mathcal{H}^{(\mathrm{tot})} \approx 0$ is satisfied. In what follows, we take the lapse and shift outside the functional derivatives under the assumption that they either have no explicit metric dependence or the constraints are satisfied. The constraints in this theory will not be those of GR, but must be modified. As noted in Ref. [156], it is not

strictly necessary that the dynamics be invariant under the choice of N and N_a, and there are sensible theories which break this, such as shape dynamics [157–159] or unimodular gravity [160–166], but it is desirable to have some local symmetry, even if it is not full diffeomorphism invariance.

We do not restrict ourselves to a particular realization of $W_h^{\alpha\beta}(z|z-\Delta;x,y)$ and $W_{p_a}^{\alpha\beta}(z|z-\Delta;x,y)$ at this point, but we assume that the realization satisfies this linear condition and then present realizations which have this property. We thus have the master equation

$$\frac{\partial\hat{\rho}(z;t)}{\partial t} = \int dx N\{\mathcal{H}^{(\mathrm{gr})},\hat{\rho}(z;t)\} + \int dx N^a\{\mathcal{H}_a^{(\mathrm{gr})},\hat{\rho}(z;t)\} - i[H^{(m)}[N,\vec{N}],\hat{\rho}(z;t)]$$
$$+ \int dxdy\mathcal{D}\Delta\left[NW_h^{\alpha\beta}(z|z-\Delta;x,y)\hat{L}_\alpha(x)\hat{\rho}(z-\Delta)\hat{L}_\beta^\dagger(y) - \frac{1}{2}NW_h^{\alpha\beta}(z;x,y)\{\hat{L}_\beta^\dagger(y)\hat{L}_\alpha(x),\hat{\rho}(z;t)\}_+ \right]$$
$$+ \int dxdy\mathcal{D}\Delta\left[N_a W_{p^a}^{\alpha\beta}(z|z-\Delta;x,y)\hat{L}_\alpha(x)\hat{\rho}(z-\Delta)\hat{L}_\beta^\dagger(y) - \frac{1}{2}N_a W_{p^a}^{\alpha\beta}(z;x,y)\{\hat{L}_\beta^\dagger(y)\hat{L}_\alpha(x),\hat{\rho}(z;t)\}_+ \right], \quad (117)$$

where we write z for the phase-space degrees of freedom when g_{ab},π^{ab} are too cumbersome and have absorbed the bare term $\lambda^{\alpha\beta}(z;x,y)$ into $W_h^{\alpha\beta}(z|z-\Delta;x,y)$. The functional integral $\mathcal{D}\Delta$ is over the phase space. Here we take the lapse N and shift N^a outside the Poisson bracket, under the assumption that the constraints (suitably modified from the deterministic theory) are satisfied [156].

The first two terms of Eq. (117) correspond to the deterministic classical evolution associated with the Lindblad operators $\hat{L}_0 = \mathbb{1}$. The dynamics associated with this classical part could be stochastic, and could *a priori* include jumping or diffusion terms as well as friction terms. We take such terms to be included in $W_h^{00}(z|z-\Delta;x)$. Because they are associated with $\hat{L}_0$, they do not need to be positive.

The next term is the commutator of the state with the matter Hamiltonian describing how quantum matter evolves in a background space-time. Finally, we have

the interaction terms between gravity and matter which must be stochastic. Two particular choices of $W^{\alpha\beta}(z;x,y)$ are given as Eqs. (120) and (125), and in both cases, the integral over Δ can be removed. As shorthand, we denote Eq. (117) by

$$\frac{\partial\hat{\rho}(g,\pi;t)}{\partial t} = \mathcal{L}_{\hat{H}_{\mathrm{ADM}}}\hat{\rho}(g,\pi;t)$$
$$= \int dx[N\mathcal{L}\hat{\rho}(g,\pi;t) + N^a\mathcal{L}_a\hat{\rho}(g,\pi;t)]. \quad (118)$$

Because $\hat{\mathcal{H}}_a^{(m)}$ has no metric dependence, the evolution along N^a can be pure Lindbladian or purely deterministic. In Ref. [156], we argue that diffeomorphism invariance requires the latter, i.e., that $W_{p^a}^{\alpha\beta}(z|z-\Delta;x,y) = W_{p^a}^{\alpha\beta}(z;x,y) = 0$. Equation (117) then simplifies to

$$\frac{\partial\hat{\rho}(z;t)}{\partial t} = \int dx\left[N\{\mathcal{H}^{(\mathrm{gr})},\hat{\rho}(z;t)\} + N^a\{\mathcal{H}_a^{(\mathrm{gr})},\hat{\rho}(z;t)\} - i[H^{(m)}[N,\vec{N}],\hat{\rho}(z;t)] \right]$$
$$+ \int dxdy N\mathcal{D}\Delta\left[W_h^{\alpha\beta}(z|z-\Delta;x,y)\hat{L}_\alpha(x)\hat{\rho}(z-\Delta;t)\hat{L}_\beta^\dagger(y) - \frac{1}{2}W_h^{\alpha\beta}(z;x,y)\{\hat{L}_\beta^\dagger(y)\hat{L}_\alpha(x),\hat{\rho}(z;t)\}_+ \right]. \quad (119)$$

This should be thought of as the fundamental dynamical equation, but we can perform a Kramers-Moyal expansion on it. Since we expect all moments to appear, the moment expansion will include terms such as $(\delta h^{\alpha\beta}/\delta g_{ab})[\delta\hat{\rho}(z;t)/\delta\pi^{ab}]$ since it transforms like a density. This moment at zeroth order in τ is needed to recover the classical limit of general relativity. We now require a positive transition matrix $W_h^{\alpha\beta}(z|z-\Delta;x,y)$ [except for the $W_h^{00}(z|z-\Delta;x,y)$ component], and that each moment transforms as a density to ensure invariance of the

dynamics under spatial diffeomorphism. An example of such a realization for the discrete master equation is given in Ref. [156]; namely, for $\alpha, \beta \neq 0$,

$$\int dy\mathcal{D}\Delta N W_h^{\alpha\beta}(z|z-\Delta;x,y)\hat{L}_\alpha(x)\hat{\rho}(z-\Delta;t)\hat{L}_\beta^\dagger(y)$$
$$= \frac{N}{\tau}\exp\left[\tau(h^{-1})(z;x)\{h(z;x),\cdot\}\right]_\gamma^\alpha$$
$$\times h^{\beta\gamma}(z;x)\hat{L}_\alpha(x)\hat{\rho}(z;t)\hat{L}_\beta^\dagger(x), \quad (120)$$

where $\exp[M]^{\alpha}_{\gamma}$ is the components of the matrix exponential of the matrix M and it is purely local in x. Here, M has components $M^k_i = [\tau(h^{-1})_{ij}(z)\{h^{jk}(z),\cdot\}]$ and $W^{\alpha\beta}(z;x,y) = (1/\tau)h^{\alpha\beta}(z;x)\delta(x,y)$, and gives a Kramers-Moyal expansion up to constants of

$$\int W^{\alpha\beta}_h(z|z-\Delta)\hat{\rho}(z-\Delta;t)\mathcal{D}\Delta$$

$$= \frac{1}{\tau}h^{\alpha\beta}(g_{ab})\hat{\rho}(z;t) + \frac{\delta h^{\alpha\beta}(z)}{\delta g_{ab}}\frac{\delta\hat{\rho}(z;t)}{\delta\pi^{ab}} - \frac{\delta h^{\alpha\beta}(z)}{\delta\pi^{ab}}\frac{\delta\hat{\rho}(z;t)}{\delta g_{ab}}$$

$$+ \tau D^{abcd,\alpha\beta}(g)\frac{\delta^2\hat{\rho}(z;t)}{\delta\pi^{ab}\delta\pi^{cd}} + \cdots, \qquad (121)$$

where the diffusion terms $D^{abcd,\alpha\beta}(g)$ and all higher-order moments are tensor densities of the appropriate weight to contract with the derivatives acting on the density matrix. The absence of functional derivatives with respect to g higher than $\delta\hat{\rho}/\delta g_{ab}$ means that this is a Brownian realization. As a result, Eq. (92) is unmodified, and we

have only diffusion in π^{ab}. Higher-order functional derivatives in π^{ab} can be suitably smeared to control divergences. If we want to ensure that our evolution equation is invariant under canonical transformations, we can express it in terms of Poisson brackets, with the second-order term being written, for example, as $\{J_o,\{J_i,\hat{\rho}(z;t)\}\}$. It may also be necessary to include the lapse, shift, and their conjugate momenta in the Poisson bracket, which is a realization we pursued in Ref. [146].

There is more freedom when it comes to $W^{00}_h(z|z-\Delta; x,y)$ which could include dispersion terms, non-Brownian terms, or even friction terms such as those of the form

$$\frac{\delta\gamma^{ab}\hat{\rho}}{\delta\pi^{ab}}, \qquad \gamma^{ab} := \eta^{abcd}\frac{\delta\mathcal{H}^{(\text{gr})}}{\delta\pi^{cd}} \qquad (122)$$

with η^{abcd} a local function of the metric. However, gauge invariance places strong restrictions on these terms, and here we leave them unspecified and denote them by the differential operator $\Gamma^{00}\hat{\rho}$. The resulting evolution law is

$$\frac{\partial\hat{\rho}(z;t)}{\partial t} = \int dx\left[N\{\mathcal{H}^{(\text{gr})},\hat{\rho}(z;t)\} + N^a\{\mathcal{H}^{(\text{gr})}_a,\hat{\rho}(z;t)\} - i[H^{(m)}[N,\vec{N}],\hat{\rho}(z;t)]\right]$$

$$+ \int dx\frac{1}{\tau}Nh^{\alpha\beta}(z)\left[\hat{L}_{\alpha}\hat{\rho}(z;t)\hat{L}^{\dagger}_{\beta} - \frac{1}{2}\{\hat{L}^{\dagger}_{\beta}\hat{L}_{\alpha},\hat{\rho}(z;t)\}_+\right] + \int dxN\Gamma^{00}\hat{\rho}(z;t)$$

$$+ \int dxN\left[\frac{\delta h^{\alpha\beta}}{\delta g_{ab}}\hat{L}_{\alpha}\frac{\delta\hat{\rho}(z;t)}{\delta\pi^{ab}}\hat{L}^{\dagger}_{\beta} + \tau D^{abcd,\alpha\beta}_{\pi}(g)\frac{\delta^2}{\delta\pi^{ab}\pi^{cd}} + \cdots\right]. \qquad (123)$$

Let us now see that in the classical limit, we recover general relativity to leading order. For convenience of presentation, we consider the case where the quantum system is uncorrelated with the classical system. The more general case is akin to Eq. (23). Recalling Eq. (9), that $\rho(z;t) := \text{tr}\hat{\rho}(z;t)$, we perform the trace on the quantum system in Eq. (123). Dropping terms of second order and higher we are left with

$$\frac{\partial\hat{\rho}(z;t)}{\partial t} = \int dxN\{\mathcal{H}^{(\text{gr})},\rho\} + \int N^a dx\{\mathcal{H}^{(\text{gr})}_a,\rho\}$$

$$+ \int dx\sqrt{g}\text{tr}N\{\hat{\mathcal{H}}^{(m)},\hat{\rho}\} + \cdots, \qquad (124)$$

which, when the constraints of GR are satisfied, is the classical Liouville equation for the gravitational degrees of freedom of general relativity sourced by quantum matter. The third term gives the backreaction we expect. We thus see that in the classical limit we recover Einstein's equations. To leading order, the evolution of the quantum fields is seen from Eq. (123) to correspond to the ordinary commutator terms (quantum field theory in curved space) plus a Lindbladian term.

A realization which gives the continuous master equation will be discussed in Ref. [50] with Sparaciari, Šoda and Weller-Davies and Ref. [146], and is given by

$$\frac{\partial\hat{\rho}}{\partial t} = \int dxN\{\mathcal{H},\hat{\rho}\} + \int dxN^a\{\mathcal{H}_a,\hat{\rho}\} - i[\hat{H}^{(m)}[N,\vec{N}],\hat{\rho}]$$

$$+ \frac{1}{2}\int dxN(\{\hat{\mathcal{H}}^{(m)},\hat{\rho}\}) - \{\hat{\rho},\hat{\mathcal{H}}^{(m)}\})$$

$$+ \frac{1}{2}\int dxdyN\lambda_{\mu\nu\hat{\sigma}\tau}(g;x,y)\left[\hat{L}^{\mu\nu}(x),[\hat{\rho},\hat{L}^{\hat{\sigma}\tau}(y)]\right]$$

$$+ \frac{1}{2}\int dxdyN\{D^{\alpha\beta}_2(x,y)J'_{\alpha}(g;x),\{J_{\beta}(g;y),\hat{\rho}\}\} \qquad (125)$$

when the constraints are satisfied. Here, $\hat{L}^{ij} := (\delta\hat{\mathcal{H}}^{(m)}/\delta g_{ij})$, $\hat{L}^{nn} := \hat{\mathcal{H}}^{(m)}$, and the pure-Lindbladian couplings $\lambda_{\mu\nu\hat{\sigma}\tau}(g;x,y)$ consist of submatrices with elements $\lambda_{ijkl} := D_{0,ijkl}(g;x,y)$, $\lambda_{ijnn}(g;x,y)$, $\lambda_{nnkl}(g;x,y)$, and the constant $\lambda_{nnnn}(g;x,y)$. Here, greek indices are akin to space-time indices $\mu = n, i$, and we use n instead of 0 since 0 already corresponds to the $\mathbb{1}$ Lindblad operator.

The condition for complete positivity can be found from Eq. (89). Denoting the generalized inverse of the matrix

$D_0(g; x, y)$ as D_0^{-1}, we require that the matrix with elements $D_0^{ijkl}(x, y)$ and $D_2^{\alpha\beta}[\delta J'_\alpha(g; x)/\delta g_{ij}][\delta J_\beta(g; y)/\delta g_{kl}] - (D_0^{-1})^{ijkl}(x, y)$ be positive semidefinite as a matrix and kernel to guarantee complete positivity of the master equation. $D_2^{\alpha\beta}(x, y)$, $J_\alpha(N, \vec{N}, g; x)$, and $D_{0,ijkl}(N, \vec{N}, g; x, y)$ are chosen so that the realization is linear in the lapse and shift. If we take the J_α to be local in the metric, the simplest case is then $J = \sqrt{g}$, so that the diffusion term is $\frac{1}{2} \int dx dy N(x) D_2(x, y) \sqrt{g(x)} \sqrt{g(y)} g^{ij}(x) g^{kl}(y) [\delta\hat{\rho}/\delta\pi^{ij}(x)\delta\pi^{kl}(y)] + \cdots$, but there are other more complicated realizations which also recover the dynamical Einstein equation as a limited case (by adding friction and additional diffusion terms, for example). The additional Lindbladian couplings corresponding to the Lindblad operator $\hat{\mathcal{H}}^{(m)}$ are not needed in order to reproduce general relativity in the classical limit or to preserve positivity, but instead appear to be required by gauge invariance. The constraint equations for this realization are derived in Ref. [146] by following the procedure outlined in Ref. [156]. Alternatively, one can include the lapse, shift, and their conjugate momenta as canonical degrees of freedom and thus part of the Poisson bracket in Eq. (125).

This realization has the clear advantage of being continuous, while the former realization has the advantage of being linear in the Hamiltonian. In terms of the evolution of the quantum system, for each classical space-time manifold, the quantum system undergoes both decoherence, as well as unitary evolution generated by the Hamiltonian, and quantum jumps which are accompanied by a backreaction on the space-time. The diffusion terms imply that the solution of Eqs. (123) and (125) [or alternatively, Eq. (125)] should be a distribution which has a variance about the solution to Einstein's equations when we perform the trace over the quantum system. This presents a testable deviation from classical general relativity, potentially observable when the other terms are small, such as at low acceleration [167]. The absence of such terms is likely to falsify this model. On the other hand, since diffusion of the metric can result in stronger gravitational fields when we might otherwise expect none to be present, it raises the possibility that diffusion may explain galaxy rotation curves [169] and galaxy formation without the need for dark matter.

In addition to the dynamical equation of motion, Eq. (117) or Eq. (123), we need to impose the constraints of general relativity. Let us recall how this is done classically. There, the Hamiltonian and momentum constraint of Eq. (102) follows from demanding that the Lagrangian be invariant under arbitrary time reparametrizations and spatial diffeomorphisms. This leads to the primary constraints on their conjugate momenta, $p_N = 0$ and $p_{N^i} = 0$. Preservation of the primary constraints is implemented in the ADM formalism by requiring that $\dot{p}_N = 0$ and $\dot{p}_{N^a} = 0$ (the supermomentum and super-Hamiltonian constraints). These are just constraints on

the classical phase space, and so we can constrain ρ to lie on the surface in phase space where the constraints are zero.

Were we to impose the constraint in an equivalent manner here, it would amount to demanding that $\hat{H}_{\text{ADM}}$ be invariant under the action of P_N and P_{N_a}. This would lead to a Hamiltonian constraint of the form

$$\mathcal{H}^{(\text{gr})} + 16\pi G \hat{\mathcal{H}}^{(m)} \stackrel{?}{\approx} 0 \qquad (126)$$

at every x which would be impossible to satisfy, since it contains both a classical term $\mathcal{H}^{(\text{gr})}$ and a local quantum operator $\hat{\mathcal{H}}^{(m)}$. However, such a constraint is not needed; rather, we should instead ask, what restrictions are required such that the equations of motion satisfy the required symmetry? In the classical case, we demand that physical degrees of freedom do not depend on the choice of lapse N and shift N^a, i.e., that these are gauge choices. This requirement necessarily leads to the constraints of general relativity. Likewise here, we should not first demand that the constraints be satisfied, but rather, require that the equations of motion be invariant under the choice of lapse and shift. This turns out to lead to modified constraints which are not of the form (126). We discuss how the constraints can be generated from gauge invariance of the equations of motion in Ref. [156]. Invariance of the equations of motion under spatial diffeomorphisms is already guaranteed, as the evolution equation involves only integrals over scalar densities.

In order to find a solution to the dynamical and constraint equations, we need the equations to be compatible. Attempts to quantize gravity have floundered on this point, as the constraints are operators and typically do not commute, so no simultaneous solution of the constraints exist. This is the case in loop quantum gravity, although it is hoped that one can find an operator ordering such that the constraints close. Likewise, string theory is background dependent, which presumably means that in the full theory, gauge invariance is broken. While it is hoped that frameworks for string field theory could result in a background-independent approach, it is not clear whether this can be maintained. Here, we also require constraint compatibility, but it takes on a different form. Because we have attempted to implement gauge invariance on the level of the equations of motion rather than as operator identities, the relevant notion of constraint compatibility is not simply the commutation of operators, but whether the evolution operators and constraints and generators "commute" with one another when applied to the state. In other words, we require for example

$$\mathcal{L}_a(x)\mathcal{L}(y)\hat{\rho}(z; t) - \mathcal{L}(y)\mathcal{L}_a(x)\hat{\rho}(z; t) \approx 0, \qquad (127)$$

and similarly, we require that the "commutation relations" for all other generators vanish on the constraint surface; the

right-hand side must be a linear combination of generators and constraint equations. This notion of constraint compatibility replaces the closure of the Dirac algebra in the classical theory.

This program is initiated in Ref. [156] in the Heisenberg representation to find gauge-invariant observables A. This constraint structure, while involved, is far simpler than the constraint structure of loop quantum gravity or canonical quantization, and provides a model in which to understand some of the conceptual challenges these theories face. While the constraints are more complicated because they are implemented on the level of equations of motion, the classical nature of the gravitational field means they do not suffer from operator ordering issues and are more tractable. Nonetheless, it may be impossible to find a realization of the theory and a symmetry principle, such that the set of constraint equations close. In that case, one could still impose the standard constraints of general relativity as initial conditions, as is done in the ADM formulation, but they may not be conserved in time. One could instead impose the standard constraints as expectation values, but again, it is possible these may also not be conserved in time. This would likely mean the theory was not invariant under diffeomorphisms of the $(3 + 1)$-dimensional space-time, but it would not render the theory inconsistent, since we formulated the theory here, not as a $3 + 1$ theory of space-time, but rather, as a Hamiltonian theory of the spatial metric. The evolution of matter is governed by the Hamiltonian and Lindblad operators, which are functionals only of the 3-metric. To be consistent, the theory needs to be invariant under spatial diffeomorphisms, which it manifestly is. However, although it would be a consistent theory, it would be unclear how to justify imposing the constraint as a mere initial condition. So while the theory may be consistent, it would require further motivation.

An alternative approach can be pursued now that classical quantum path integral methods have recently been developed [170]. Since the path integral is computed in configuration space, this allows for the construction of manifestly diffeomorphism-invariant theories. These have since been proposed together with Weller-Davies [142]. One, based on the trace of Einstein's equations, is different from Einstein's general relativity, but does at least demonstrate that diffeomorphism-invariant classical quantum theories are possible in principle. A theory which contains all components of the Einstein equations is also proposed, but it is not yet known whether dynamics which deviate from general relativity are suitably suppressed.

VI. DISCUSSION

The general form of hybrid classical quantum dynamics is presented here as Eq. (14), which includes discrete evolution and a continuous form [71]. Based on this, we exhibit a theory of classical general relativity coupled to quantum field theory which is invariant under spatial diffeomorphisms. The theory is consistent: The dynamics is completely positive, norm preserving, and linear in the density matrix. We specify the dynamics up to a choice of the positive-semidefinite matrix $W_h^{\alpha\beta}(z|z - \Delta; x, y)$. The choice is constrained by gauge invariance, and thus must satisfy further criteria, as discussed in Ref. [156]. Two particular choices of $W_h^{\alpha\beta}(z|z - \Delta; x, y)$, are the model of Eq. (120) and that corresponding to Eq. (125). These are further considered in Ref. [156] and Refs. [50,146], respectively, along with the constraint equations which correspond to them. In Ref. [71], it is shown that the first realization leads to discrete jumps in phase space, while the second realization follows from demanding that the dynamics be continuous in phase space. The first moment in their Kramers-Moyal expansion corresponds to general relativity in the ADM formulation, and thus both recover classical general relativity as a limiting case. A proposal based on the path integral is introduced in Ref. [142].

We require that the metric remains classical even when backreacted upon by quantum fields, and this has two important consequences. First, the interaction is necessarily stochastic, which is important, since a principle motivation behind the theory are attempts to resolve the black-hole information problem. That there exists a fully quantum theory of gravity has been an argument against theories which have fundamental information loss, in the sense that given any such dynamics one can always find a purification of the state onto some environment, such that the full theory is unitary. In this case, Coleman has shown that when we trace out the environment, the dynamics is equivalent to a unitary theory with unknown coupling constants [62]. This renders the information loss trivial: In a universe with a finite number of random coupling constants, an observer has only to perform a sufficient number of experiments to determine these coupling constants, after which time the theory will behave arbitrarily close to a unitary one from the point of view of this experienced experimentalist. However, the argument in Ref. [62] need not apply here. Any purification which keeps some degrees of freedom classical would need to obey an unnatural condition, namely, that it remain of the form

$$\hat{\rho}_{AB} = \sum |g, \pi\rangle_A \otimes |g, \pi\rangle_B \otimes |\hat{\rho}_{\text{matter}}(g, \pi)\rangle_{AB} \quad (128)$$

with system A encoding the classical degrees of freedom and system B being the purifying system [171]. For these degrees of freedom to remain classical, one further needs to impose the condition that one can measure only in the basis where $|g, \pi\rangle_A$ is diagonal. If one lifts this restriction on measurements, or allows dynamics more general than the one preserving the form of the state of Eq. (128), then it is far from clear that the dynamics would still be completely positive and trace preserving. The dynamics here needs only to give positive probabilities on measurements which

commute with the basis representing the classical degrees of freedom. It does not need to give positive probabilities for all measurements. Thus, we cannot conclude that this theory has consistent dynamics on the purified state. Or to put it another way, it is unlikely to be the case that the dynamics considered here can be viewed as unitary quantum dynamics with a local environment which is traced out. The fact that the quantum system couples to a classical one, thus provides a potential loophole to the no-go theorem of Ref. [62] ruling out nontrivial information destruction, while as argued in Sec. I, information destruction provides the loophole out of the no-go argument of Feynman and Aharonov against quantum systems interacting with classical ones. The two no-go theorems annihilate each other.

There may appear to be some tension between the claim that the theory may evade the no-go argument of Coleman, and the idea that the theory presented here could be an effective theory which emerges as the classical limit of a fully quantum theory. However, this is not necessarily in contradiction. In Ref. [176], we consider fully quantum descriptions, which effectively act like a classical quantum description. The classical system corresponds to a quantum system decohered into a family of coherent states, and for some potentials, this description is valid for long times. Since this description is fully quantum, Coleman's argument would imply that it corresponds to a unitary theory on an enlarged Hilbert space with unknown coupling constants, and that there exist measurements which would allow an experimenter to determine these constants. However, in the classical quantum description of this fully quantum theory, we do not allow arbitrary measurements on the "classical" system—only measurements of q and p are allowed. Coleman's argument allows the experimenter to perform measurements in an arbitrary basis, and it appears unlikely one could use fixed measurements to determine the coupling constants.

The second consequence of having a quantum classical interaction is that the measurement postulate of quantum theory is no longer needed. One has fundamental decoherence of the quantum field by the classical metric. In this view, classicality is not an emergent or effective property of large quantum systems, but rather, fundamental. In the discrete theory discussed here, $h^{\alpha\beta}$ is diagonal and unique for each Lindblad operator. As with the Stern-Gerlach example, the accompanying jump in phase space unambiguously determines which Lindblad operator was applied to the quantum state. If we know the initial pure state of the quantum system, then by monitoring the classical system, we know which sequence of Lindblad operators were applied to the quantum state and at what times. As such, the quantum state conditioned on the classical degrees of freedom remains pure. Likewise, in the continuous case, we have since found that there is a very natural class of theories in which the quantum state remains

pure conditioned on the classical trajectory [29]. This is ironic given the motivation of obtaining dynamics which allowed for the destruction of information in black-hole evaporation. The dynamics here, while stochastic, can leave the quantum state pure. It is the classical degrees of freedom which gain entropy. This suggests that in black-hole evaporation, the associated entropy is the entropy of the classical space-time. Since the formalism introduced here allows one to study the backreaction of matter fields on the gravitational field, there is hope that we can better understand black holes, as well as other systems where gravity and quantum effects are important.

Let us next address a number of open questions and challenges that the theory faces, starting with the objection to theories of information destruction due to Banks, Peskin, and Susskind (BPS) [60]. BPS argued that a local interaction which leads to decoherence will lead to anomolous heating to such a degree that it will contradict observation. Although the BPS argument was not made in the context of classical quantum theories, it does apply to Lindbladian evolution of the quantum state $\hat{\sigma}$ with Lindblad operators being local field operators, and our theory does contain such terms. In particular, BPS considered an evolution equation

$$\frac{\partial\hat{\sigma}(t)}{\partial t} = -i[\hat{H}^{(m)}, \hat{\sigma}(t)]$$

$$+ \int dx dx' D_0^{\alpha\beta}(x-x')\left(\hat{L}_\alpha(x)\hat{\sigma}(t)\hat{L}_\beta^\dagger(x')\right.$$

$$\left. -\frac{1}{2}\{\hat{L}_\beta^\dagger(x')\hat{L}_\alpha(x), \hat{\sigma}(t)\}_+\right) \quad (129)$$

and couplings $D_0^{\alpha\beta}(x-x')$. The Hamiltonian of the matter fields $\hat{H}^{(m)}$ will not be conserved, and when $D_0^{\alpha\beta}(x-x')$ has short range (decays rapidly with $|x-x'|$) there may be significant anomalous heating. In essence, local operators $\hat{L}_\alpha(x)$ do not commute with the Hamiltonian, and in the limit where $D_0^{\alpha\beta}(x-x')$ is a delta function, the average rate of energy production contains terms which diverge. This is easily seen in the Heisenberg representation, since when $D_0^{\alpha\beta}(x-x')$ is symmetric in x, x', and α, β,

$$\frac{d\hat{H}^{(m)}}{dt} = \frac{1}{2}\int dx dx' D_0(x-x')[\hat{L}_\alpha(x), [\hat{H}^{(m)}, \hat{L}_\beta^\dagger(x')]]. \quad (130)$$

If the $\hat{L}_\alpha(x)$ contain only a finite number of derivatives of local field operators, then the commutator gives terms proportional to $\delta(x-x')$, or kth derivatives $\nabla^k\delta(x-x')$. These terms will diverge if $D_0(\epsilon)$ diverges as $\epsilon \to 0$. For example, if $D_0(x-x') \propto \delta(x-x')$.

This anomalous heating should trouble us if it contradicts observation, but it need not trouble us on a fundamental level. If the dynamics is Lindbladian, there is no reason to expect energy conservation; the dynamics is not

unitary and so Noether's theorem does not apply. The generator of time translations is not the Hamiltonian but the Lindbladian [177]. In fact, there is no way to canonically define energy, since by choosing a different basis of Lindblad operators, the Hamiltonian will change. In the case of classical quantum systems, the notion of energy becomes even harder to define, since the Hamiltonian is a hybrid object. On top of this, in general relativity there is no local definition of energy or conservation law for energy integrated over an infinitesimal spatial surface [180]. From a physical point of view, this is arguably the more relevant quantity, and so the question of energy conservation is not a matter of principle, but rather, consistency with experiment. One must ensure that the vacuum is stable enough that it does not contradict experimental observation.

The vacuum can be made stable by turning down the Lindbladian coupling so that $D_0(\epsilon)$ decreases as $\epsilon \to 0$. Since $d\hat{H}^{(m)}/dt$ goes as $D_0(\epsilon)$, we can regularize it by discretizing space-time, and scaling $D_0(\epsilon)$ with ϵ. In the pure-Lindbladian case, there is no reason not to take the Lindbladian coupling to be arbitrarily small, in which case the effect is weak in flat space and only strong when the 3-metric becomes singular, since we expect $D_0(x - x')$ to contain terms proportional to $\sqrt{g(x)}$ and $\sqrt{g(x')}$. Seen in this light, the problem raised by BPS is merely one of having to introduce another scale.

A similar issue occurs in spontaneous collapse models [67–69], but in and of itself, this does not present a problem, since again, one can tune down the Lindbladian coupling $D_0(\epsilon)$ as small as one wishes. Those who advocate for spontaneous collapse models typically require $D_0(x - x')$ to be large enough to account for the level of decoherence we see around us. This requirement on D_0 is not needed here for three reasons. First, environmental decoherence is expected to account for most of the decoherence we observe. Second, the collapse of the wave function is a byproduct which occurs only when sufficient correlations has build up between the gravitational field and the quantum state together with its correlated environment. And third, decreasing $D_0(\epsilon)$ requires more gravitational diffusion [50], which causes additional decoherence once the gravitational field is integrated out [30,92]. The heating effect of this secondary decoherence is a source of concern, but appears to be milder than in BPS [30,50], although whether this is within experimental limits requires further investigation since at short distances, we are operating in a regime where relativistic effects become important.

The question of whether the couplings can be tuned down as we decrease ϵ is related to the question of whether the theory is regularizable [170]. In the case of a scalar field with a single Lindblad operator $\phi(x)$, the Lindbladian dynamics is both renormalizable and Lorentz invariant [140,141], and we can choose the coupling constant to generate as little anomalous heating as we wish, since we

do not require the amount of entropy production to be large. Understanding whether the theory can be regularized, remains an important open question for the program presented here. The question is complicated by the fact that relativistic corrections in the gravitational field become important at short distances [168] and by the fact that some aspects of the heating appear to be gauge artifacts [181].

Other strategies have been considered, but they may make the theory less compelling. One may be able to make the anomalous heating small by introducing a small violation of locality as has been attempted in the case of Lindbladians in the context of field theories [66], or as is done in spontaneous collapse models where the coupling is modified at a length scale at the order of the proton radius. It is worth emphasizing that if one reduces heating by allowing $D_0^{\alpha\beta}(x - x')$ to have finite range, this does not necessarily violate locality. If $\hat{L}_\alpha(x)$ are local operators, then it is possible for local observables to still evolve locally even though $D_0^{\alpha\beta}(x - x')$. The price we pay if $D_0^{\alpha\beta}(x - x')$ does not fall off fast enough with $|x - x'|$ is that correlations can be created over finite distances [60,65]. While this seems to suggest some correlated environment should be evoked, it does not lead to super-luminal signaling or inconsistencies. It will however naively violate Lorentz invariance, since one has spatial correlations but no temporal ones. This could be rectified by taking $D_0^{\alpha\beta}(x - x')$ to not depend on the spatial distance $|x - x'|$, but rather the space-time distance. For example, one could take it to be the Green's function of the Laplace-Beltrami operator. This would then suggest a non-Markovian evolution reminiscent of an effective theory rather than a fundamental one. Another alternative is to modify gravity at short distances. Since gravity is tested only to within the milometer scale, such modifications have often been considered, but would likely necessitate some additional structure such as extra dimensions [182].

Whether such measures are required is unclear. In the context of the theories considered here, the situation is significantly different from the pure-Lindbladian evolution considered by BPS. First, the action of the Lindblad operators on the quantum field is accompanied by a back-action on the classical degrees of freedom, and in particular, the gravitational degrees of freedom. While the argument of BPS suggests there will be an increase in the expectation value of the energy density of the matter Hamiltonian density $\hat{\mathcal{H}}^{(m)}$, this can be accompanied by a decrease in the energy of the classical system due to this backreaction. In this case, this is the kinetic energy density $G_{ijkl}\pi^{ij}\pi^{kl}$ of the pure gravity part of the super-Hamiltonian, $\mathcal{H}^{(gr)}$. In the case of a discrete realization of this theory, such as Eq. (120), the action of the Lindblad operator is accompanied by a jump in $\pi^{ij}(x)$. Likewise, the diffusion term of the continuous realization, Eq. (125) acts on the kinetic term of $\mathcal{H}^{(gr)}(x)$ in the Heisenberg representation producing an energy density flux of $D_2^{ijkl}(x,x)G_{ijkl}(x)$ which can be negative

and state independent, and also needs to be regularized. For example, consider

$$\lim_{x-x'\to\ell} D_2^{ijkl}(x,x') = D_2 \frac{\sqrt{g}N}{2}(g^{ik}g^{jl} + g^{il}g^{jk} - 2\beta g^{ij}g^{kl})$$

(131)

with D_2 some constant which depends on the cutoff ℓ, and $\beta < 1/3$ to ensure D_2^{ijkl} is positive semidefinite. If we wish

$D_2^{ijkl}(x,x')$ to include only the final term in Eq. (131) proportional to $g^{ij}g^{kl}$, then we can take $D_2 \to D_2/\beta$ and $\beta \to -\infty$.

Using the diffusion coefficient of Eq. (131), we can compute the apparent flux of gravitational kinetic energy density using the continuous realization of Eq. (125) in the Heisenberg representation. Ignoring any boundary terms, this gives

$$\frac{dH^{(\mathrm{gr})}[N,\vec{N}]}{dt} = \{H^{(\mathrm{gr})}[N,\vec{N}], \hat{H}_{\mathrm{ADM}}[N,\vec{N}]\} + \frac{1}{2}\int dx dx' D_2^{ijkl}(x,x')\frac{\delta^2}{\delta\pi^{ij}(x)\delta\pi^{kl}(x')}H^{(\mathrm{gr})}[N,\vec{N}]$$

$$= \{H^{(\mathrm{gr})}[N,\vec{N}], \hat{H}^{(m)}[N,\vec{N}]\} + \int dx N^2 D_2 \sqrt{g}\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk} - 2\beta g^{ij}g^{kl})G_{ijkl}$$

$$= \{H^{(\mathrm{gr})}[N,\vec{N}], \hat{H}^{(m)}[N,\vec{N}]\} + \left(d^2 - \frac{d}{2}\right)(1+\beta)D_2\int dx N^2,$$

(132)

where d is the number of spatial dimensions. We assume that the lapse and shift $N,\vec{N}$ depend only on the metric and not its conjugate momenta. The first term is just the change in energy due to the backreaction of the matter Hamiltonian, and is compensated for by the action of $H^{(\mathrm{gr})}[N,\vec{N}]$ on $\hat{H}^{(m)}[N,\vec{N}]$ which results in an equal and opposite energy flux $-\{H^{(\mathrm{gr})}[N,\vec{N}], \hat{H}^{(m)}[N,\vec{N}]\}$ into the matter degrees of freedom. It is the second term which is due to the diffusion. Taking $\beta < 1$ results in an apparent negative flux of gravitational kinetic energy. It can at least partially compensate for the heating of the matter field given in Eq. (130). The question is a subtle one; for example, in the weak-field limit, what initially appears to be constant anomalous energy production in the gravitational degrees of freedom turns out to be a gauge artifact [181]. The dependence of Eq. (132) on the lapse suggests this might be the case more generally.

Whether this backreaction is enough to restore equilibrium or slow the production of energy in the matter sector of the theory is an open question. It may merely be that the total Hamiltonian is conserved on average, with heating occurring in the matter degrees of freedom, and a negative flux of kinetic energy occurring in the gravitational degrees of freedom. To what extent this would be observable has since been partially taken up in Ref. [50], where it is shown that the decoherence (and heating) in the matter degrees of freedom can be made small, but only at the expense of a large diffusion in the gravitational degrees of freedom. However, because gravity is so weak, the effect appears to be within current experimental bounds, although they do constrain $D_2^{ijkl}(x,x')$. The question then becomes whether the anomalous heating and diffusion are within experimental bounds, while at the same time, respecting various local symmetry properties such as diffeomorphism invariance or some weaker version.

Related to this is the question of whether the Hamiltonian and momentum constraint is preserved in time. In the deterministic theory, this says that the total energy and momentum density is everywhere zero. These constraints are however modified in the theory considered here. Since conservation of the constraints in general relativity enforces diffeomorphism invariance, the issue of diffusion of gravitational kinetic energy and anomalous heating is linked with whether diffeomorphism invariance or a related local symmetry will be respected. Whether the theory satisfies some local symmetry then becomes a question of satisfying some potentially modified constraint equations, and whether one can satisfy them all at the same time. This turns on the question of the compatibility of the constraint equations, i.e., whether the algebroid of constraints [183] closes. Of course in the CQ theory, the constraint is not of the form Eq. (126), and the full dynamics are subtle. A fuller understanding of the algebroid of constraint equations thus remains an open issue, with some initial progress reported in Ref. [156] where these issues are further explored (cf. Ref. [142]).

There is some reason to believe $3 + 1$ diffeomorphism invariance or something related to it can be respected, given the diffeomorphism-invariant realization of Ref. [142] based on the trace of Einstein's equations, and using the path integral formulation of CQ dynamics [170]. In Ref. [142], we also present a diffeomorphism-invariant theory which gives all of Einstein's equations, although we have not proven that the dynamics preserves positivity of initial states. The challenge then is to see if the realization is consistent with experimental constraints such as those on anomalous heating and gravitational diffusion, and is ideally also renormalizable in the matter degrees of freedom. While realizations exist which are renormalizable, and realizations exist which have small amounts of

anomalous heating, the question of whether a realization exists which satisfies all these requirements remains open.

The classical quantum theory has a number of experimental signatures, some of which have since been quantified in Ref. [50]. It predicts a form of gravitational decoherence, which already a number of experiments are looking for [184–186]. It predicts stochastic, finite-sized jumps, or continuous diffusion of the gravitational field which might be detectable in the lab with current technology. The theory also predicts diffusion which might be observable in astrophysical systems and in cosmology. The theory is highly constrained, especially if τ the parameter in Eq. (17) is taken to be of order $\hbar$, since increasing the coherence time results in larger jumps and more diffusion. Finally, the locally correlated realizations considered here predict a null result to recently proposed experiments to test the quantum nature of gravity [187–190].

Here, space-time is treated as fundamentally classical, but one could instead view this theory as a candidate for an effective theory which results from taking the classical limit of the gravitational degrees of freedom of a fully quantum theory of gravity. There are a number of ways of taking this partial classical limit. The extent and the conditions under which this gives the equations of motion presented here is taken up in Ref. [176]. We find that care should be taken, since an effective theory might violate our assumptions of complete positivity at short timescales or of being time local. Nonetheless we find that there is a regime, and a choice of parameters, such that Eq. (14) describes the consistent evolution of interacting quantum systems in the limit where a subsystem behaves classically.

Whether the theory presented here is taken to be fundamental or not, it provides a consistent way to explore the regime where space-time is classical in a manner which does not suffer from the pathologies of the semiclassical equations. This has been lacking in our investigation into how space-time reacts to quantum systems. This gives hope that the ability to consistently study backreaction effects finds application on large-distance scales, in cosmology, and in black-hole evaporation.

APPENDIX A: CONDITIONS FOR COMPLETE POSITIVITY OF A CQ MAP

Here we extend a theorem due to Kossakowski [194] for a cq map $\mathcal{E}(t)$ to be completely positive on a separable Hilbert space; see also Ref. [50] for a more complete proof using indicator functions. Let us consider all complete sets of projectors $P_r(z)$ on the doubled Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_S$ and classical degrees of freedom z.

Theorem 1. $\mathcal{L}$ is a generator of continuous-time, completely positive, Markovian evolution of $\hat{\rho}(z;t)$ which preserves $\int dz \mathrm{tr} \hat{\rho}(z;t)$ if and only if for all sets of orthogonal projectors $\{P_r(z)\}$:

$$\mathrm{tr} P_s(z) \mathcal{L}_{z|z'} \otimes \mathbb{1}(P_r(z')) \geq 0, \quad \forall \, r \neq s, z, z', \quad \text{(A1)}$$

$$\mathrm{tr} P_s(z) \mathcal{L}_{z|z'} \otimes \mathbb{1}(P_s(z')) \geq 0, \quad \forall \, s, z \neq z', \quad \text{(A2)}$$

$$\int_z \mathrm{tr} \mathcal{L}_{z|z'} \otimes \mathbb{1}(P_r(z')) = 0, \quad \forall \, r, z', \quad \text{(A3)}$$

where $\mathcal{L}\hat{\rho}(z;t) = \int dz' \mathcal{L}_{z|z'}\hat{\rho}(z';t)$.

As a consequence of the above, we automatically get

$$\mathrm{tr} \int dz' \delta(z-z') P_s(z) \mathcal{L}_{z|z'} \otimes \mathbb{1}(P_s(z')) \leq 0 \quad \forall \, s, z \quad \text{(A4)}$$

since the probability conservation condition, Eq. (A3) ensures that outward flux from any $P_s(z)$ contributes negatively to the probability rate. The proof of Theorem 1 follows from writing

$$\mathcal{L}_{z|z'} := \lim_{\delta t \to 0} \frac{\mathcal{E}_{z|z'}(\delta t) - \mathbb{1}\delta(z-z')}{\delta t}, \quad \text{(A5)}$$

where $\mathcal{E}(t)\hat{\rho}(z;t) = \int dz' \mathcal{E}_{z|z'}(t)\hat{\rho}(z';t)$, and using the fact that the map $\mathcal{E}_{z|z'}$ is completely positive, probability-conserving map if and only if

$$\mathrm{tr} P_s(z) \mathcal{E}_{z|z'} \otimes \mathbb{1}(P_r(z')) \geq 0 \quad \forall \, \{P_s(z)\}, \{P_r(z')\}, z, z' \quad \text{(A6)}$$

and

$$\sum_z \text{tr}\mathcal{E}_{z|z'} \otimes \mathbb{1}(P_r(z')) = 1 \quad \forall \{P_r(z)\}, r, z'. \quad (A7)$$

It follows that $\mathcal{E}_{z|z'}$ is trace nonincreasing

$$\mathcal{E}_{z|z'} \otimes \mathbb{1}(P_r(z')) \leq \mathbb{1}. \quad (A8)$$

Equation (A2) is an additional constraint on the quantum system not found in Ref. [194], although if one considers the projector to also be on the classical degree of freedom, then it becomes subsumed by Eq. (A1). From Kraus's theorem, we know that this implies that $\Lambda^{\alpha\beta}(z|z';t)$ in Eq. (34) needs to be positive semidefinite in α, β for all z, z'.

APPENDIX B: HEISENBERG REPRESENTATION

Equations (14) and (17) tell us how the classical quantum state evolves. As with both quantum and classical mechanics, there is an alternative "Heisenberg" picture in which the state does not evolve but the operators do. One can easily derive the time evolution of operators in this representation, since it follows from demanding that for all $\hat{\rho}(z;t)$,

$$\frac{d}{dt}\int dz \text{tr}\hat{A}_S(z;t)\hat{\rho}_S(z) = \frac{d}{dt}\int dz \text{tr}\hat{A}_H(z;t)\hat{\rho}_H(z), \quad (B1)$$

where the subscript S indicates the Schrödinger representation, and H the Heisenberg representation. Using this and Eq. (14) for the evolution of $\hat{\rho}_S(z)$ leads to the Heisenberg evolution equation for the operator $\hat{A}(z;t)$ over phase space

$$\begin{aligned}
\frac{d\hat{A}_H(z;t)}{dt} &= \frac{\partial\hat{A}_H(z;t)}{\partial t} + i[\hat{H}(z), \hat{A}_H(z;t)] \\
&+ \int d\Delta W^{\alpha\beta}(z+\Delta|z)\hat{L}_\beta^\dagger\hat{A}_H(z+\Delta,t)\hat{L}_\alpha - \frac{1}{2}W^{\alpha\beta}(z)\{\hat{L}_\beta^\dagger\hat{L}_\alpha, \hat{A}_H(z;t)\}_+ \\
&= \frac{\partial\hat{A}_H(z;t)}{\partial t} + i[\hat{H}(z), \hat{A}_H(z;t)] + W^{\alpha\beta}(z)\left[\hat{L}_\beta^\dagger\hat{A}_H(z;t)\hat{L}_\alpha - \frac{1}{2}\{\hat{A}_H(z;t), \hat{L}_\beta^\dagger(x)\hat{L}_\alpha\}_+\right] \\
&- \hat{L}_\beta^\dagger\{h^{\alpha\beta}, \hat{A}_H(z;t)\}\hat{L}_\alpha] + \cdots,
\end{aligned} \quad (B2)$$

where we use cyclicity of operators under the trace and integration by parts. We include the purely classical term in W^{00}, but for deterministic evolution, it would be $-\{H(z), \hat{A}_H(z;t)\}$. For a field theory, and in general relativity, we are usually interested in the evolution of diffeomorphism-invariant operators such as

$$\hat{A}_H(z) = \int dx\sqrt{g}\hat{A}(z;x), \quad (B3)$$

where $A(z;x)$ is a local scalar operator. In such a case, we have

$$\begin{aligned}
\frac{d\hat{A}_H(z;t)}{dt} &= \frac{\partial\hat{A}_H(z;t)}{\partial t} + i[\hat{H}(z), \hat{A}_H(z;t)] \\
&+ \int dx\left[d\Delta W^{\alpha\beta}(z+\Delta|z;x,y)\hat{L}_\beta^\dagger(y)\hat{A}_H(z+\Delta,t)\hat{L}_\alpha(x) - \frac{1}{2}W(z;x,y)\{\hat{L}_\beta^\dagger(y)\hat{L}_\alpha(x), \hat{A}_H(z;t)\}_+\right] \\
&= \frac{\partial\hat{A}_H(z;t)}{\partial t} + i[\hat{H}(z), \hat{A}_H(z;t)] + \int W^{\alpha\beta}(z;x,y)\left[\hat{L}_\beta^\dagger(y)\hat{A}_H(z;t)\hat{L}_\alpha(x) - \frac{1}{2}\{\hat{L}_\beta^\dagger(y)\hat{L}_\alpha(x), \hat{A}_H(z;t)\}_+\right]dx \\
&- \int dx\hat{L}_\beta^\dagger(x)\{h^{\alpha\beta}(z;x), \hat{A}_H(z;t)\}\hat{L}_\alpha(x) + \cdots.
\end{aligned} \quad (B4)$$

Note that care must be taken here. The Heisenberg representation allows one to compute the change of the expectation value of an observable $\hat{A}$ with time. However, if the evolution is nonunitary, then it is not necessarily the case that $\langle\hat{A}^n(t)\rangle = \langle\hat{A}(t)\rangle^n$. For example, if there is dispersion, than typically $\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2$ will increase with time. Thus, knowing $\hat{A}(t)$ does not necessarily allow one to deduce the expectation value other moments of the operator.

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