


Nonreciprocal Frustration: Time Crystalline Order-by-Disorder Phenomenon and a Spin-Glass-like State

Ryo Hanai^{*}

*Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-8502, Japan*

and Asia Pacific Center for Theoretical Physics, Pohang 37673, Korea

 (Received 1 October 2022; revised 22 September 2023; accepted 22 December 2023; published 26 February 2024)

Active systems are composed of constituents with interactions that are generically nonreciprocal in nature. Such nonreciprocity often gives rise to situations where conflicting objectives exist, such as in the case of a predator pursuing its prey, while the prey attempts to evade capture. This situation is somewhat reminiscent of those encountered in geometrically frustrated systems where conflicting objectives also exist, which result in the absence of configurations that simultaneously minimize all interaction energies. In the latter, a rich variety of exotic phenomena are known to arise due to the presence of accidental degeneracy of ground states. Here, we establish a direct analogy between these two classes of systems. The analogy is based on the observation that nonreciprocally interacting systems with antisymmetric coupling and geometrically frustrated systems have in common that they both exhibit marginal orbits, which can be regarded as a dynamical system counterpart of accidentally degenerate ground states. The former is shown by proving a Liouville-type theorem. These “accidental degeneracies” of orbits are shown to often get “lifted” by stochastic noise or weak random disorder due to the emergent “entropic force” to give rise to a noise-induced spontaneous symmetry breaking, in a similar manner to the order-by-disorder phenomena known to occur in geometrically frustrated systems. Furthermore, we report numerical evidence of a nonreciprocity-induced spin-glass-like state that exhibits a short-ranged spatial correlation (with stretched exponential decay) and an algebraic temporal correlation associated with the aging effect. Our work establishes an unexpected connection between the physics of complex magnetic materials and nonreciprocal matter, offering a fresh and valuable perspective for comprehending the latter.

DOI: [10.1103/PhysRevX.14.011029](https://doi.org/10.1103/PhysRevX.14.011029)

Subject Areas: Condensed Matter Physics, Soft Matter, Statistical Physics

I. INTRODUCTION

There is a current surge of interest in the physics of nonreciprocally interacting active systems [1,2]. Nonreciprocal interaction refers to an asymmetry in the interaction between two or more entities in which the action and reaction are not equal. This phenomenon arises generically whenever the system is coupled to a non-equilibrium environment, and is, therefore, ubiquitous in nature [3–26]. The importance of nonreciprocal interaction has been extensively acknowledged in various scientific disciplines, ranging from active matter [3–11], ecology [12–17], social science [18], neuroscience [19–22], and

robotics [23], to open quantum systems [24–27]. More recently, researchers have found that nonreciprocal interaction significantly impacts the collective behavior of many-body systems [9–11,26,28–33]. The effects include the emergence of odd elasticity [10,33], nonreciprocal phase transitions [11,28,29,34], and long-ranged order in two spatial dimensions [31,32].

In the presence of nonreciprocal interactions, conflicting objectives often arise. As a simple example, consider a case where agent A attracts agent B while B repulses A. In such a situation, no configurations can satisfy both agents, as A seeks to be close to B while B desires the opposite. This situation is, to some extent, analogous to situations encountered in geometrically frustrated systems. Geometrically frustrated systems are defined as systems that cannot satisfy the constituents’ “desire” to minimize all interaction energy at every bond [35,36]. [See Figs. 1(a1) and 1(b1) for a typical example of a frustration-free and geometrically frustrated system, respectively.] In other words, these systems do not have any configurations that can make all constituents “happy”

^{*}ryo.hanai@yukawa.kyoto-u.ac.jp

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.

simultaneously, similar to the situation in nonreciprocally interacting systems. This means that at least some constituents must compromise for global optimization. As there can be many ways to achieve this, geometrical frustrated systems often exhibit accidentally degenerate ground states [Figs. 1(a2) and 1(b2)]. This not only makes the system extremely sensitive to external perturbations

but also gives rise to various exotic phenomena, such as order-by-disorder phenomena (OBDP) [37–41], spin glass [42–48], spin ice [49], and quantum [50] and classical [39,40] spin liquids, in and out of equilibrium [51–55].

This raises the question of whether nonreciprocally interacting systems can also give rise to phenomena similar to those induced by geometrical frustration. Phrased

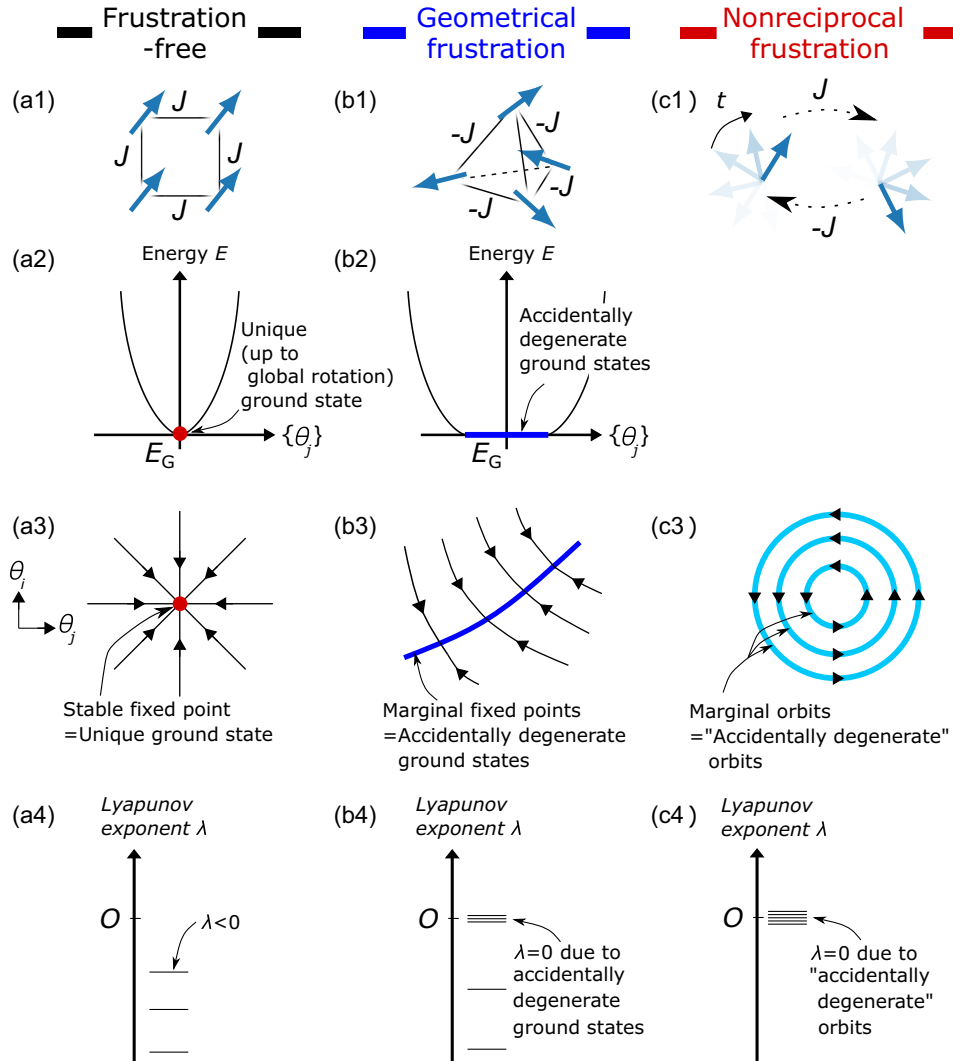


FIG. 1. Geometrical and nonreciprocal frustration and the emergence of “accidental degeneracy” of orbits. (a1)–(c1) Examples of systems with (a1) no frustration, (b1) geometrical frustration, and (c1) antisymmetric nonreciprocal interactions. (a2),(b2) Schematic description of energy E as a function of the spin configuration $\{\theta_j\}$. Here, E_G is the ground state energy and we have omitted the degeneracy that trivially arises from the global rotation symmetry, for clarity of the figure. Note that the energy E is not defined for the nonreciprocal case. (a3)–(c3) Orbits. (a4)–(c4) Lyapunov exponents λ . (a) In frustration-free systems, since fixing the angle of one spin would determine all other spin configurations to minimize the energy of the system, the ground state is unique up to global symmetry. As a result, the system converges into a unique stable fixed point [red point in (a3)], which gives negative Lyapunov exponents $\lambda < 0$. (b) Geometrically frustrated systems, on the other hand, often exhibit accidentally degenerate ground states because of the underconstrained degrees of freedom. The presence of the degenerate ground states implies that there is a direction in which a restoring force (torque) is absent. This means that the accidentally degenerate ground states correspond to marginal fixed points in the language of dynamical systems [blue line in (b3)] that have zero Lyapunov exponent(s) $\lambda = 0$. (c) In nonreciprocally frustrated systems with perfect nonreciprocity $J_{ij} = -J_{ji}$, the spins start a chase and run away motion that corresponds to marginal orbits with zero Lyapunov exponents $\lambda = 0$, arising due to the Liouville-type theorem [Eq. (3)]. These orbits can be regarded as the dynamical counterpart of the ground state accidental degeneracy of geometrically frustrated systems that also have zero Lyapunov exponents $\lambda = 0$.

differently, is there a counterpart of accidentally degenerate ground states in nonreciprocal systems that were the origin of these exotic phenomena? At first glance, it seems highly unlikely, due to a crucial difference from geometrically frustrated systems: the absence of the notion of energy in nonreciprocal systems [11]. Consequently, accidentally degenerate ground states cannot be defined. Moreover, nonreciprocally interacting agents typically start a “chase and run away” motion that cannot be described in terms of energy minimization [Fig. 1(c1)], unlike in geometrically frustrated systems where they settle in a compromised configuration. As such, the two types of frustration appear to have no further connection in their phenomenology beyond the vague resemblance mentioned earlier.

Despite these fundamental differences, in this paper, we establish a *direct* analogy between the two types of frustration. This is achieved by pointing out a crucial common feature shared between geometrically frustrated systems and nonreciprocally interacting systems with antisymmetric coupling: the presence of marginal orbits characterized by zero Lyapunov exponents that do not originate from symmetry [see Figs. 1(a3)–1(c3) and 1(a4)–1(c4)]. In the case of geometric frustration [Fig. 1(b1)], the accidentally degenerate ground state [Fig. 1(b2)] corresponds to marginal *fixed points* [Fig. 1(b3)] in the dynamical system language. The presence of marginal orbits in nonreciprocally interacting cases, on the other hand, is supported by a Liouville-type theorem that holds in the antisymmetric coupling limit. In contrast to the marginal fixed points in the geometrically frustrated systems, the marginal orbits in the nonreciprocal systems are generally time dependent [Fig. 1(c3)], reflecting their nonequilibrium nature. The emerging marginal orbits in the latter can therefore be viewed as the dynamical counterparts of the accidentally degenerate ground states.

We show that these “accidentally degenerate” orbits often get “lifted” by stochastic noise or quenched disorder. This leads to the emergence of OBDP, where noise or quenched disorder *induce* order instead of destroying it, which is opposite from what one usually expects. This effect is attributed to the emergence of the “entropic (disorder-induced) force” that naturally arises in the presence of accidental degeneracy combined with stochasticity. We show that this entropic (disorder-induced) force can trigger noise-induced (disorder-induced) spontaneous symmetry breaking via nonreciprocal phase transition [11,34]. In addition, we provide numerical evidence that a spin-glass-like state emerges in a randomly coupled spin chain with nonreciprocal interaction but has no geometric frustration. We observe in this model a power-law decay of a time correlation function with a clear sign of aging, while the spatial correlation function is found to be short-ranged (stretched exponential decay). These findings establish an unexpected connection between the seemingly unrelated fields of complex magnetic materials and nonreciprocal

matter. Our results may have applications in the field of active matter and biological systems and offer a novel design principle for the robotic metamaterial.

The paper is organized as follows. In Sec. II, we draw a direct analogy between geometrically frustrated systems and nonreciprocally interacting systems with antisymmetric coupling by proving that marginal orbits, which can be regarded as a dynamical counterpart of accidental degeneracy, generically arise in both classes of systems. This is shown by proving a Liouville-type theorem that holds in this limit. In Sec. III, we demonstrate that this accidental degeneracy of orbits typically gets lifted by stochastic noise or quenched disorder, giving rise to time crystalline OBDP. In Sec. IV, we show numerically that a state analogous to a spin glass state emerges when nonreciprocity is introduced in their coupling in a one-dimensional randomly coupled spin chain. In Sec. V, we summarize our paper and discuss the outlook.

II. EMERGENCE OF ACCIDENTAL DEGENERACY OF ORBITS

In this paper, for concreteness, we focus on dissipatively coupled classical XY spin systems with their spin angle $\theta = (\theta_1, \dots, \theta_N)$ dynamics governed by

$$\dot{\theta}_i = - \sum_{j=1}^N J_{ij} \sin(\theta_i - \theta_j), \quad (1)$$

which generically has a nonreciprocal coupling $J_{ij} \neq J_{ji}$. The concepts we introduce below, however, should be valid for a more general class of systems (see Table I and Appendix A for other candidate systems). The effect of stochastic noise will be addressed later. This dynamical system [Eq. (1)] is invariant under global rotation $\theta_i \rightarrow \theta_i + \chi$ (where χ is a real constant).

Let us first briefly review the reciprocal coupling case $J_{ij} = J_{ji}$ with and without geometrical frustration. In such systems, Eq. (1) can be rewritten using a derivative of an energy $E(\theta)$ as $\dot{\theta}_i = -\partial E(\theta)/\partial \theta_i$, where

$$E(\theta) = - \sum_{i,j} J_{ij} \cos(\theta_i - \theta_j). \quad (2)$$

As a result, the system is driven toward the (local) minimum of the energy E .

Frustration-free systems are systems that have ground states that minimize the energy E of Eq. (2) *term by term* [35] [Fig. 1(a)]. In this case, the ground state configuration is uniquely determined when one of the spin angles is fixed. Therefore, these systems only have a ground state degeneracy that is trivially due to the rotation symmetry of the dynamical system [Fig. 1(a2)]. The dynamical system Eq. (1) therefore has a unique stable fixed point [Fig. 1(a3)] when regarding the orbits identical up to global rotation as

TABLE I. List of nonreciprocally interacting systems that satisfy the Liouville-type theorem in the antisymmetric limit.

Model	Candidate systems	Proof
Nonreciprocal XY model	Robotic metamaterials	Appendix A 1
Nonreciprocal Heisenberg model	Robotic metamaterials	Appendix A 2
Oscillators with phase-delayed interactions	Biased Josephson junctions arrays [56,57], microscopic rotors [4]	Appendix A 3
Nonreciprocally interacting particles	Complex plasma [3], chemically or optically active colloids [5–8]	Appendix A 4

the same orbit. In this case, all Lyapunov exponents would be negative $\lambda_i < 0$ [Fig. 1(a4)] except for the zero modes arising from the global rotation symmetry [i.e., the Nambu-Goldstone mode, which we omitted in Fig. 1(a4)].

In contrast, in geometrically frustrated systems [Fig. 1(b)], there are no configurations that simultaneously minimize all the interaction terms [35]. In such a situation, the ground state configuration is often underconstrained [36,39,40], causing the emergence of an accidental degeneracy of ground states [Fig. 1(b2)] that does not stem from their underlying symmetry. Which ground state the system ultimately converges to depends on its initial condition. No restoring force would be applied in the direction in plane of the accidentally degenerate ground state manifold. In the language of dynamical systems, this implies the existence of marginal fixed points [Fig. 1(b3)] that indicate the presence of zero Lyapunov exponent(s) $\lambda = 0$ [Fig. 1(b4)] (in addition to the zero Lyapunov exponent trivially arising from the Nambu-Goldstone mode).

We show below that the nonreciprocally interacting system with antisymmetric coupling $J_{ij} = -J_{ji}$ (which we refer to below as *perfectly nonreciprocal*) has exactly the same feature: the existence of marginal orbits with zero Lyapunov exponents $\lambda = 0$ [Fig. 1(c)]. In this situation, the distribution function $\rho(\theta)$ is found to stay constant along any trajectory (see Appendix A for the proof), i.e.,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \sum_i \frac{\partial\rho}{\partial\theta_i} \dot{\theta}_i = 0, \quad (3)$$

in a similar manner to Liouville's theorem of Hamiltonian systems. Note that a similar theorem holds for nonreciprocally interacting Heisenberg models, oscillators with phase-delayed interactions [58] (that well describe biased Josephson junctions arrays [56,57] and microscopic rotors [4]), and nonreciprocally interacting particles (that describe, e.g., complex plasma [3] and chemically [5,6] and optically active colloidal matter [7,8]), as shown in Appendix A. (See Table I. See also Ref. [59] for a similar relation known in the context of evolutionary game theories.) The conservation of phase volume $dV = \rho \prod_i d\theta_i$ of Eq. (3) means that the dynamics are dissipationless and the sum of all Lyapunov exponents is zero, $\sum_{i=1}^N \lambda_i = 0$. In the absence of chaos $\lambda_i \leq 0$, this makes all Lyapunov exponents vanish $\lambda_i = 0$ [Fig. 1(c4)], which, generically, implies the emergence of marginal orbits

described schematically in Fig. 1(c3). Which orbit the system ends up taking depends on the initial condition, in an identical situation to the geometrically frustrated case.

We interpret these marginal orbits as the emergence of accidental degeneracy caused by nonreciprocal frustration. This degeneracy is accidental, in the sense that they do not originate from the global symmetry or topology of the dynamical system [Eq. (1)], in direct analogy to those of geometrical frustration. The difference lies in both its physical origin and the consequence: in the nonreciprocal (geometrical) frustration case, the degeneracy comes from Liouville's theorem (underconstrained degrees of freedom [36,39,40]) and the resulting marginal orbits are typically time dependent (static).

Take a two-spin perfectly nonreciprocal system $J_{12} = -J_{21} = J_-$ as the simplest example [11,60]. One can readily find an analytical solution to the center-of-mass angle $\Theta = (\theta_1 + \theta_2)/2$ and the difference $\Delta\theta = \theta_1 - \theta_2$ for a given initial condition $\theta_{i=1,2}(t=0)$ as

$$\Theta(t) = -J_- t \sin[\Delta\theta(0)], \quad \Delta\theta(t) = \Delta\theta(0). \quad (4)$$

As expected, the system exhibits marginal periodic orbits, where the speed and direction of the drift of the center-of-mass angle Θ are determined by the initial condition of $\Delta\theta$ that stays constant. The numerical solution of a three-spin perfectly nonreciprocal system is depicted in Fig. 2 as another example, where we similarly find marginal periodic orbits.

Accidental degeneracy is usually associated with fine-tuning of parameters. Here, in nonreciprocally frustrated

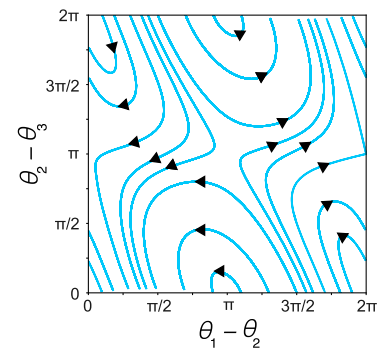


FIG. 2. Marginal orbits in perfectly nonreciprocal three-spin system. We set $J_{12} = -J_{21} = 3$, $J_{23} = -J_{32} = -1$, and $J_{31} = -J_{13} = 2$.

systems, the emergence of marginal orbits relies on the fine-tuning of the coupling to be perfectly nonreciprocal $J_{ij} = -J_{ji}$. Once the coupling strength deviates from this limit, the marginal orbits would generically turn into (un)stable orbits, corresponding to the “lifting” of degeneracy. This situation is in parallel to the geometrical frustration case where the degeneracy is often contingent on the coupling strength being identical $J_{ij} = J$ [36].

So far, we have considered cases where *all* spins are perfectly nonreciprocally interacting $J_{ij} = -J_{ji}$, where we have shown that a Liouville-type theorem Eq. (3) holds in such cases. This means that there is absolutely no dissipation occurring in the system: *Any* initial state will exhibit a marginal orbit that conserves the phase volume in this case. In some sense, this is similar to systems with a *constant* energy $E(\theta) = \text{const}$, where *all* states are trivially in the ground state manifold. This stands in contrast to generic geometrically frustrated systems, where only a small subset of states reside in the ground state manifold. In such systems, a typical initial state would *relax* to a state that corresponds to a marginal fixed point.

In what follows, we show that there is a class of nonreciprocal systems where a generic initial state relaxes to a marginal orbit, making the analogy to geometrically frustrated systems even more direct. Namely, we consider a system that is separated into communities that interact nonreciprocally between different communities but ferromagnetically within the same community:

$$\dot{\theta}_i^a = \sum_b \sum_j J_{ij}^{ab} \sin(\theta_j^b - \theta_i^a). \quad (5)$$

Here, a, b label the community and i, j label the spins in the community, and the intracommunity coupling is ferromagnetic $J_{ij}^{aa} > 0$. In such a situation, the spins in the intracommunities would eventually align $\theta_i^a = \phi_a$ to give

$$\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a), \quad (6)$$

in the long time limit, which has an identical form to Eq. (1) when the intercommunity coupling $j_{ab} = \sum_j J_{ij}^{ab}$ ($a \neq b$) is i independent. Therefore, following the same logic as before, the system would exhibit marginal orbits with zero Lyapunov exponents in the perfectly nonreciprocal *intercommunity* coupling $j_{ab} = -j_{ba}$. The difference from the systems considered before is the presence of dissipative processes toward this attractor due to the ferromagnetic *intracommunity* interactions $J_{ij}^{aa} > 0$. In the next section, we will show that such a relaxation process combined with the presence of marginal orbits indeed plays a crucial role in the emergence of a counterintuitive phenomenon called the OBDP.

III. TIME CRYSTALLINE ORDER-BY-DISORDER PHENOMENA

Having established that nonreciprocal interaction gives an alternative route from geometrical frustration to generating accidental degeneracy of orbits (i.e., marginal orbits), we now investigate their impact on the many-body properties of the system. In geometrically frustrated systems, a paradigmatic example of a phenomenon emerging from such accidental degeneracy is the OBDP [37–41]. As the degeneracy generated by frustration is not protected by symmetry nor topology, it is fragile, not only against external perturbations but also against disorders such as thermal noise or weak random potential. As a result, the degeneracy often gets lifted and ends up, perhaps counterintuitively, in a more ordered state than that of the clean system. This is known as the OBDP.

In this section, we show that an analogous phenomenon arises in the nonreciprocally interacting many-body systems as well, with the peculiarity that the emerging ordered state is typically time periodic, also known as a time crystal [61,62]. To illustrate the idea, let us first briefly review the concept of OBDP in the geometrically frustrated systems in equilibrium systems. As we have seen, at zero temperature $T = 0$ (no noise), geometrically frustrated systems often exhibit accidental degeneracy in their ground states [Figs. 3(a) and 3(c)]. Mathematically, this can be described as the ground state energy E_G being independent of the system’s configuration within the accidentally degenerate ground state manifold, which is parametrized by ϕ [i.e., $E_G(\phi) = \text{const}$].

Now, let us introduce thermal noise, corresponding to a system at finite temperature $T > 0$ [Figs. 3(b) and 3(d)]. In this case, the system converges to a state that minimizes the free energy $F = E - TS$, where S represents entropy. Although the energy $E = E_G$ remains constant within the ground state manifold by definition, the fluctuation properties are typically configuration dependent, resulting in a configuration-dependent entropy $S(\phi)$. As a consequence, the accidental degeneracy is generically lifted entropically, driving the system toward the ground state with maximum entropy ϕ_* , which is “selected” [Fig. 3(b)]. The selected state often exhibits a long-range order, giving rise to the counterintuitive phenomenon of OBDP, where thermal noise *induces* order [37–41]. We remark that, when the degeneracy originates from symmetry, the entropy S cannot be configuration dependent within the ground state degeneracy manifold because the symmetry guarantees their equivalence. This shows how the origin of degeneracy being *accidental* is a key element for the emergence of OBDP.

In the language of dynamical systems, this can be translated into the dynamics of the (thermal averaged) parameter ϕ described by

$$\dot{\phi} = -\frac{\partial F(\phi)}{\partial \phi} = -\frac{\partial E}{\partial \phi} + T \frac{\partial S(\phi)}{\partial \phi} = T \frac{\partial S(\phi)}{\partial \phi}. \quad (7)$$

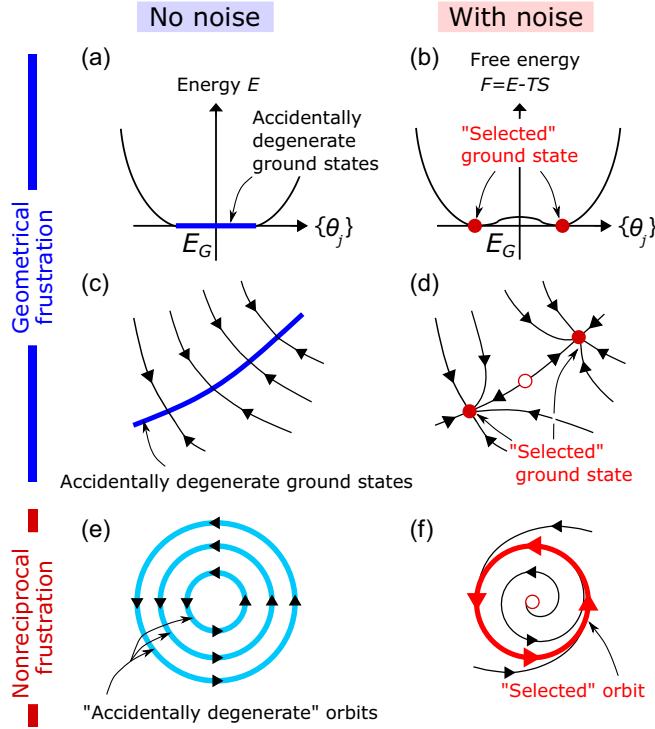


FIG. 3. The concept of order-by-disorder phenomena in equilibrium and their generalization to nonreciprocally interacting systems. (a),(b) Energy and free-energy profiles in a geometrically frustrated system (a) without noise (i.e., zero temperature $T = 0$) and (b) with noise (i.e., finite temperature $T > 0$) in a geometrically frustrated system. (c),(d) Orbits in a geometrically frustrated system without (c) and with (d) thermal noise. At $T = 0$, the system exhibits ground state degeneracy, while the presence of thermal noise at $T > 0$ lifts this degeneracy through entropic forces. (e),(f) Orbits in a nonreciprocally interacting system without (e) and with (f) noise. Similarly to the geometrically frustrated system counterpart, “entropic force” selects one of the orbits to give rise to OBDP.

The term $f_S \equiv T \partial S(\phi) / \partial \phi$ is an entropic force (or entropic torque, in the context of spin systems) induced by thermal noise, which drives the system toward the state of maximum entropy [i.e., the “selected” ground state; see Figs. 3(c) and 3(d)]. Crucially, the energy term $f_E \equiv -\partial E / \partial \phi$ vanishes because of the property that the system is marginal, which makes the entropic force $f_S \propto T$ dominant even at the weak thermal noise limit $T \rightarrow 0^+$.

We will demonstrate in this section that this concept can be generalized to nonreciprocally interacting systems [Figs. 3(e) and 3(f)]. We will show that finite noise strength in a nonreciprocally interacting system can give rise to a conceptually similar entropic force that drives a specific orbit toward stability, thereby triggering “orbit selection” among the accidentally degenerate orbits. In parallel to the case of geometrical frustration, we will demonstrate that this entropic force tends to favor an ordered phase, leading to noise-induced symmetry breaking—a characteristic feature of OBDP. Importantly, with this concept extended

to a broader class of dynamical systems, noise can now trigger phase transitions (bifurcations) beyond the conventional equilibrium paradigm. We will showcase this by demonstrating a noise-induced nonreciprocal phase transition [11] in our studied model, which has no counterpart in equilibrium systems.

A. All-to-all coupled models

To set the stage, we consider an all-to-all coupled system where the spins are grouped into a few communities (labeled by $a, b = A, B, C, \dots$) that each consist of N_a spins and are now subject to Gaussian white noise η_i^a ,

$$\dot{\theta}_i^a = -\sum_b \frac{j_{ab}}{N_b} \sum_{j=1}^{N_b} \sin(\theta_i^a - \theta_j^b) + \eta_i^a, \quad (8)$$

where $\langle \eta_i^a(t) \rangle = 0$, $\langle \eta_i^a(t) \eta_j^b(t') \rangle = \sigma \delta_{ab} \delta_{ij} \delta(t - t')$. We consider the case where the intracommunity couplings are reciprocal and ferromagnetic $j_{aa} > 0$, while the intercommunity couplings may be nonreciprocal $j_{ab} \neq j_{ba}$ ($a \neq b$). This is an example of systems described by Eq. (5). The former causes the intracommunity spins to order ferromagnetically at sufficiently weak noise strength, which is characterized by the order parameter $\psi_a(t) = (1/N_a) \sum_{i=1}^{N_a} e^{i\theta_i^a(t)} = r_a(t) e^{i\phi_a(t)}$ [63]. Note that, for the reciprocal case $j_{ab} = j_{ba}$, this setup corresponds to an equilibrium system at finite temperature $T = \sigma / (2k_B)$ (where k_B is the Boltzmann constant).

In the absence of noise $\sigma = 0$, as discussed in the final part of Sec. II, all of the spins in the same community would eventually align ($\theta_i^a = \phi_a$) to give perfect magnetization $r_a = 1$. As a result, the spins in the same community will collectively behave as a macroscopic object that follows the same dynamics as Eq. (1) [as we have discussed in Eq. (6)]. Therefore, these macroscopic angles $\phi(t) = [\phi_A(t), \phi_B(t), \dots]$ exhibit marginal, accidentally degenerate orbits when the intercommunity couplings j_{ab} are chosen to have geometrical or nonreciprocal frustration. For example, in a geometrically frustrated system consisting of four communities ($a, b = A, B, C, D$) that interacts antiferromagnetically $j_{ab} = -j < 0$ ($a \neq b$) [Fig. 4(a)], the system relaxes to the accidentally degenerate ground states parametrized by a relative angle α illustrated in the inset of Fig. 4(a) (see Refs. [39,40] and Appendix B). Similarly, systems with nonreciprocal frustration with $j_{ab} = -j_{ba}$ exhibit time-dependent, marginal orbits $\phi(t)$. [See Figs. 2 and 4(b).]

Below, we show that this accidental degeneracy generically gets lifted by the stochastic noise, irrespective of whether the degeneracy is originated from geometrical or nonreciprocal frustration. In the presence of noise, θ_i^a fluctuates around the macroscopic spin angle ϕ_a . At sufficiently weak noise strength, the distribution of $\delta\theta_i^a = \theta_i^a - \phi_a$ takes a Gaussian distribution (see Appendix B),

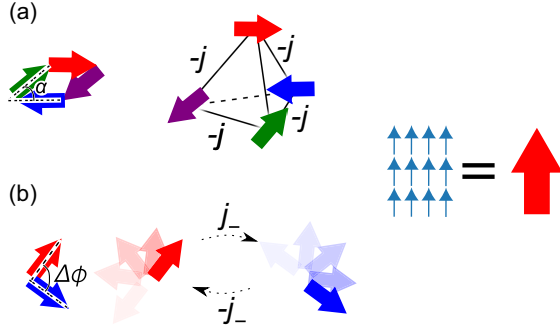


FIG. 4. Accidental degeneracy of orbits in geometrically and nonreciprocally frustrated all-to-all coupled many-body systems. The thick arrows represent the macroscopic angles ϕ_a that are composed of a macroscopic number of spins represented by smaller solid arrows. A geometrically frustrated four-community system illustrated in (a) exhibits an accidental degeneracy parametrized by a relative angle α . Similarly, a nonreciprocally frustrated two-community system illustrated in (b) exhibits marginal orbits parametrized by a relative angle $\Delta\phi$. These degeneracies are shown to get lifted by introducing disorder to the system.

$$\rho_i^a[t, \delta\theta_i^a; \phi(t)] = \frac{1}{\sqrt{\pi}w_a[t; \phi(t)]} e^{-(\delta\theta_i^a)^2/w_a^2[t; \phi(t)]}, \quad (9)$$

with its width w_a given by

$$w_a^2[t; \phi(t)] = 2\sigma \int_0^t d\tau e^{-2\int_\tau^t dt' \sum_b j_{ab} \cos[\phi_a(\tau') - \phi_b(\tau')]}$$

for an initial condition with a perfectly magnetized state $\delta\theta_i^a(t=0) = 0$. In many cases we will consider below, $\phi_a - \phi_b$ converges to a constant value in the long time limit, in which the width $w_a^2(t \rightarrow \infty, \phi)$ is given by

$$w_a^2(t \rightarrow \infty, \phi) = \frac{\sigma}{\sum_b j_{ab} \cos(\phi_a - \phi_b)}. \quad (10)$$

Crucially, the width $w_a(\phi)$ of the fluctuations depends on which accidentally degenerate orbit $\phi(t)$ the system happened to take. This is in stark contrast to the degenerate states arising from global symmetry, where all the degenerate states are guaranteed to have the same fluctuation properties by symmetry. The configuration-dependent fluctuation seen above is therefore a salient feature of the accidentally degenerate states. Note how the ferromagnetic intracommunity coupling $j_{aa} > 0$ is playing a crucial role in preventing the width w_a^2 from becoming negative, ensuring the stability of the orbits.

As a result, macroscopic angle dynamics,

$$\dot{\phi}_a(t) = -\sum_b j_{ab}^*[\phi(t)] \sin[\phi_a(t) - \phi_b(t)] + \bar{\eta}_a(t), \quad (11)$$

are now governed by ϕ -dependent, renormalized coupling:

$$j_{ab}^*[\phi(t)] = j_{ab} \frac{r_b[\phi(t)]}{r_a[\phi(t)]} \langle \cos^2 \delta\theta_i^a \rangle_{\phi(t)}, \quad (12)$$

where $\langle h(\delta\theta_i^a) \rangle_{\phi(t)} = \int d\delta\theta_i^a \rho_i^a[t, \delta\theta_i^a; \phi(t)] h(\delta\theta_i^a)$ (see Appendix B for derivation). Here, we have assumed that the system self-averages, $\langle h(\delta\theta_i^a) \rangle_{\phi(t)} = (1/N_a) \sum_{i=1}^{N_a} h(\delta\theta_i^a)$. In Eq. (12), $\bar{\eta}_a \approx (1/N_a) \sum_{i=1}^{N_a} \eta_i^a$ is the noise acting on the macroscopic angle ϕ_a that obeys $\langle \bar{\eta}_a(t) \rangle = 0$ and

$$\langle \bar{\eta}_a(t) \bar{\eta}_b(t') \rangle \approx \frac{\sigma}{N_a} \delta_{ab} \delta(t - t'). \quad (13)$$

As a result of $\phi_a(t)$ being a macroscopic quantity, the noise strength on this quantity vanishes as one takes the thermodynamic limit $N_a \rightarrow \infty$.

The renormalization of the coupling gives rise to an additional torque to the deterministic limit [Eq. (6)], which can be regarded as the entropic torque generalized to dynamical systems that are not necessarily written in terms of free energy [cf. Eq. (7)]. As we will see, this entropic contribution determines the macroscopic features of systems with geometrical or nonreciprocal frustration that have marginal orbits.

First consider the geometrically frustrated system introduced above, which consists of four communities that antiferromagnetically interact [$j_{ab} = -j < 0$ ($a \neq b$)]. This system has an accidentally degenerate ground state manifold parametrized by an angle α [Fig. 4(a)]. In this situation, the effective coupling turns out to be ϕ independent $j_{ab}^*(\phi(\alpha)) = -j^* < 0$ on this manifold (Appendix B). Therefore, this many-body problem maps to that of a four-spin system on a tetrahedron lattice, but importantly, at a very low but *finite* temperature $T \sim \sigma/N_a \rightarrow 0^+ > 0$. As pointed out in Ref. [40], under such stochasticity, the probability to realize the angle α is given by the Boltzmann distribution [where $F(\alpha)$ is a free energy and $S(\alpha)$ is the entropy at configuration α] (see Appendix B 1 a for derivation),

$$\rho(\alpha) \propto e^{-F(\alpha)/(k_B T)} = e^{S(\alpha)/k_B} \sim |\sin(\alpha)|^{-1}, \quad (14)$$

for $\sin^2 \alpha \gg \sigma/(N_a j^*) \rightarrow 0$ that is found to be overwhelmingly concentrated to the collinear configuration $\alpha_* = 0, \pi$. In other words, the entropic effects “select” the collinear configuration $\alpha_* = 0, \pi$ among the degenerate ground states (or the marginal fixed points in the dynamical system language), giving rise to an OBDP.

We show below that a similar orbit selection takes place in nonreciprocally frustrated systems as well [Fig. 4(b)], due to the entropic force f_S that is analogous to those arising in Eq. (7). To be explicit, let us consider the case of two communities $a = A, B$ that are nonreciprocally coupled ($j_{AB} \neq j_{BA}$). In the deterministic case $\sigma = 0$, since the dynamics of the order parameter is given by Eq. (6),

the angle difference $\Delta\phi = \phi_A - \phi_B$ and the center-of-mass angle $\Phi = (\phi_A + \phi_B)/2$ dynamics is governed by

$$\Delta\dot{\phi} = -(j_{AB} + j_{BA}) \sin \Delta\phi, \quad (\sigma = 0), \quad (15)$$

$$\dot{\Phi} = -\frac{j_{AB} - j_{BA}}{2} \sin \Delta\phi. \quad (\sigma = 0). \quad (16)$$

Hence, for the perfectly nonreciprocal case $j_{AB} = -j_{BA} = j_-$, where the Liouville-type theorem [Eq. (3)] holds, the angle difference is initial state dependent $\Delta\phi(t) = \Delta\phi(0)$. The accidentally degenerate orbits are parametrized by $\Delta\phi$ in this case [Fig. 4(b)].

We will now show that the orbit selection occurs in the presence of noise $\sigma > 0$ due to the emergence of entropic torque. From Eq. (11), $\Delta\phi$ and Φ dynamics is governed by the equation of motion determined by the renormalized coupling $j_{ab}^*(\Delta\phi)$ [Eq. (12)]:

$$\Delta\dot{\phi} = -[j_{AB}^*(\Delta\phi) + j_{BA}^*(\Delta\phi)] \sin \Delta\phi, \quad (17)$$

$$\dot{\Phi} = -\frac{j_{AB}^*(\Delta\phi) - j_{BA}^*(\Delta\phi)}{2} \sin \Delta\phi. \quad (18)$$

Here, we have dropped the macroscopic noise $\bar{\eta}_a(t)$, which is justified in the thermodynamic limit $N_a \rightarrow \infty$. [See Eq. (13).] Because of the $\Delta\phi$ dependence of the renormalized couplings $j_{ab}^*(\Delta\phi)$, the Liouville-type theorem no longer holds, and the angle difference $\Delta\phi$ exhibits *stable* fixed points even in the nonreciprocal limit $j_{AB} = -j_{BA}$; the orbit selection occurs. In particular, there are two candidates for stable fixed points of $\Delta\phi$ [Eq. (17)] that correspond to different phases of matter. One is a phase that satisfies

$$\sin \Delta\phi_* = 0, \quad (19)$$

which corresponds to a static phase $\dot{\Phi} = 0$ that has an aligned ($\Delta\phi_* = 0$) or an antialigned ($\Delta\phi_* = \pi$) configuration.

The other is a phase that only emerges in the presence of noise $\sigma > 0$, which satisfies

$$j_{AB}^*(\Delta\phi_*) = -j_{BA}^*(\Delta\phi_*). \quad (20)$$

Generically, $\Delta\phi_* \neq 0, \pi$, corresponding to a time-dependent phase $\dot{\Phi} \neq 0$ that is referred to as a chiral phase in Ref. [11]. Importantly, while the static phase is invariant under the parity operation,

$$(\phi_A, \phi_B) \rightarrow (-\phi_A, -\phi_B), \quad (21)$$

the chiral phase spontaneously breaks it. Note that the renormalized coupling satisfies $j_{ab}^*(\Delta\phi) = j_{ab}^*(-\Delta\phi)$,

which follows from the property $\rho_i^a(\delta\theta_i^a; \Delta\phi) = \rho_i^a(\delta\theta_i^a; -\Delta\phi)$ [that can be obtained from Eqs. (9) and (12)]. Therefore, if $\Delta\phi = \Delta\phi_*$ were found to be a stable fixed point of Eq. (17), then $\Delta\phi = -\Delta\phi_*$ must also be a stable fixed point, which transforms one to the other via parity operation Eq. (21).

Take a perfectly nonreciprocal system $j_{AB} = -j_{BA} = j_-$ that has an identical intracommunity ferromagnetic coupling strength between the two communities $j_{AA} = j_{BB} = j_0$ (taken to be $j_0 > |j_-|$ to ensure stability $w_a^2 > 0$) as an example. In this case, Eq. (17) reads (see Appendix B)

$$\Delta\dot{\phi} \simeq \frac{j_0 j_-^2 \sigma^2}{2} \frac{\cos \Delta\phi}{(j_0^2 - j_-^2 \cos^2 \Delta\phi)^2} \sin \Delta\phi \quad (22)$$

at sufficiently weak noise level, which has stable fixed points at $\Delta\phi_* = \pm\pi/2$ that corresponds to a chiral phase, satisfying Eq. (20). The fixed points $\Delta\phi_* = 0, \pi$ are unstable. All these features are consistent with the numerical result presented in Fig. 5. This clearly shows that the noise $\sigma > 0$ has induced an ‘‘entropic torque’’ that stabilizes a spontaneous parity broken phase, in a similar manner to the geometrically frustrated case; cf. Eq. (7). Derivation of this entropic torque induced by nonreciprocity is one of the main results of this paper.

This property has an important implication for a more general case, i.e., when one is away from the perfectly nonreciprocal case $j_{AB} \neq -j_{BA}$. Figure 6 shows the noise strength dependence of $\Delta\phi_*$ when $j_{AB} = 0.35 \neq -j_{BA} = 0.25$. While at small noise strength σ , the parity-symmetric static phase $\Delta\phi_* = 0$ is realized, a parity-broken chiral phase $\Delta\phi_* \neq 0$ emerges at *higher* noise level. This noise-induced spontaneous symmetry breaking is a salient feature of OBDP.

A qualitative understanding of this counterintuitive phenomenon can be obtained from our formalism as follows. In the deterministic case, when the reciprocal part of the intercommunity coupling is positive (negative) $j_+ \equiv (j_{AB} + j_{BA})/2 > 0$ (< 0), Eq. (15) tells us that a static phase with an aligned configuration $\Delta\phi_* = 0(\pi)$ would be realized, in agreement with Fig. 6 at $\sigma = 0$. As one turn on the noise strength $\sigma > 0$, Eq. (17) reads

$$\Delta\dot{\phi} = \left[-2j_+ + \frac{j_0 j_-^2 \sigma^2}{2} \frac{\cos \Delta\phi}{(j_0^2 - j_-^2 \cos^2 \Delta\phi)^2} \right] \sin \Delta\phi, \quad (23)$$

where we have assumed small reciprocity $|j_+| \ll |j_-|$, j_0 and restricted ourselves to be near the phase transition point. (See Appendix B for derivation.) This equation is to be compared with its geometrical frustration counterpart, Eq. (7). Here, the first term proportional to the reciprocal piece $j_+ > 0$ (< 0) describes the torque that tries to make the angles (anti)aligned, which can be considered as an analog of the first term of Eq. (7). This torque competes with the entropic torque induced by nonreciprocal

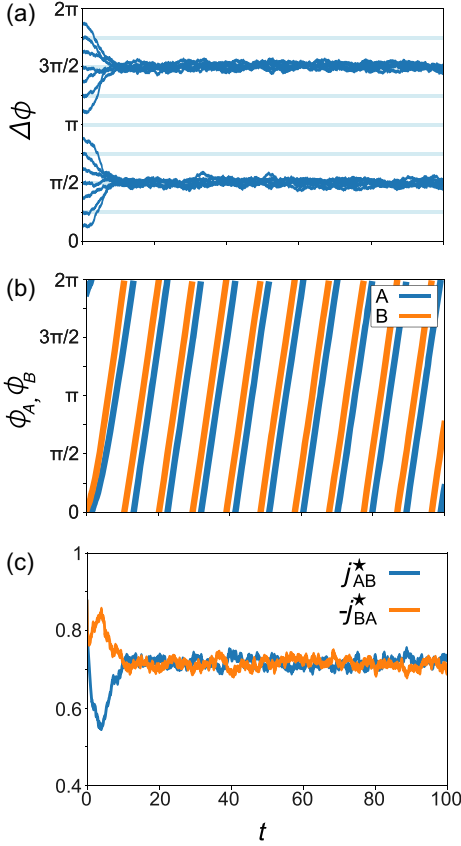


FIG. 5. Time crystalline order-by-disorder phenomena induced by nonreciprocal frustration. Time evolution of (a) the angle difference $\Delta\phi = \phi_A - \phi_B$, (b) ϕ_A, ϕ_B , and (c) the effective coupling strength $j_{ab}^*(\phi)$ [Eq. (12)]. In (a), the solid (thin) line represents the dynamics in the presence (absence) of noise. We set $j_{AA} = j_{BB} = 3$, $j_{AB} = -j_{BA} = 1$, the noise strength $\sigma = 1.5$, and the number of spins $N_A = N_B = 2000$. The system “selects” $\Delta\phi_* = \pm\pi/2$ that satisfies Eq. (20) to give rise to the time-dependent phase (chiral phase), all in agreement with our analytical analysis in the main text.

frustration [the second term that is identical to the right-hand side of Eq. (22)].

When the noise is weak and the first term is dominant over the second, Eq. (23) has only one stable fixed point $\Delta\phi_* = 0(\pi)$. However, once the noise strength σ exceeds a critical value,

$$\sigma > \sigma_c = 2\sqrt{\frac{|j_+|}{j_0} \frac{j_0^2 - j_-^2}{|j_-|}}, \quad (24)$$

the entropic torque [the second term of Eq. (23)] makes the system bifurcate to a chiral phase with $\phi_* \neq 0, \pi$. This signals the emergence of a spontaneous parity breaking seen in Fig. 6. This result implies a phase diagram schematically depicted in Fig. 7.

We remark that the phase transition observed above is an instance of a nonreciprocal phase transition [11], which has

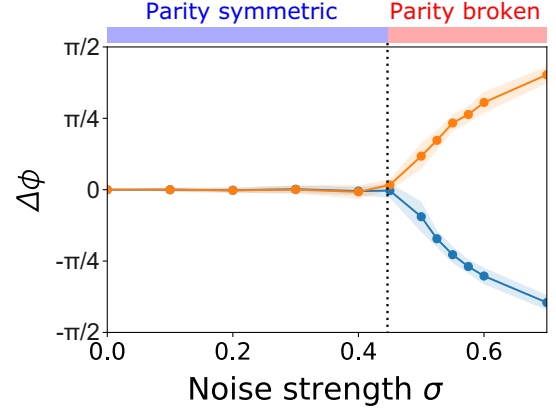


FIG. 6. Noise-induced spontaneous parity breaking. The computed noise strength dependence of the angle difference $\Delta\phi = \phi_A - \phi_B$ of the order parameter $\psi_a = r_a e^{i\phi_a}$ in the steady state for $j_{AB} = 0.35 \neq -j_{BA} = 0.25$. Here, the solid lines and the shaded area represent the average and the variance of $\Delta\phi$, respectively, which were calculated using the data after reaching a steady state (in the time range $100 < t < 400$) with an initial condition $\theta_i^A(t=0) = \pi/4 (= -\pi/4)$ and $\theta_i^B(t=0) = 0$ for blue (orange) plots. The transition from $\Delta\phi = 0$ (parity-symmetric static phase) to $\Delta\phi \neq 0$ (parity-broken chiral phase) as σ increases indicates that the noise has induced the spontaneous symmetry breaking, which is a salient feature of OBDP and is opposite from what is usually expected. We set $j_{AA} = j_{BB} = 1$, and $N_A = N_B = 2000$.

no equilibrium counterpart. Nonreciprocal phase transition is marked by a spectral singularity called an exceptional point, where the eigenvectors coalesce [64] to a zero mode at the critical point [11,65]. This can be seen by linearizing Eqs. (23) and (18) around the static phase $\Delta\phi = \Delta\phi_* + \delta\Delta\phi = \delta\Delta\phi$,

$$\begin{aligned} \begin{pmatrix} \delta\Phi \\ \delta\Delta\dot{\phi} \end{pmatrix} &= \begin{pmatrix} 0 & -2j_- + \frac{j_0\sigma}{j_0^2 - j_-^2} + \frac{(2j_0^2 + j_-^2)\sigma^2}{4(j_0^2 - j_-^2)^2} \\ 0 & -2j_+ + \frac{j_0j_-^2\sigma^2}{2(j_0^2 - j_-^2)} \end{pmatrix} \begin{pmatrix} \delta\Phi \\ \delta\Delta\phi \end{pmatrix} \\ &= \hat{L} \begin{pmatrix} \delta\Phi \\ \delta\Delta\phi \end{pmatrix}. \end{aligned} \quad (25)$$

At the transition point $\sigma = \sigma_c$ [Eq. (24)], the (2,2) component of the Jacobian \hat{L} vanishes, giving a defective matrix [64] with zero eigenvalues:

$$\hat{L}_c = \begin{pmatrix} 0 & -2j_- + 2\sqrt{j_-j_+} - \frac{j_+j_-}{j_0} - \frac{2j_0j_+}{j_-} \\ 0 & 0 \end{pmatrix}. \quad (26)$$

This shows that the eigenvectors with zero eigenvalues coalesce at the critical point, which is a salient feature of nonreciprocal phase transitions [11,28,29,65–67].

It is also worth mentioning that there are cases where the “entropic effects” favor a static state, even in the perfectly nonreciprocal case $j_+ = 0$. In Appendix B, we show both

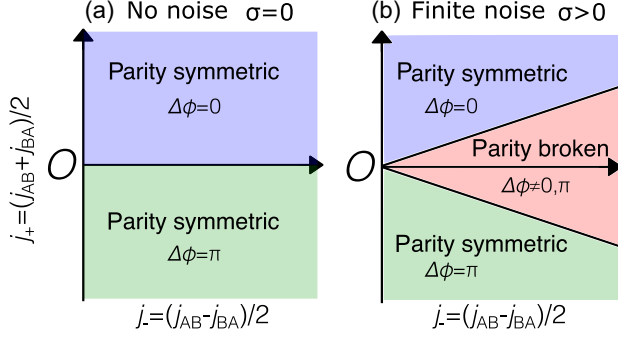


FIG. 7. Schematic phase diagram. (a),(b) Schematic steady-state phase diagram in the absence of noise $\sigma = 0$ (a) and in the presence of noise $\sigma > 0$ (b). The intracommunity couplings $j_{AA} > 0$ and $j_{BB} > 0$ are assumed to be large compared to intercommunity couplings j_{AB} and j_{BA} .

analytically and numerically that systems with $j_{AA} \gg j_{BB}$ and $j_{AA} \ll j_{BB}$ “select” a static state, $\Delta\phi_* = \pi$ and $\Delta\phi_* = 0$, respectively (see Fig. 16 in Appendix B). Physically, this is due to the property that, when $j_{AA} \gg j_{BB}$ ($j_{AA} \ll j_{BB}$), the width of the fluctuation of the A(B) community w_A (w_B) is smaller because the A(B) community gets stiff, leading to stronger suppression of $|j_{AB}^*(\Delta\phi)|$ [$|j_{BA}^*(\Delta\phi)|$]. [See Eq. (B9).] Similar effects can be seen when the noise strengths are different between the two communities, i.e., when the noise is characterized by $\langle \eta_a(t) \rangle = 0$, $\langle \eta_a(t) \eta_b(t') \rangle = \sigma_a \delta_{ab} \delta(t - t')$, with $\sigma_A \neq \sigma_B$. When $\sigma_A \gg \sigma_B$ ($\ll \sigma_B$) with $j_{AA} = j_{BB}$ leads to “selecting” the (anti) aligned static configuration $\Delta\phi_* = 0(\pi)$. These illustrate how the entropic torques that determine which state the system selects are strongly affected by the fluctuation properties of the degenerate states.

So far, we have analyzed the simplest system with two communities under stochastic noise. However, the underlying mechanism of OBDP is not restricted to such a specific case. We show below that an orbit selection occurs for a nonreciprocal *three-community* system,

$$\dot{\theta}_i^a = \omega_i^a - \sum_b \frac{j_{ab}}{N_b} \sum_{j=1}^{N_b} \sin(\theta_i^a - \theta_j^b), \quad (27)$$

with a random torque ω_i^a ($a, b = A, B, C$) distributed in a Lorentz distribution function,

$$P_a(\omega_i^a) = \frac{1}{\pi} \frac{\Delta}{(\omega_i^a)^2 + \Delta^2}, \quad (28)$$

as a source of quenched disorder. The width Δ characterizes the strength of the quenched disorder. This is the Kuramoto model [11,63,68] generalized to multiple communities. According to Refs. [69,70], the order parameter dynamics of this system are governed by

$$\dot{\psi}_a = -\Delta\psi_a + \frac{1}{2} \sum_b j_{ab} (\psi_b - \psi_a^2 \psi_b^*). \quad (29)$$

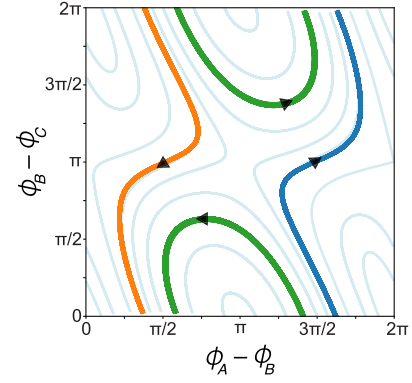


FIG. 8. Order-by-disorder phenomena in a nonreciprocally coupled three-community system with random torque. Solid (thin) lines represent the trajectories for different initial conditions in the presence (absence) of random torque, where one can see that certain orbits are selected by the disorder. The angle difference $\phi_a - \phi_b$ dynamics are computed using the Ott-Antonsen ansatz [69,70] [Eq. (29) in Appendix B]. We set the coupling $j_{AA} = j_{BB} = j_{CC} = 4$, $j_{AB} = -j_{BA} = 3$, $j_{BC} = -j_{CB} = -1$, $j_{CA} = -j_{AC} = 2$, the torque distribution width $\Delta = 0.1$.

Figure 8 shows the order parameter dynamics of this system. Among the marginal orbits in the absence of disorder $\Delta = 0$ shown in Fig. 2 (and the thin line of Fig. 8), certain orbits are selected to be stable (solid lines in Fig. 8), signaling the occurrence of OBDP.

B. Spatially extended models

Systems that exhibit the nonreciprocal frustration-induced OBDP are not restricted to all-to-all coupled models analyzed in the previous section. To demonstrate the generality of our finding, we now consider a nonreciprocal *XY-spin* system on a d -dimensional hypercubic lattice with nearest- and next-nearest-neighbor interactions. (The $d = 2$ case is illustrated in Fig. 9.) As in the model considered in the previous section, this model is composed of two communities of spins on different sublattices $a = A, B$. While the spins on the same sublattice interact ferromagnetically via the next-nearest-neighbor interaction $j_{AA}, j_{BB} > 0$, the spins on different sublattices interact nonreciprocally by the nearest-neighbor interaction $j_{AB} \neq j_{BA}$. This system can be regarded as a natural extension of the all-to-all coupled model studied in the previous section generalized to a spatially extended d -dimensional system. The stochastic equation of motion is given by

$$\begin{aligned} \dot{\theta}_x^A = & \frac{1}{2d} \sum_{\hat{a}} j_{AA} \sin(\theta_{x+\hat{a}}^A - \theta_x^A) \\ & + \frac{1}{2d} \sum_{\hat{b}} j_{AB} \sin(\theta_{x+\hat{b}}^B - \theta_x^A) + \eta_x^A, \end{aligned} \quad (30)$$

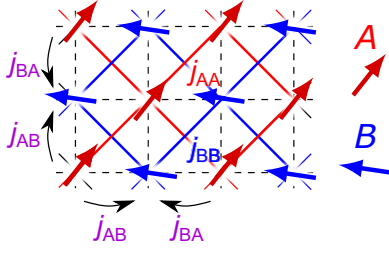


FIG. 9. Nonreciprocal XY-spin system on a hypercubic lattice. In this model, XY spins on a hypercubic lattice interact with their next-nearest-neighbor (nearest-neighbor) spins in a reciprocal (nonreciprocal) manner. The spins are divided into two communities, namely the sublattices A and B. Within each sublattice, the spins interact reciprocally via next-nearest-neighbor interactions j_{AA} and j_{BB} , while the spins on different sublattices interact nonreciprocally through nearest-neighbor interactions j_{AB} and j_{BA} . The figure illustrates the case of a two-dimensional spatial dimension ($d = 2$), but the model can be readily extended to higher spatial dimensions.

$$\begin{aligned} \dot{\theta}_x^B = & \frac{1}{2d} \sum_{\hat{a}} j_{BB} \sin(\theta_{x+\hat{a}}^B - \theta_x^B) \\ & + \frac{1}{2d} \sum_{\hat{b}} j_{BA} \sin(\theta_{x+\hat{b}}^A - \theta_x^B) + \eta_x^B. \end{aligned} \quad (31)$$

At small reciprocal regime $|j_+| \ll |j_-|, j_0$, one finds

$$\Delta\phi \simeq \left[-2j_+ + \frac{j_-^2 \sigma^2 \cos \Delta\phi}{4j_0} \sum_{k,k'} \frac{j_0^2 k'^2 - 2j_-^2 \cos^2 \Delta\phi}{(4j_-^2 \cos^2 \Delta\phi - j_0^2 k^2)(4j_-^2 \cos^2 \Delta\phi - j_0^2 k'^2) k^2 k'^2} \right] \sin \Delta\phi, \quad (32)$$

which has a similar form to the all-to-all coupling case [Eq. (23)]. Especially when $j_+ = 0$ (perfectly nonreciprocal limit), one can readily check that Eq. (32) has a stable fixed point at $\Delta\phi_* = \pm\pi/2$ corresponding to the chiral phase (as in the previous section). Therefore, the second term describes the entropic torque that drives the system to the spontaneous parity-broken phase, which competes with the reciprocal coupling [first term of Eq. (32)] that drives the system to the static, parity-symmetric phase. This triggers a nonreciprocal phase transition [11] from a static phase $\Delta\phi_* = 0, \pi$ to the chiral phase $\Delta\phi_* \neq 0, \pi$ as the noise strength is *increased*, again, signaling the OBDP.

We briefly note that the fluctuations are known to exhibit an anomalous enhancement that diverges for spatial dimensions below $d = 4$ at the phase transition point of a nonreciprocal phase transition [65]. A recent study [67] has further demonstrated that these significant fluctuations induce a discontinuous transition for spatial dimensions below $d = 4$. To fully capture this physics, a more advanced analysis beyond the lowest-order perturbative approach employed in this study is necessary, which

Here, the angle of spin in the position x on a sublattice A(B) is denoted by $\theta_x^{A(B)}$ and \hat{a} and \hat{b} are the vectors that point to the next-nearest- and nearest-neighbor sites, respectively. $\eta_x^a(t)$ is a white noise characterized by $\langle \eta_x^a(t) \rangle = 0$ and $\langle \eta_x^a(t) \eta_x^b(t') \rangle = \sigma \delta_{ab} \delta_{x,x'} \delta(t-t')$.

In parallel to the previous section, we first consider the deterministic case, $\sigma = 0$. In this case, as before, there is nothing that disturbs the spins from aligning within the same sublattice. As a result, all the angles in the same sublattices align, $\theta_x^A = \phi_A$ and $\theta_x^B = \phi_B$, where, as before, we have introduced the order parameter $\psi_a = r_a e^{i\phi_a} = N^{-1} \sum_x e^{i\theta_x^a}$ (where N is the number of spins at each sublattice). In such a case, Eqs. (30) and (31) reduce to the same dynamics as the two-spin system [Eq. (6)]. Therefore, in the case of perfect nonreciprocity $j_{AB} = -j_{BA}$, the accidental degeneracy of orbits arises, parametrized again by the relative angle $\Delta\phi = \phi_A - \phi_B$.

In the presence of the noise $\sigma > 0$, OBDP occurs. Because of the property that the distribution of fluctuations around the steady state $\delta\theta_x^a = \theta_x^a - \phi_a$ is strongly dependent on the orbit of the order parameter $\Delta\phi$, the order parameter dynamics follows the same form as Eqs. (17) and (18), with the renormalized coupling that has a different expression from the all-to-all coupled case. Their explicit form is reported in Appendix B.

is beyond the scope of this paper. We emphasize, however, that our noise-induced spontaneous symmetry-breaking scenario itself would be unaffected by this fluctuation physics, as the OBDP physics is relevant outside the critical regime.

IV. NONRECIPROCALITY-INDUCED SPIN-GLASS-LIKE STATE

Another striking phenomenon arising from geometrical frustration is the emergence of spin glasses [42–48], which occurs ubiquitously in geometrically frustrated systems with random interactions. In such a situation, a macroscopic number of fixed points and saddle points are generated to make the energy landscape bumpy. This makes it extremely difficult for the system to find its global minimum, resulting in slow dynamics characterized by a power-law decay (or slower [48]) of time correlation functions and the aging phenomena [45,46,48] associated with no long-ranged spatial order.

A natural question is whether such glassy states can be generated by nonreciprocal interaction. It is tempting to

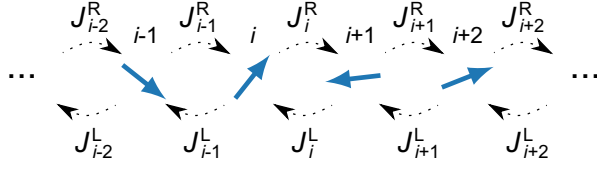


FIG. 10. One-dimensional spin chain with random asymmetric nearest-neighbor coupling.

expect the negative, as they induce the chase and run away dynamics that may cause the glass to melt. Indeed, there are a number of works that support this view [15,17,20–22,71–74] including the works in the context of neural [20,21,72–74] and ecological systems [15,17]. However, the above studies analyzed (mostly all-to-all coupled) models that already contained geometrical frustration in the reciprocal limit, making the exact role of nonreciprocal frustration unclear.

To unambiguously study the effect of nonreciprocal frustration alone, it is important for us to consider models that have *no* geometrical frustration in the reciprocal limit. For this purpose, we consider a one-dimensional XY spin chain that follows Eq. (1) that consists of N spins with

nearest-neighbor interaction $J_{ij} = J_i^R \delta_{i+1,j} + J_i^L \delta_{i,j+1}$ in an open boundary condition (Fig. 10),

$$\dot{\theta}_i = J_i^R \sin(\theta_{i+1} - \theta_i) + J_i^L \sin(\theta_{i-1} - \theta_i), \quad (33)$$

with $J_i^{L/R}$ being randomly distributed according to

$$p(J_i^{L/R}) \propto \begin{cases} e^{-(J_i^{L/R})^2/(2\sigma_J^2)} & |J_i^{L/R}| \geq J_c \\ 0 & |J_i^{L/R}| < J_c. \end{cases} \quad (34)$$

Here, $\sigma_J^2 = \langle (J_i^R)^2 \rangle = \langle (J_i^L)^2 \rangle$ characterizes the randomness of the coupling and has introduced a cutoff J_c (which we set $J_c = 0.1\sigma_J$ throughout) to prevent the coupling from completely vanishing. We also introduce an asymmetry parameter γ defined by $\langle J_i^L J_i^R \rangle \equiv \gamma \sigma_J^2$ that parametrizes the asymmetry (nonreciprocity) of the coupling. For example, $\gamma = 1$ corresponds to the reciprocal limit (where all spins satisfy $J_{ij} = J_{ji}$), while $\gamma = -1$ corresponds to the anti-symmetric limit (where all spins satisfy $J_{ij} = -J_{ji}$). $\gamma = 0$ corresponds to the case where J_{ij} and J_{ji} are independent.

This model has a crucial advantage in that no geometrical frustration exists in the reciprocal case $\gamma = 1$, and

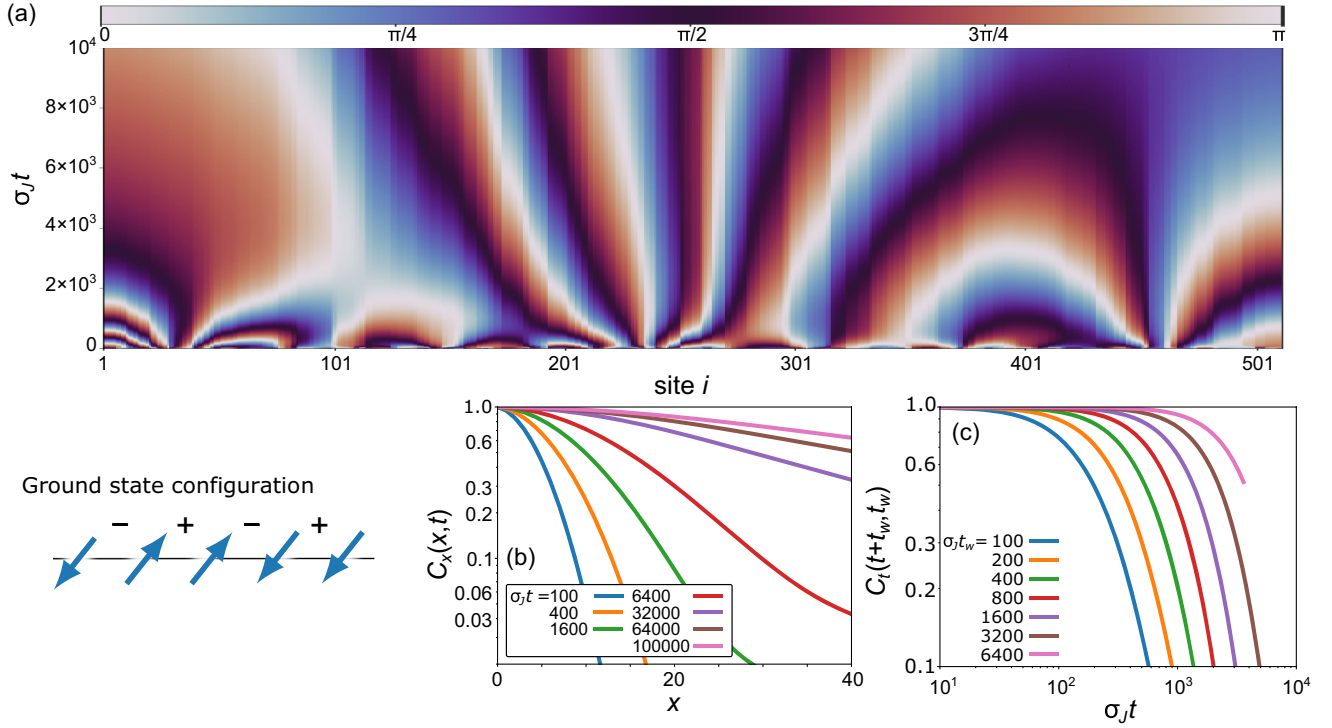


FIG. 11. Domain-wall annihilation dynamics in reciprocal one-dimensional random spin chain. The reciprocal coupling $J_i^R = J_i^L$ ($J_{ij} = J_{ji}$) case. As shown in the bottom left-hand panel, the ground state configuration of the reciprocally coupled spin chain (where the signs represent the sign of the reciprocal coupling at each bond) exhibits nematic order that is unique up to global rotation, implying the absence of geometrical frustration. (a) Typical trajectory of (nematic) angles $\varphi_i = \theta_i \pmod{\pi}$. We set $N = 2^9 = 512$, and the initial conditions were taken randomly from a uniform distribution $\theta_i = [0, 2\pi)$. (b) Spatial correlation function $C_x(x, t)$. (c) Time correlation function $C_t(t_w + t, t_w)$. In (b) and (c), we have averaged over 400 trajectories of random initial conditions and configurations of coupling strengths and have set $N = 2^{10} = 1024$. The domain-wall annihilation dynamics of this one-dimensional chain give rise to slow relaxation (that shows aging phenomena) toward a long-ranged nematically ordered state.

therefore, frustration can only arise through nonreciprocal interactions. This can be seen from the fact that the ground state configuration of the reciprocal system is uniquely determined once one fixes one of the spins (Fig. 11, bottom left-hand panel). Since reciprocal coupling favors either alignment or antialignment of spins, the ground state in the reciprocal limit exhibits a nematic order characterized by a complex order parameter $\psi_2 = (1/N) \sum_{i=1}^N e^{2i\theta_i}$.

Figure 11(a) shows a typical trajectory of $\varphi_i = \theta_i \pmod{\pi}$ (which regards the angles of the arrow pointing to opposite directions as being identical, thus making it useful to measure nematicity) in the reciprocal case, $J_i^R = J_i^L$. Here, we have set the initial state to be random. As seen, the dynamics are governed by the annihilation dynamics of the initially created (nematic) domain walls toward the nematic long-range ordered state. This is captured in the spatial correlation function,

$$C_x(x, t) = \left| \frac{1}{N-x} \sum_{i=1}^{N-x} \psi_{2,i+x}(t) \psi_{2,i}^*(t) \right|, \quad (35)$$

that is converging toward the long-range ordered state $C_x(x, t \rightarrow \infty) \rightarrow 1$ [Fig. 11(b)]. Here, $\psi_{2,i}(t) = e^{2i\theta_i(t)}$ is a complex representation of nematic direction at site i , and $\overline{(\dots)}$ represents the average over random initial conditions, with a different configuration of J_{ij} taken for each run. The temporal correlation function,

$$C_t(t_w + t, t_w) = \left| \frac{1}{N} \sum_{i=1}^N \psi_{2,i}(t_w + t) \psi_{2,i}^*(t_w) \right|, \quad (36)$$

on the other hand, converges to a constant in this phase [Fig. 11(c)]. Note that, because of the slow domain-wall annihilation process, the temporal correlation function $C(t + t_w, t_w)$ exhibits an aging behavior, i.e., the feature that the system takes more time to decorrelate as the waiting time t_w proceeds even at timescales a lot larger than microscopical timescales $t_w (\gg \sigma_J^{-1}, J_c^{-1})$ [Fig. 11(c)].

Now let us turn to the nonreciprocal case $\gamma = 0$, where we sample the couplings J_i^R and J_i^L independently. In this case, as seen in Fig. 12(a), we observe the formation of

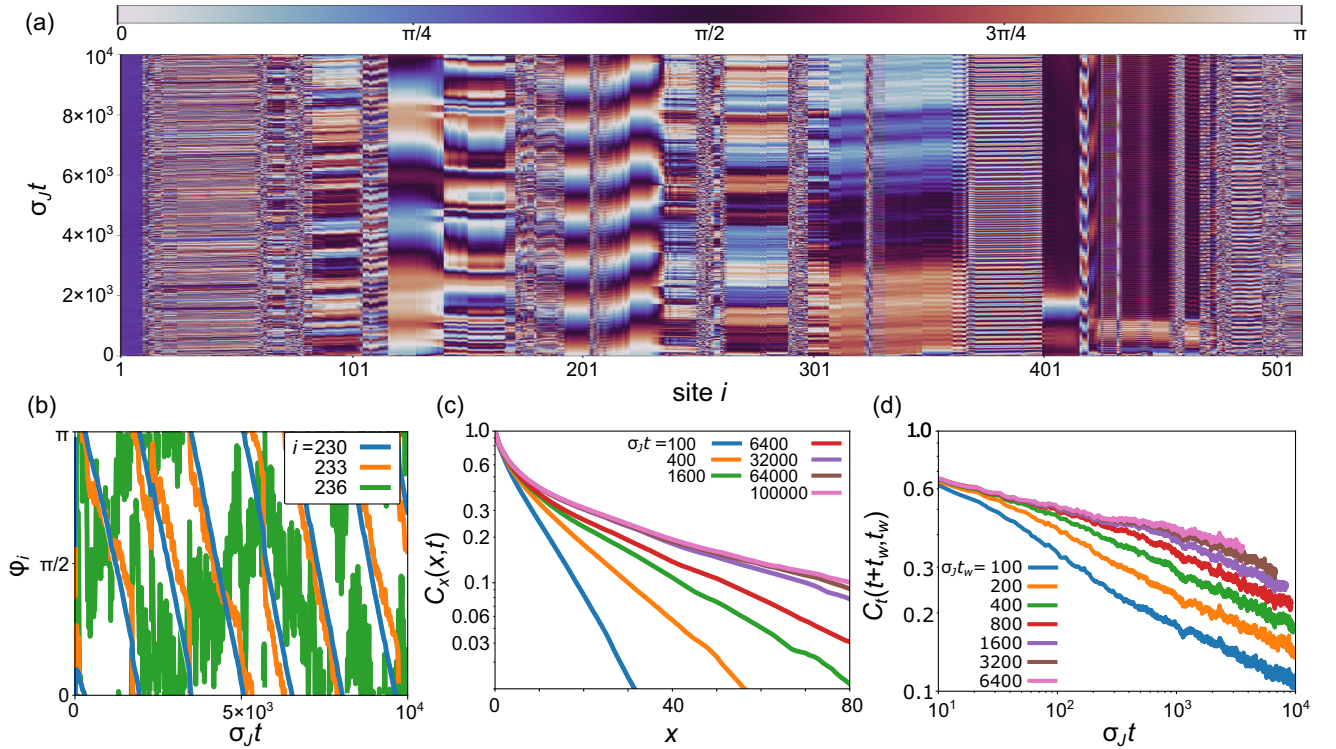


FIG. 12. Nonreciprocal frustration-induced spin-glass-like state in an asymmetric random spin chain. The asymmetric coupling case ($J_{ij} \neq J_{ji}$) with the coupling to the left J_i^L and the right J_i^R sampled independently ($\gamma = 0$). (a),(b) Typical trajectory of (nematic) angles $\varphi_i = \theta_i \pmod{\pi}$ of a one-dimensional random nonreciprocal spin chain. Here, we have set $N = 2^9 = 512$, and the initial condition was taken randomly from a uniform distribution $\theta_i = [0, 2\pi)$. (b) Line-cut data of the trajectory at site $i = 230, 233, 236$ of (a). (c) Spatial correlation function $C_x(x, t)$. (d) Time correlation function $C_t(t_w + t, t_w)$. Note that both axes are plotted on a logarithmic scale. We have averaged over 500 trajectories for $\sigma_J t \leq 6.4 \times 10^3$ in (c) and 400 trajectories for $\sigma_J t \geq 3.2 \times 10^4$ in (c) and for all plots in (d). We have set $N = 2^{10} = 1024$. This system exhibits slow dynamics characterized by power-law decay and aging phenomena and a short-ranged spatial correlation (which exhibits stretched exponential decay; see Fig. 13), where the latter property is in stark contrast to the nematically ordered state seen in the reciprocal case (Fig. 11). These properties are reminiscent of a spin glass.

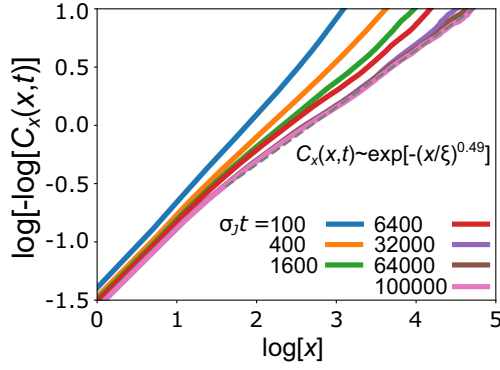


FIG. 13. Stretched exponential decay of spatial correlation function in an asymmetric random spin chain. This is identical to Fig. 12(c) but plotted in a different scale. The dashed line is a fitted curve $C_x(x,t) = e^{-(x/\xi)^\alpha}$ to the $\sigma_j t = 10^5$ result, which gives $\alpha \simeq 0.49$.

domains that are *locally* nematically ordered, in which many of them are almost time periodic [see, e.g., $i = 230$ in Fig. 12(b)], but others seem to be interrupted ($i = 233$) by the nearby chaotic domain ($i = 236$). These behaviors are vastly different from the reciprocal case of Fig. 11(a) dominated by domain-wall annihilation dynamics.

Figures 12(c) and 12(d) show the spatial and time correlation function of this asymmetric spin chain, respectively. Strikingly, the time correlation function exhibits a power-law decay $C_t(t_w + t, t_w) \sim t^{-\alpha}$ at large t with a clear sign of aging, while the state is converging toward a

short-ranged spatially correlated state [which exhibits a stretched exponential decay $C_x(x,t) \sim e^{-(|x|/\xi)^\alpha}$ with $\alpha = 0.49$, see Fig. 13], in stark contrast to the reciprocal case. These features are reminiscent of a spin glass, except that the time correlation function does not seem to converge to a finite value at $t \rightarrow \infty$ [42] (at least up to $\sigma_j t = 10^5$), implying that the state does not completely freeze to a static state.

Figure 14 shows the spatial and temporal correlation function for various values of asymmetry parameter γ . Here, to make the convergence to the nematic state in the nematic order phase faster, we have chosen the initial states to be close to the nematic phase. We observe that even at very weak nonreciprocity ($\gamma = 0.9$), qualitatively the same feature to the spin-glass-like phase observed at $\gamma = 0$ in Fig. 12 is also seen.

This slow decay observed in asymmetric cases is qualitatively different from the disordered state that occurs when stochastic noise $\eta_i(t)$ is added to Eq. (33). Figure 15 shows the spatial and temporal correlation functions in the presence of a small Gaussian white noise [where $\langle \eta_i(t) \rangle = 0$, $\langle \eta_i(t) \eta_j(t') \rangle = \sigma \delta_{ij} \delta(t - t')$]. As seen, both the spatial and temporal correlation function at the long time limit exhibit an exponential decay in the disordered phase, in qualitative difference from our spin-glass-like state in Fig. 12.

Summarizing, we have found numerical evidence that nonreciprocal coupling can induce a phase reminiscent of a spin glass, which is qualitatively different from both the

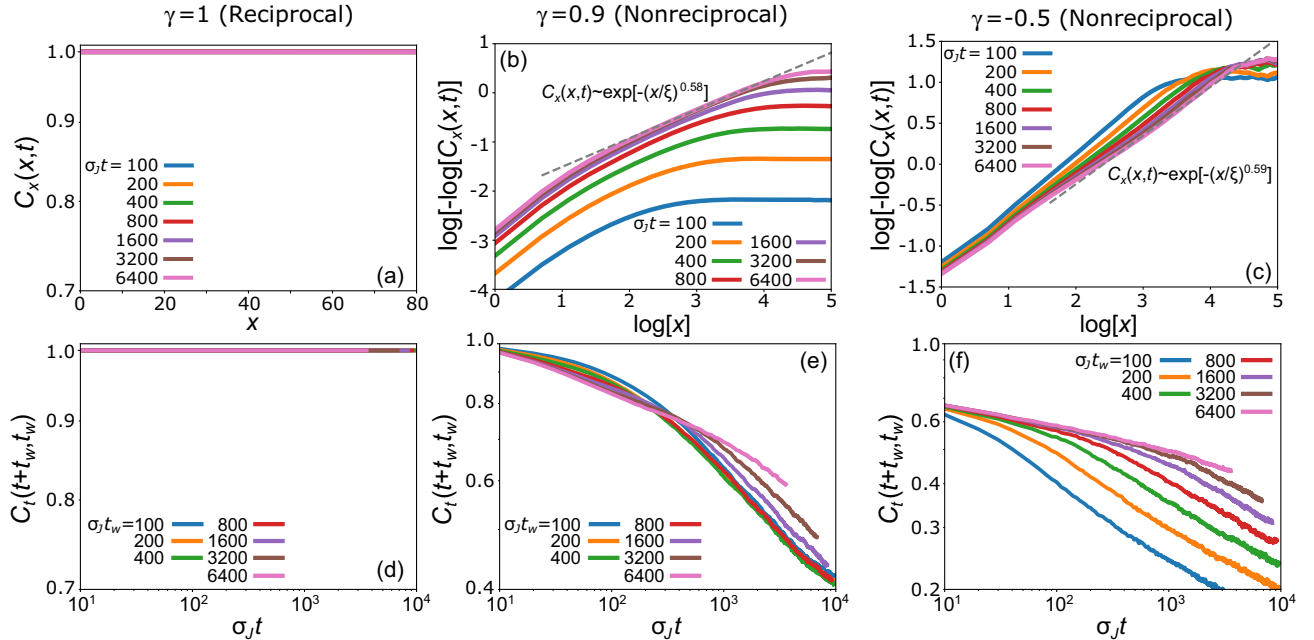


FIG. 14. Asymmetry parameter γ dependence of the spatial and time correlation functions. (a)–(c) Spatial correlation function $C_x(x,t)$ and (d)–(f) temporal correlation function $C_t(t+t_w, t_w)$. (a),(d) $\gamma = 1$. (b),(e) $\gamma = 0.9$. (c),(f) $\gamma = -0.5$. We set $N = 500$ in (a) and (d) and $N = 1000$ in (b), (c), (e), and (f) and averaged over 500 trajectories. The initial state is set to be close to a nematic phase $\theta_i = \sum_{j=1}^i \pi[1 + \text{sgn}(J_j^R)]/2 + 10^{-3}\eta_i$, where $\eta_i = [0, 2\pi)$ is a uniformly distributed random variable.

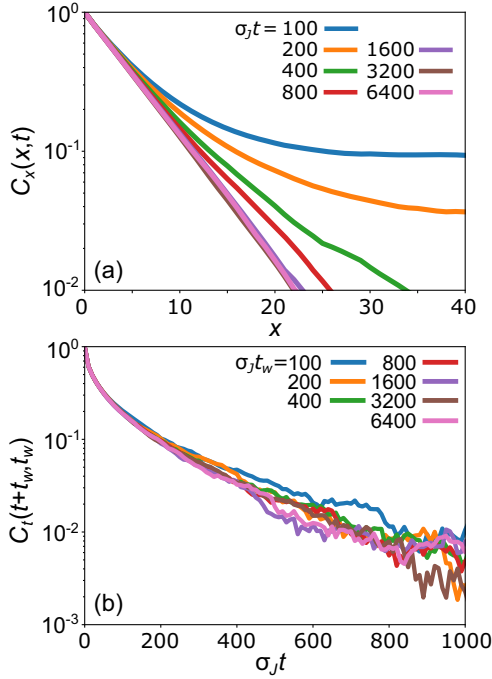


FIG. 15. Correlation function in the disordered state. (a) Spatial correlation function $C_x(x, t)$ and (b) temporal correlation function $C_t(t + t_w, t_w)$. We set $\gamma = 1$ and the noise strength $\sigma = 0.1$. We have averaged over 500 trajectories. Similarly to Fig. 14, we have set the initial state to be close to a nematic phase $\theta_i = \sum_{j=1}^i \pi[1 + \text{sgn}(J_j^R)]/2 + 10^{-3}\eta_i$, where $\eta_i = [0, 2\pi)$ is a uniformly distributed random variable.

disordered state and the nematically ordered state. For convenience, we have summarized the different behaviors of the correlation function in different phases in Table II. In contrast to both of these phases, the spin-glass-like state exhibits an algebraic decay in the temporal correlation function while exhibiting a short spatial correlation.

We remark that a similar slow decay to the one observed in our spin-glass-like phase has been observed in one-dimensional coupled logistic maps in their discrete-time evolution, as pioneered by Kaneko [75,76]. In his model, each site is itself a logistic map that exhibits bifurcations to limit cycles or chaos, and these sites are coupled with their neighboring sites. At a phenomenological level, we observe interesting similarities between our model and Kaneko's model: In the former, by regarding each domain seen in Fig. 12(a) as a chaotic or periodic element, each element seems to be attempting to align with the nearby domains,

somewhat analogous to the latter situation. However, there are also clear differences, e.g., the randomness is explicitly encoded in the former from random coupling (similarly to the original spin-glass problem) while they are generated spontaneously from chaos in the latter. The connection between the two models deserves further investigation.

V. CONCLUSION AND OUTLOOK

We have shown that nonreciprocal interaction may generate marginal orbits (accidental degeneracy) similar to those in geometrically frustrated systems, establishing a direct analogy between the two classes of systems. We have shown that the emergence of this accidental degeneracy can give rise to a dynamical counterpart of order-by-disorder phenomena and spin glasses. Our results offer an unexpected bridge between complex magnetic materials with geometrical frustration and nonreciprocal systems.

There are many possible directions for further extensions. For example, in Sec. III (where we studied order-by-disorder phenomena induced by nonreciprocal frustration), for simplicity, we have focused on models that have accidental degeneracy of orbits parametrized by just one or two parameters in the deterministic limit. This is in a similar situation to (geometrically frustrated) 2D $J_1 - J_2$ XY model [38]. However, various geometrically frustrated models have a *macroscopic* number of ground state degeneracy. They either exhibit order by disorder [37,39,40] or classical spin liquids that lack long-ranged order in the low-temperature limit $T \rightarrow 0$, depending on the number of unconstrained degrees of freedom and number of directions of gapless excitations [39,40]. We plan to study systems with such macroscopic numbers of degeneracy induced by nonreciprocal frustrations [77] to find the criterion for the emergence of order by disorder similar to those given in Refs. [39,40] for geometrically frustrated systems.

For the spin-glass-like state that was studied in Sec. IV, the purpose of focusing on a one-dimensional spin chain with nearest-neighbor interactions was to make sure that geometrical frustration is absent and therefore the glassy dynamics observed in the simulation can be safely attributed to nonreciprocal effects. In two or higher dimensions, the two types of frustrations (geometrical and nonreciprocal) may coexist. It would be interesting to ask how the conventional spin glass [that is characterized by a nonzero Edwards-Anderson order parameter [42], i.e., $q_{EA} = \lim_{t \rightarrow \infty} \lim_{t_w \rightarrow \infty} C_t(t_w + t, t) \neq 0$] evolves to the nonreciprocal spin-glass-like state (that exhibits a

TABLE II. Phases in an asymmetric random spin chain and the behaviors of correlation function.

Parameter	Phase	Spatial correlation function	Temporal correlation function
Reciprocal ($\gamma = 1$) and deterministic	Nematically ordered	Long-ranged order	Converges to a constant (slowly)
Nonreciprocal ($\gamma < 1$) and deterministic	Spin-glass-like	Stretched exponential decay	Power-law decay with aging
Stochastic	Disordered	Exponential decay	Exponential decay without aging

vanishing $q_{\text{EA}} = 0$ but still exhibits aging phenomena found in this work), and what are the properties of the phase transitions between them, if any.

Another possible direction is to extend our work to open quantum systems [25,78]. In Ref. [25], a recipe to realize nonreciprocal interaction via reservoir engineering has been proposed and nonreciprocal hopping has already been implemented experimentally [79]. It would be interesting to ask whether states reminiscent of quantum spin liquids with long-ranged entanglement can appear by nonreciprocal frustration.

Finally, in this work, we have focused on how far we can push the *analogies* between geometrical and nonreciprocal frustrations. Given that we have established such analogies, an interesting next step would be to ask what the fundamental *differences* between the two are, other than the rather obvious difference that the final states are usually time dependent in the latter.

ACKNOWLEDGMENTS

R. H. thanks Mark Bowick, Fridtjof Brauns, Aashish Clerk, Michel Fruchart, Ramin Golestanian, Hisao Hayakawa, Peter B. Littlewood, M. Cristina Marchetti, Alexander McDonald, Roderich Moessner, Vincenzo Vitelli, Cheyne Weis, and Frédéric van Wijland for discussions. He also thanks Samuel Begg for the critical reading of the manuscript. This work was supported by an appointment to the JRG program at the APCTP through the Science and Technology Promotion Fund and Lottery Fund of the Korean Government and by Grant-in-Aid for Grant-in-Aid for Research Activity Start-up from JSPS in Japan (Grant No. 23K19034), and by National Research Foundation (NRF) funded by the Ministry of Science of Korea (Grant No. RS-2023-00249900).

APPENDIX A: LIOUVILLE-TYPE THEOREM FOR PERFECTLY NONRECIPROCAL SYSTEMS

1. XY model

Here we provide the proof for the Liouville-type theorem [Eq. (3) in the main text] for the XY model:

$$\dot{\theta}_i = -\sum_j J_{ij} \sin(\theta_i - \theta_j). \quad (\text{A1})$$

We also provide its generalization to more general nonreciprocal models in later subsections. The continuity equation of the distribution function ρ for the XY model [Eq. (A1)] is given by

$$\frac{\partial \rho}{\partial t} = -\sum_i \frac{\partial(\rho \dot{\theta}_i)}{\partial \theta_i} = -\sum_i \left[\frac{\partial \rho}{\partial \theta_i} \dot{\theta}_i + \rho \frac{\partial \dot{\theta}_i}{\partial \theta_i} \right]. \quad (\text{A2})$$

In the perfectly nonreciprocal case $J_{ij} = -J_{ji}$, the second term of Eq. (A2) can be shown to vanish as

$$\rho \sum_i \frac{\partial \dot{\theta}_i}{\partial \theta_i} = \rho \sum_{ij} [J_{ij} \cos(\theta_i - \theta_j)] = 0, \quad (\text{A3})$$

where in the last equality we have used the property that J_{ij} is antisymmetric and $\cos(\theta_i - \theta_j)$ is symmetric. This gives

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial \theta_i} \dot{\theta}_i = 0, \quad (\text{A4})$$

proving the Liouville-type theorem.

2. Heisenberg model

In the above proof, note how we have used only the property that the derivative of the right-hand side of the dynamical system Eq. (A1) is antisymmetric. This suggests that the Liouville-type theorem holds more generally. For example, the Heisenberg spin $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$ (with $|\mathbf{S}_i|^2 = 1$) systems that is described by the Landau-Lifshitz equation [80],

$$\dot{\mathbf{S}}_i = -\sum_{j=1}^N J_{ij} [\mathbf{S}_i \times \mathbf{S}_j + \alpha \mathbf{S}_i \times (\mathbf{S}_i \times \mathbf{S}_j)], \quad (\text{A5})$$

can be shown to satisfy the Liouville-type theorem,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^N \sum_{\mu=x,y,z} \frac{\partial \rho}{\partial S_i^\mu} \dot{S}_i^\mu, \quad (\text{A6})$$

when the coupling is antisymmetric $J_{ij} = -J_{ji}$. This can be shown by noting that

$$\frac{\partial \rho}{\partial t} = -\sum_{i,\mu} \frac{\partial(\rho \dot{S}_i^\mu)}{\partial S_i^\mu} = -\sum_{i,\mu} \left[\frac{\partial \rho}{\partial S_i^\mu} \dot{S}_i^\mu + \rho \frac{\partial \dot{S}_i^\mu}{\partial S_i^\mu} \right]. \quad (\text{A7})$$

Rewriting Eq. (A5) as (where $\epsilon_{\mu\nu\sigma}$ is the Levi-Civita symbol)

$$\dot{S}_i^\mu = -\sum_j J_{ij} \left[\sum_{\nu\sigma} \epsilon_{\mu\nu\sigma} S_i^\nu S_j^\sigma + \alpha \left[\sum_\nu S_i^\nu S_j^\nu S_i^\mu - S_j^\mu \right] \right], \quad (\text{A8})$$

one can see that

$$\begin{aligned} \sum_{i,\mu} \frac{\partial \dot{S}_i^\mu}{\partial S_i^\mu} &= \sum_{i,j,\mu} J_{ij} \frac{\partial}{\partial S_i^\mu} \left[\sum_{\nu\sigma} \epsilon_{\mu\nu\sigma} S_i^\nu S_j^\sigma + \alpha \left[\sum_\nu S_i^\nu S_j^\nu S_i^\mu - S_j^\mu \right] \right] \\ &= \sum_{i,j,\mu} J_{ij} \left[\sum_{\nu\sigma} \epsilon_{\mu\nu\sigma} \delta_{\mu\nu} S_j^\sigma + \alpha \sum_\nu [\delta_{\mu\nu} S_j^\nu S_i^\mu + S_i^\nu S_j^\nu] \right] \\ &= \sum_{i,j,\mu} J_{ij} \alpha \left[S_i^\mu S_j^\mu + \sum_\nu S_i^\nu S_j^\nu \right] \\ &= 0 \end{aligned} \quad (\text{A9})$$

holds, where again, we have used the property that J_{ij} is antisymmetric $J_{ij} = -J_{ji}$ in the last line. Combining Eqs. (A7) and (A9) proves the Liouville-type theorem for perfectly nonreciprocal Heisenberg system [Eq. (A6)].

3. Coupled oscillators with phase delay

We consider a system composed of coupled oscillators with a phase delay [58,81]:

$$\dot{\theta}_i = \omega_i + \sum_j J_{ij} \sin(\theta_j - \theta_i + \alpha_i). \quad (\text{A10})$$

Here, $0 \leq \alpha_i \leq \pi/2$ is the phase delay, ω_i is a natural frequency, and the coupling constant $J_{ij} = J_{ji}$ is symmetric. This model is relevant for the physics of biased Josephson junctions arrays [56,57] and microscopic rotors [4] that has been derived from microscopic models.

The phase delay $\alpha_i \neq 0$ drives the coupling to be nonreciprocal. In the nonreciprocal limit $\alpha_i = \pi/2$,

$$\dot{\theta}_i = \omega_i - \sum_j J_{ij} \cos(\theta_i - \theta_j), \quad (\text{A11})$$

the Liouville-type theorem,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial \theta_i} \dot{\theta}_i = 0, \quad (\text{A12})$$

holds. This can be shown from the relation

$$\rho \sum_i \frac{\partial \dot{\theta}_i}{\partial \theta_i} = \rho \sum_{i,j} J_{ij} \sin(\theta_i - \theta_j) = 0 \quad (\text{A13})$$

that gives

$$\frac{\partial \rho}{\partial t} = - \sum_i \frac{\partial(\rho \dot{\theta}_i)}{\partial \theta_i} = - \sum_i \frac{\partial \rho}{\partial \theta_i} \dot{\theta}_i, \quad (\text{A14})$$

and, hence, Eq. (A12) is proven.

4. Nonreciprocally interacting particles

Another example is the nonreciprocally interacting particles that are realized in systems such as complex plasma [3] and chemically [5,6] and optically active colloidal matter [7,8]. The position of the particle $\mathbf{r}_i = (x_i, y_i, z_i)$ of an interacting system is given by

$$\dot{\mathbf{r}}_i = \sum_j \mathbf{f}_{ij}(|\mathbf{r}_i - \mathbf{r}_j|), \quad (\text{A15})$$

where the force $\mathbf{f}_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$ acting on the particle i from the interaction with the particle j is assumed to be a

function of the interparticle distance $|\mathbf{r}_i - \mathbf{r}_j|$. The force can in general split into reciprocal and antireciprocal contributions,

$$\mathbf{f}_{ij} = \mathbf{f}_{ij}^r(|\mathbf{r}_i - \mathbf{r}_j|) + \mathbf{f}_{ij}^a(|\mathbf{r}_i - \mathbf{r}_j|), \quad (\text{A16})$$

where $\mathbf{f}_{ij}^r(|\mathbf{r}_i - \mathbf{r}_j|) = -\mathbf{f}_{ji}^r(|\mathbf{r}_i - \mathbf{r}_j|)$ and $\mathbf{f}_{ij}^a(|\mathbf{r}_i - \mathbf{r}_j|) = \mathbf{f}_{ji}^a(|\mathbf{r}_i - \mathbf{r}_j|)$ are the reciprocal and antireciprocal contributions, respectively.

In the antireciprocal case $\mathbf{f}_{ij} = \mathbf{f}_{ij}^a(|\mathbf{r}_i - \mathbf{r}_j|)$, Liouville-type theorem holds. Similar to the spin systems, the continuity equation reads [where $\nabla_i = (\partial_{x_i}, \partial_{y_i}, \partial_{z_i})$]

$$\frac{\partial \rho}{\partial t} = - \sum_i \nabla_i \cdot (\rho \dot{\mathbf{r}}_i) = - \sum_i [(\nabla_i \rho) \cdot \dot{\mathbf{r}}_i + \rho \nabla_i \cdot \dot{\mathbf{r}}_i]. \quad (\text{A17})$$

The last term of Eq. (A17) can be shown to vanish:

$$\rho \sum_i \nabla_i \cdot \dot{\mathbf{r}}_i = \rho \sum_{ij} [\nabla_i \cdot \mathbf{f}_{ij}^a(|\mathbf{r}_i - \mathbf{r}_j|)] = 0, \quad (\text{A18})$$

since

$$\begin{aligned} \sum_{ij} \nabla_i \cdot \mathbf{f}_{ij}^a(|\mathbf{r}_i - \mathbf{r}_j|) &= \sum_{ij} \nabla_j \cdot \mathbf{f}_{ji}^a(|\mathbf{r}_j - \mathbf{r}_i|) \\ &= \sum_{ij} \nabla_j \cdot \mathbf{f}_{ij}^a(|\mathbf{r}_i - \mathbf{r}_j|) \\ &= - \sum_{ij} \nabla_i \cdot \mathbf{f}_{ij}^a(|\mathbf{r}_i - \mathbf{r}_j|). \end{aligned} \quad (\text{A19})$$

This proves the desired Liouville-type theorem,

$$\frac{\partial \rho}{\partial t} + \sum_i (\nabla_i \rho) \cdot \dot{\mathbf{r}}_i = 0, \quad (\text{A20})$$

for nonreciprocally interacting systems with antisymmetric coupling.

APPENDIX B: ORDER-BY-DISORDER PHENOMENA

1. All-to-all coupled model

We provide here the details of the analysis of order-by-disorder phenomena occurring in both geometrically and nonreciprocally frustrated systems. For concreteness, we consider the dynamics of all-to-all coupled XY model grouped into a few communities $a = A, B, C, \dots$, following the Langevin equation,

$$\dot{\theta}_i^a = - \sum_b \frac{J_{ab}}{N_b} \sum_{j=1}^{N_b} \sin(\theta_i^a - \theta_j^b) + \eta_i^a, \quad (\text{B1})$$

where $\langle \eta_i^a(t) \rangle = 0$, $\langle \eta_i^a(t) \eta_j^b(t') \rangle = \sigma \delta_{ab} \delta_{ij} \delta(t - t')$. The all-to-all coupled nature allows us to rewrite Eq. (B1) in a single spin picture,

$$\dot{\theta}_i^a = - \sum_b j_{ab} r_b \sin(\theta_i^a - \phi_b) + \eta_i^a, \quad (\text{B2})$$

by introducing the order parameter $\psi_a = (1/N_a) \sum_{i=1}^{N_a} e^{i\theta_i^a} = r_a e^{i\phi_a}$.

As emphasized in the main text, in the absence of stochasticity, the order parameter dynamics can take different orbits $\phi(t) = [\phi_A(t), \phi_B(t), \dots]$ when the inter-community coupling is taken to be geometrically or non-reciprocally frustrated, depending on their initial condition. We will show below that this ‘‘accidental degeneracy’’ of orbits is generically lifted by the presence of noise.

To proceed, we consider the dynamics of fluctuations $\delta\theta_i^a = \theta_i^a - \phi_a$ caused by noise. Assuming weak noise strength, we linearize the stochastic equation of motion as

$$\delta\dot{\theta}_i^a \approx - \sum_b j_{ab} \cos[\phi_a(t) - \phi_b(t)] \delta\theta_i^a + \eta_i^a. \quad (\text{B3})$$

As Eq. (B3) is linear, the probability distribution function $\rho_i^a(\delta\theta_i^a)$ can be computed analytically through a standard approach of mapping the Langevin equation to the Fokker-Planck equation [82] as [83,84]

$$\rho_i^a[t, \delta\theta_i^a; \phi(t)] = \frac{1}{\sqrt{\pi} w_a[t; \phi(t)]} e^{-(\delta\theta_i^a)^2 / w_a^2[t; \phi(t)]}, \quad (\text{B4})$$

with its width w_a given by

$$w_a^2[t; \phi(t)] = 2\sigma \int_0^t d\tau e^{-2\int_\tau^t d\tau' \sum_b j_{ab} \cos[\phi_a(\tau') - \phi_b(\tau')]}$$

when an initial condition is a perfectly magnetized state, $\delta\theta_i^a(t=0) = 0$. Especially in the case where $\Delta\phi_{ab} = \phi_a - \phi_b$ converges to a constant value (which occurs, e.g., in a geometrically frustrated system and two-community perfectly nonreciprocal system), the steady-state distribution has the width [82]

$$w_a^2(t \rightarrow \infty, \phi) = \frac{\sigma}{\sum_b j_{ab} \cos \Delta\phi_{ab}}. \quad (\text{B5})$$

Let us now write down the order parameter dynamics that are affected by the above fluctuations induced by noise. From

$$\dot{\psi}_a = (\dot{r}_a + r_a i \dot{\phi}_a) e^{i\phi_a} = \frac{i}{N_a} \sum_{i=1}^{N_a} \dot{\theta}_i^a e^{i\theta_i^a}, \quad (\text{B6})$$

one obtains

$$\begin{aligned} \dot{\phi}_a &= - \sum_b \frac{j_{ab}}{N_a} \sum_{i=1}^{N_a} \frac{r_b}{r_a} \sin(\theta_i^a - \phi_b) \cos(\theta_i^a - \phi_a) + \bar{\eta}_a \\ &= - \sum_b j_{ab}^*[\phi(t)] \sin(\phi_a - \phi_b) + \bar{\eta}_a \end{aligned} \quad (\text{B7})$$

that is governed by the renormalized couplings,

$$j_{ab}^*[\phi(t)] = j_{ab} \frac{r_b[\phi(t)]}{r_a[\phi(t)]} \langle \cos^2 \delta\theta_i^a \rangle_{\phi(t)}, \quad (\text{B8})$$

which are, crucially, ϕ dependent. Here, the effective noise for the macroscopic angle ϕ_a is given by $\bar{\eta}_a = 1/(r_a N_a) \sum_{i=1}^{N_a} \eta_i^a \cos \delta\theta_i^a \approx (1/N_a) \sum_{i=1}^{N_a} \eta_i^a$ that follows $\langle \bar{\eta}_a \rangle \approx 0$, $\langle \bar{\eta}_a(t) \bar{\eta}_b(t') \rangle \approx (\sigma/N_a) \delta_{ab} \delta(t - t')$, and $\langle h(\delta\theta_i^a) \rangle_{\phi(t)} = \int d\delta\theta_i^a \rho_i^a[t, \delta\theta_i^a; \phi(t)] h(\delta\theta_i^a)$ is the noise average. In the second line of Eq. (B7), we have assumed that the system self-averages, i.e., $\langle h(\delta\theta_i^a) \rangle_{\phi(t)} = (1/N_a) \sum_{i=1}^{N_a} \times h[\delta\theta_i^a(t)]$ and used the property $\rho_i^a(\delta\theta_i^a) = \rho_i^a(-\delta\theta_i^a)$. As one sees by comparing with the deterministic case [Eq. (6) in the main text], we find that the bare couplings j_{ab} has been replaced by the renormalized, ϕ -dependent coupling $j_{ab}^*(\phi)$. For later use, we expand Eq. (B7) in terms of $\delta\theta_i^a$, giving

$$\begin{aligned} j_{ab}^*(\phi(t)) &= j_{ab} \frac{\langle \cos \delta\theta_i^b \rangle_{\phi(t)}}{\langle \cos \delta\theta_i^a \rangle_{\phi(t)}} \langle \cos^2 \delta\theta_i^a \rangle_{\phi(t)} \\ &\simeq j_{ab} \frac{1 - \frac{1}{2!} \langle (\delta\theta_i^b)^2 \rangle_{\phi} + \frac{1}{4!} \langle (\delta\theta_i^b)^4 \rangle_{\phi}}{1 - \frac{1}{2!} \langle (\delta\theta_i^a)^2 \rangle_{\phi} + \frac{1}{4!} \langle (\delta\theta_i^a)^4 \rangle_{\phi}} \\ &\quad \times \left[1 - \langle (\delta\theta_i^a)^2 \rangle_{\phi} + \frac{1}{3} \langle (\delta\theta_i^a)^4 \rangle_{\phi} \right] \\ &\simeq j_{ab} \left[1 - \frac{1}{4} [w_a^2(\phi) + w_b^2(\phi)] \right. \\ &\quad \left. + \frac{1}{32} [5w_a^4(\phi) + 2w_a^2(\phi)w_b^2(\phi) + w_b^4(\phi)] \right] \\ &\simeq j_{ab} \left[W(\phi) + \frac{w_a^4(\phi)}{8} \right], \end{aligned} \quad (\text{B9})$$

where

$$\begin{aligned} W(\phi) &= 1 - \frac{1}{4} [w_a^2(\phi) + w_b^2(\phi)] \\ &\quad + \frac{1}{32} [4w_a^4(\phi) + 2w_a^2(\phi)w_b^2(\phi) + w_b^4(\phi)]. \end{aligned} \quad (\text{B10})$$

Here, we have used the relation $r_a(\phi) = \langle \cos \delta\theta_i^a \rangle_{\phi}$ in the first line of Eq. (B9), expanded in terms of $\delta\theta_i^a$ in the second line of Eq. (B9), and the result of the Gaussian integral,

$$\langle (\delta\theta_a)^2 \rangle_{\phi} = \frac{w_a^2(\phi)}{2}, \quad \langle (\delta\theta_a)^4 \rangle_{\phi} = \frac{3w_a^4(\phi)}{4}, \quad (\text{B11})$$

in the third line of Eq. (B9).

Below, we will show that Eq. (B7) generically exhibit an OBDP in both geometrically and nonreciprocally frustrated systems.

**a. Geometrically frustrated case:
Communities on a tetrahedron lattice**

Consider first a geometrically frustrated system that is composed of four communities, which is all-to-all antiferromagnetically coupled $j_{ab} = -j < 0$ [Fig. 4(a) in the main text]. We set the intracommunity ferromagnetic coupling strength to be identical $j_{aa} = j_0 > 0$ with $a, b = A, B, C, D$ and $N_a = N$, for simplicity. In the absence of noise, the system is driven toward its energy minimum that is accidentally degenerate because of the geometrical frustration. To see this, define $\mathbf{S}_a = (S_a^x, S_a^y) = (\cos \phi_a, \sin \phi_a)$ and observe that [36,39,40] the energy E can be written as

$$E(\phi) = j \sum_{a,b} \mathbf{S}_a \cdot \mathbf{S}_b = j \left(\sum_a \mathbf{S}_a \right)^2 + \text{const.} \quad (\text{B12})$$

The ground state is given by the configuration that makes $\sum_a \mathbf{S}_a$ vanish. As illustrated in Fig. 4(a) in the main text, for the case considered here, the ground state is accidentally degenerate and is parametrized by an angle α and β as

$$\begin{aligned} \phi_{A^*} &= \beta, & \phi_{B^*} &= \pi + \beta, \\ \phi_{C^*} &= \alpha + \beta, & \phi_{D^*} &= \alpha + \pi + \beta, \end{aligned} \quad (\text{B13})$$

where the angle β parametrizes the degeneracy trivially arising from the rotation symmetry, while α parametrizes the accidental degeneracy arising from geometrical frustration. The labels of the communities can be permuted.

Now, in the presence of noise ($\sigma > 0$), the width is given by [see Eq. (B5)]

$$\begin{aligned} w_a^2(\phi) &= \frac{\sigma}{j_0 + j[\cos \pi + \cos \alpha + \cos(\alpha + \pi)]} \\ &= \frac{\sigma}{j_0 + j}, \end{aligned} \quad (\text{B14})$$

which is independent of the configuration α and is identical for all communities. As a result, from Eq. (B9), one finds that $j_{ab}^*(\phi) = -j^* = \text{const} < 0$ on the ground state manifold, giving the macroscopic angle dynamics ($a, b = A, B, C, D$),

$$\dot{\phi}_a = j^* \sum_{b(\neq a)} \sin(\phi_b - \phi_a) + \bar{\eta}_a, \quad (\text{B15})$$

where $\langle \bar{\eta}_a(t) \rangle = 0$ and $\langle \bar{\eta}_a(t) \bar{\eta}_b(t') \rangle = (\sigma/N) \delta_{ab} \delta(t - t')$. Since this system obeys the fluctuation-dissipation theorem [82], the system is mapped to a problem of four spins at

very low but *finite* temperature $T \sim \sigma/N$. As we will derive below (which is pointed out in Ref. [40], Sec. IV), the distribution function for realizing the angle α in such a system is given by Eq. (14) in the main text, reproduced below for convenience,

$$\rho_{ss}(\alpha) \propto \frac{1}{|\sin \alpha|}, \quad (\text{B16})$$

in the regime $\sin^2 \alpha \gg \sigma/(N|j^*|) \rightarrow 0$. This shows that the probability distribution is overwhelmingly concentrated to a collinear configuration $\alpha_* = 0$ or $\alpha_* = \pi$, which is nothing but an OBDP. This is attributed to the property that, while the energy in generic configurations varies quadratically in displacement from the ground state configuration, there exists a special direction of displacement around the collinear configuration $\alpha = 0, \pi$ that the energy varies *quartically* [40], making the fluctuations in the collinear configuration large and therefore the entropy large.

The first step to derive Eq. (B16) is to linearize the equation of motion around the ground state configuration ϕ_* [given by Eq. (B13)]:

$$\delta \dot{\vec{\phi}} = \hat{L}(\alpha) \delta \vec{\phi} + \vec{\eta}, \quad (\text{B17})$$

where

$$\hat{L}(\alpha) = j^* \begin{pmatrix} -1 & 1 & -\cos \alpha & \cos \alpha \\ 1 & -1 & \cos \alpha & -\cos \alpha \\ -\cos \alpha & \cos \alpha & -1 & 1 \\ \cos \alpha & -\cos \alpha & 1 & -1 \end{pmatrix} \quad (\text{B18})$$

characterizes the fluctuation dynamics, $\delta \vec{\phi} = (\delta \phi_A, \delta \phi_B, \delta \phi_C, \delta \phi_D)^T = (\phi_A, \phi_B, \phi_C, \phi_D)^T - (\phi_{A^*}, \phi_{B^*}, \phi_{C^*}, \phi_{D^*})^T$ is the fluctuation, and $\vec{\eta} = (\bar{\eta}_A, \bar{\eta}_B, \bar{\eta}_C, \bar{\eta}_D)$ is the noise.

There are two zero (marginal) modes, corresponding to fluctuation within the degenerate ground state manifold. One is the Nambu-Goldstone mode $\delta \vec{\phi} = (1, 1, 1, 1)^T$ corresponding to global rotation [therefore changing the parameter β in Eq. (B13)], while the other $\delta \vec{\phi} = (1, 1, -1, -1)^T$ corresponds to changing the parameter α . The finite noise that continuously excites these zero modes gives rise to the diffusion of the probability distribution of α and β . However, in the thermodynamic limit $N \rightarrow \infty$, the noise level and the diffusion constant are negligibly small. In what follows, we will consider the timescales where this diffusion process is negligible.

On top of the two zero modes, there are two relaxational modes: one mode $(1, -1, -1, 1)^T$ with a relaxation rate $\lambda_1 = -2j^*(1 - \cos \alpha)$ and another mode $(1, -1, 1, -1)^T$ with a relaxation rate $\lambda_2 = -2j^*(1 + \cos \alpha)$. The

steady-state distribution of these fluctuation modes (denoted by $\delta\phi_1$ and $\delta\phi_2$) is then given by

$$\rho_{ss}(\delta\phi_1, \delta\phi_2; \alpha) \propto e^{-4j^*N(1-\cos\alpha)\delta\phi_1^2/\sigma} e^{-4j^*N(1+\cos\alpha)\delta\phi_2^2/\sigma}. \quad (\text{B19})$$

Integrating out these fluctuations, we arrive at the distribution function to realize the angle α as

$$\begin{aligned} \rho_{ss}(\alpha) &= \int_{-\pi}^{\pi} d\delta\phi_1 \int_{-\pi}^{\pi} d\delta\phi_2 \rho_{ss}(\delta\phi_1, \delta\phi_2; \alpha) \\ &\approx \int_{-\infty}^{\infty} d\delta\phi_1 \int_{-\infty}^{\infty} d\delta\phi_2 \rho_{ss}(\delta\phi_1, \delta\phi_2; \alpha) \\ &\propto \int_{-\infty}^{\infty} d\delta\phi_1 e^{-4j^*N(1-\cos\alpha)\delta\phi_1^2/\sigma} \\ &\quad \times \int_{-\infty}^{\infty} d\delta\phi_2 e^{-4j^*N(1+\cos\alpha)\delta\phi_2^2/\sigma} \\ &\sim \frac{1}{\sqrt{1-\cos\alpha}\sqrt{1+\cos\alpha}} = \frac{1}{|\sin\alpha|}. \end{aligned} \quad (\text{B20})$$

Here, note that the approximations that are employed are justified when $(\sigma/N) \ll |\lambda_{1,2}|$ or $\sin^2\alpha \gg \sigma/(N|j^*|) \rightarrow 0$. This derives Eq. (B16).

b. Nonreciprocally frustrated case: Two-community stochastic XY model

We now turn to the nonreciprocally frustrated case. From Eq. (B7), the dynamics of the angle difference $\Delta\phi$ that characterizes the spontaneous parity breaking is given by

$$\begin{aligned} \Delta\dot{\phi} &= -[j_{AB}^*(\Delta\phi) + j_{BA}^*(\Delta\phi)] \sin\Delta\phi \\ &\simeq -\left[2j_+ W(\Delta\phi) + \frac{j_-}{4} [w_A^4(\Delta\phi) - w_B^4(\Delta\phi)]\right] \sin\Delta\phi, \end{aligned} \quad (\text{B21})$$

with the width

$$w_A^2(\Delta\phi) = \frac{\sigma}{j_{AA} + j_- \cos\Delta\phi}, \quad (\text{B22})$$

$$w_B^2(\Delta\phi) = \frac{\sigma}{j_{BB} - j_- \cos\Delta\phi}. \quad (\text{B23})$$

Note that the second term in Eq. (B21) (i.e., the entropic torque that arises only when the nonreciprocity j_- and the noise $\sigma > 0$ are both turned on) becomes dominant in the perfectly nonreciprocal case $j_+ = 0$.

We will focus here on the strong nonreciprocal case, where the reciprocal component of the intercommunity coupling $|j_+|$ is smaller than the nonreciprocal part $|j_-|$; i.e., $|j_+| \ll |j_-|$. In particular, we will be focusing on the regime where the nonreciprocity-induced entropic force [the second term of Eq. (B21)] can become comparable to

the reciprocal component [the first term Eq. (B21)] at small noise strength. By expanding Eq. (B21) with respect to j_+ , the $\Delta\phi$ dynamics reads

$$\Delta\dot{\phi} \simeq \left[-2j_+ + \frac{j_0 j_-^2 \sigma^2}{2} \frac{\cos\Delta\phi}{(j_0^2 - j_-^2 \cos^2\Delta\phi)^2}\right] \sin\Delta\phi, \quad (\text{B24})$$

for the identical intracommunity coupling case $j_{AA} = j_{BB} = j_0$, deriving Eq. (23) [and Eq. (22) for the case of $j_+ = 0$] in the main text. As thoroughly discussed in the main text, Sec. III, the nonreciprocity-induced entropic torque triggers a nonreciprocal phase transition [11] to a chiral phase $\Delta\phi_* \neq 0, \pi$ when the noise strength σ exceeds a critical value.

It is worth noting that the nonreciprocal frustration-induced torque does not always prefer the chiral phase (that spontaneously breaks parity). To make the discussion simple, let us consider below the perfectly nonreciprocal case $j_+ = 0$. When the intracommunity coupling of community A(B) is sufficiently large compared to B(A), i.e., $j_{AA} \gg j_{BB}$ ($j_{AA} \ll j_{BB}$), the community A(B) becomes stiff such that the fluctuations of the community A(B) get strongly suppressed to give $w_A^2(\Delta\phi) < w_B^2(\Delta\phi)$ [$w_A^2(\Delta\phi) > w_B^2(\Delta\phi)$] for arbitrary $\Delta\phi$ [see Eqs. (B22) and (B23)]. As a result, no configuration can satisfy the condition that $j_{AB}^*(\Delta\phi) + j_{BA}^*(\Delta\phi) \simeq -(j_-/4)[w_A^4(\Delta\phi) - w_B^4(\Delta\phi)]$ must vanish in the chiral phase [see Eq. (20) in the main text]. In this case, the effective coupling always satisfies $j_{AB}^*(\Delta\phi) < |j_{BA}^*(\Delta\phi)|$ [$j_{AB}^*(\Delta\phi) > |j_{BA}^*(\Delta\phi)|$] for arbitrary $\Delta\phi$ when $j_- > 0$. This leads the system to select $\Delta\phi_* = \pi(0)$ for $j_- > 0$, corresponding to a static phase $\dot{\Phi}_*(t) = 0$, even in the absence of bare reciprocal coupling $j_+ = 0$. All these features are demonstrated numerically in Figs. 16(b) and 16(c).

Similar features are obtained when the noise strength is different between different communities, i.e., when the noise is characterized by $\langle \eta_a(t) \rangle = 0$, $\langle \eta_a(t) \eta_b(t') \rangle = \sigma_a \delta_{ab} \delta(t-t')$ with $\sigma_A \neq \sigma_B$. In this case, the width of fluctuation of each community is given by

$$w_A^2(\Delta\phi) = \frac{\sigma_A}{j_{AA} + j_- \cos\Delta\phi}, \quad (\text{B25})$$

$$w_B^2(\Delta\phi) = \frac{\sigma_B}{j_{BB} - j_- \cos\Delta\phi}, \quad (\text{B26})$$

leading to a similar situation to the above when the noise strength is sufficiently different between the two communities. For example, when $\sigma_A \gg \sigma_B$ ($\sigma_A \ll \sigma_B$) and $j_{AA} = j_{BB} = j_0$, $w_A^2(\Delta\phi) > w_B^2(\Delta\phi)$ [$w_A^2(\Delta\phi) < w_B^2(\Delta\phi)$] would always be satisfied for arbitrary $\Delta\phi$, leading to a static phase with (anti)aligned configuration $\Delta\phi_* = 0$ ($\Delta\phi_* = \pi$).

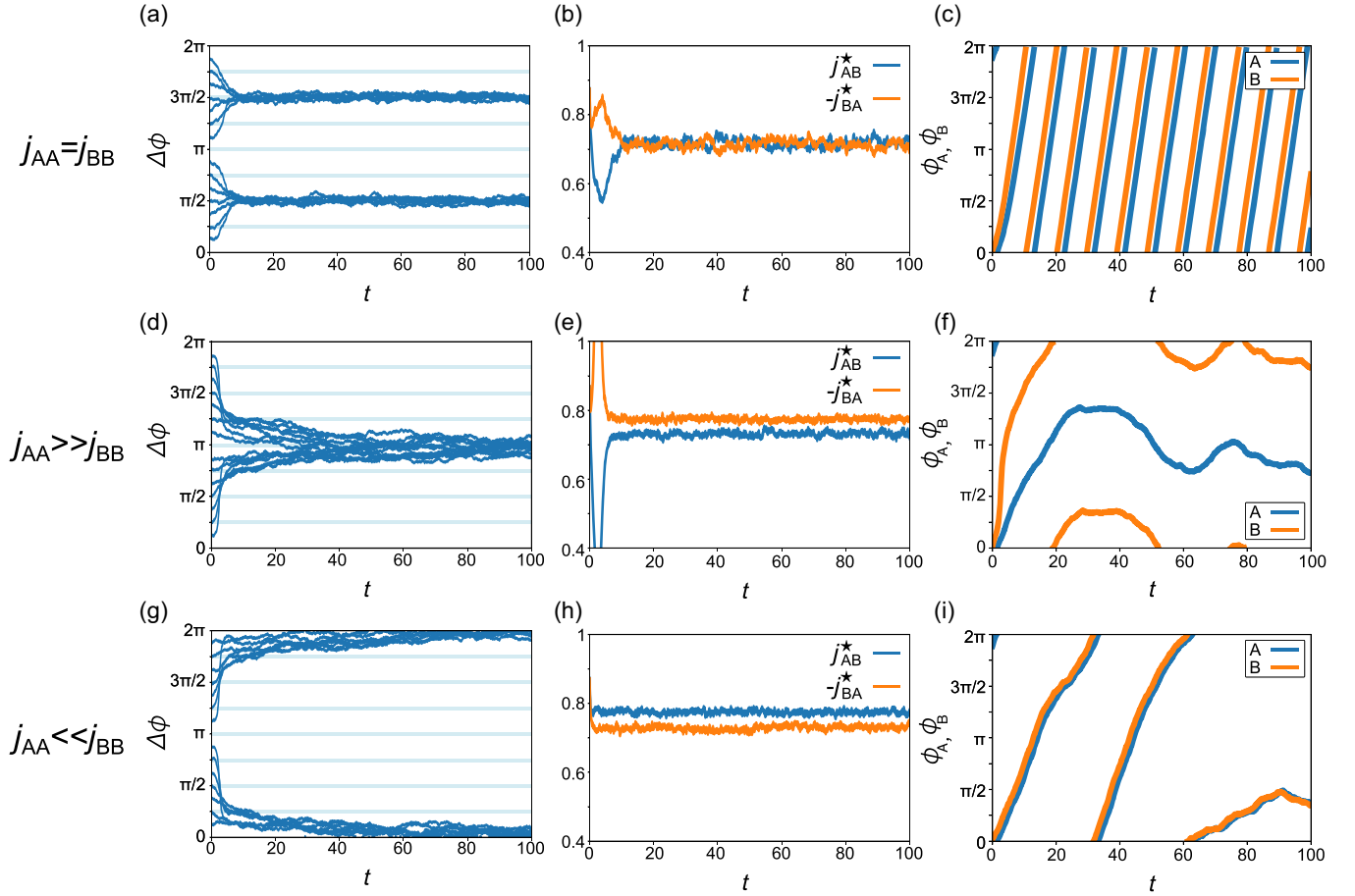


FIG. 16. Time crystalline order-by-disorder phenomena with different intracommunity coupling strength. The intracommunity coupling strength is set to (a)–(c) $j_{AA} = j_{BB} = 3$, (d)–(f) $j_{AA} = 5$, $j_{BB} = 2$, (g)–(i) $j_{AA} = 2$, $j_{BB} = 5$. (a), (d), (g) Angle difference $\Delta\phi$ dynamics, where a solid (thin) line represents the dynamics in the presence (absence) of noise. (b), (e), (h) Effective coupling $j_{ab}^*(\phi)$. (c), (f), (i) Phase ϕ_a dynamics. While in (a)–(c), the chiral phase with $\Delta\phi_* = \pm\pi/2$ that satisfies $j_{AB}^*(\Delta\phi_*) = -j_{BA}^*(\Delta\phi_*)$ is selected, in (d)–(f) [(g)–(i)], as the effective coupling $j_{BA}^*(\Delta\phi)$ [$j_{AB}^*(\Delta\phi)$] is more strongly renormalized than $j_{AB}^*(\Delta\phi)$ [$j_{BA}^*(\Delta\phi)$], one always finds $j_{AB}^*(\Delta\phi) < -j_{BA}^*(\Delta\phi)$ [$j_{AB}^*(\Delta\phi) > -j_{BA}^*(\Delta\phi)$] that stabilizes the antialigned [aligned] phase characterized by the phase difference $\Delta\phi_* = \pi$ [$\Delta\phi_* = 0$]. These results are all consistent with our analytical analysis [Eq. (B21)]. We set the noise strength $\sigma = 1.5$, the number of spins $N_A = N_B = 2000$, and the intercommunity coupling strength $j_{AB} = -j_{BA} = 1$.

2. Spatially extended model

We consider here the spatially extended model governed by Eqs. (30) and (31) (and illustrated in Fig. 9) in the main text. In the presence of noise $\sigma > 0$, the angle fluctuates around its mean value as $\delta\theta_x^a = \theta_x^a - \phi_a$, where the macroscopic angle ϕ_a is defined from the order parameter $\psi_a = r_a e^{i\phi_a} = (1/N) \sum_x e^{i\theta_x^a}$ dynamics for the sublattice $a = A, B$. In the Fourier space,

fluctuations $\delta\theta_{a,k}$ obey

$$\partial_t \begin{pmatrix} \delta\theta_{A,k} \\ \delta\theta_{B,k} \end{pmatrix} = \hat{L}_k(\Delta\phi) \begin{pmatrix} \delta\theta_{A,k} \\ \delta\theta_{B,k} \end{pmatrix} + \begin{pmatrix} \eta_{A,k} \\ \eta_{B,k} \end{pmatrix}, \quad (\text{B27})$$

where noise is characterized by $\langle \eta_{a,k}(t) \rangle = 0$, $\langle \eta_{a,k}(t) \eta_{b,k'}(t') \rangle = \sigma(2\pi)^{d+1} \delta_{a,b} \delta(t-t') \delta^d(\mathbf{k} + \mathbf{k}')$, and the dynamical matrix is given in the case of $d = 2$ as

$$\begin{aligned} \hat{L}_k(\Delta\phi) &= \begin{pmatrix} j_{AA}(\cos k_x \cos k_y - 1) - j_{AB} \cos \Delta\phi & (j_{AB}/2) \cos \Delta\phi (\cos k_x + \cos k_y) \\ (j_{BA}/2) \cos \Delta\phi (\cos k_x + \cos k_y) & j_{BB}(\cos k_x \cos k_y - 1) - j_{BA} \cos \Delta\phi \end{pmatrix} \\ &\simeq \begin{pmatrix} -\frac{j_{AA}}{2} k^2 - j_{AB} \cos \Delta\phi & j_{AB} \cos \Delta\phi \left(1 - \frac{k^2}{2}\right) \\ j_{BA} \cos \Delta\phi \left(1 - \frac{k^2}{2}\right) & -\frac{j_{BB}}{2} k^2 - j_{BA} \cos \Delta\phi \end{pmatrix}. \end{aligned} \quad (\text{B28})$$

The extension to higher spatial dimension $d > 2$ is straightforward, and the final expression is applicable for arbitrary spatial dimensions. As in the previous sections, through a standard approach of mapping the Langevin equation to the Fokker-Planck equation [82], the distribution function $\rho_{ss}(\{\delta\theta_{A,k}, \delta\theta_{B,k}\}; \Delta\phi) = \prod_k \rho_{ss,k}(\delta\theta_{A,k}, \delta\theta_{B,k}; \Delta\phi)$ can be obtained as a product of Gaussian distribution,

$$\rho_{ss,k}(\delta\theta_{A,k}, \delta\theta_{B,k}; \Delta\phi) = \frac{1}{\sqrt{(2\pi) \det \hat{\Xi}_k}} \exp\left[-\frac{1}{2} [\hat{\Xi}_k^{-1}(\Delta\phi)]_{ab} \delta\theta_{k,a} \delta\theta_{-k,b}\right], \quad (\text{B29})$$

where the correlation matrix is given by [where $j_{\pm} = (j_{AB} \pm j_{BA})/2$]

$$\hat{\Xi}_k(\Delta\phi) \simeq \frac{\sigma}{\mathbf{k}^2 (2j_+ \cos \Delta\phi + j_0 \mathbf{k}^2) [4j_0 j_+ \cos \Delta\phi - 4\cos^2 \Delta\phi (j_-^2 - j_+^2) + j_0^2 \mathbf{k}^2]} \times \begin{pmatrix} j_0^2 \mathbf{k}^4 - 2\cos \Delta\phi j_0 (j_- - 2j_+) \mathbf{k}^2 + 4\cos^2 \Delta\phi (j_-^2 + j_+^2) & 2\cos \Delta\phi [2\cos \Delta\phi (j_-^2 + j_+^2) + j_- j_+ \mathbf{k}^2] \\ 2\cos \Delta\phi [2\cos \Delta\phi (j_-^2 + j_+^2) + j_- j_+ \mathbf{k}^2] & j_0^2 \mathbf{k}^4 + 2\cos \Delta\phi j_0 (j_- - 2j_+) \mathbf{k}^2 + 4\cos^2 \Delta\phi (j_-^2 + j_+^2) \end{pmatrix}. \quad (\text{B30})$$

We have restricted ourselves to the case with $j_{AA} = j_{BB} = j_0$ for simplicity. This gives the variance of fluctuations as

$$\langle \delta\theta_x^a \delta\theta_x^b \rangle_{\Delta\phi} = \sum_k \int_{-\infty}^{\infty} d\delta\theta_{A,k} \int_{-\infty}^{\infty} d\delta\theta_{B,k} \rho_{ss,k}(\delta\theta_{A,k}, \delta\theta_{B,k}; \Delta\phi) \delta\theta_{a,k} \delta\theta_{b,-k} = \frac{1}{4} \sum_k \Xi_{k,ab}(\Delta\phi). \quad (\text{B31})$$

We briefly note that, while generically the correlation matrix is inversely proportional to \mathbf{k}^2 ($\hat{\Xi}_k \propto \mathbf{k}^{-2}$), correlation matrix behave as $\hat{\Xi}_k \propto \mathbf{k}^{-4}$ in the case of perfect nonreciprocity $j_+ = 0$. This feature, which implies the diverging fluctuations $\langle (\delta\theta_x^a)^2 \rangle \sim \int dk k^{d-1} |\delta\theta_{k,a}|^2 \sim \int dk k^{d-1} k^{-4} \rightarrow \infty$ at $d < 4$ (implying the destruction of long-range order at $d < 4$) in the vicinity of a critical point, is a characteristic of critical exceptional point, which is the salient feature of nonreciprocal phase transitions [65,67].

Equipped with the distribution of fluctuations, we now consider their impact on the macroscopic phase ϕ_a dynamics. The dynamics of the order parameter reads

$$\dot{\psi}_a = (\dot{r}_a + i r_a \dot{\phi}_a) e^{i\phi_a} = i \sum_x \dot{\theta}_x^a e^{i\theta_x^a}. \quad (\text{B32})$$

Plugging in the governing equation (30) and (31) in the main text, we arrive at the same form as Eq. (11) in the main text, reproduced below for convenience:

$$\dot{\phi}_a(t) = -\sum_b j_{ab}^* (\phi(t)) \sin[\phi_a(t) - \phi_b(t)] + \bar{\eta}_a(t). \quad (\text{B33})$$

Here, the renormalized coupling is given by

$$W(\Delta\phi) = 1 - \frac{1}{2} \left[\langle (\delta\theta_x^A)^2 \rangle_{\Delta\phi} + \langle (\delta\theta_x^B)^2 \rangle_{\Delta\phi} - 2 \langle \delta\theta_x^A \delta\theta_{x+\hat{b}}^B \rangle_{\Delta\phi} \right] + \frac{1}{24} \left[\langle (\delta\theta_x^A)^4 \rangle_{\Delta\phi} + \langle (\delta\theta_x^B)^4 \rangle_{\Delta\phi} + 12 \langle (\delta\theta_x^A)^2 (\delta\theta_x^B)^2 \rangle_{\Delta\phi} - 4 \langle (\delta\theta_x^A)^3 \delta\theta_{x+\hat{b}}^B \rangle_{\Delta\phi} - 4 \langle \delta\theta_x^A (\delta\theta_{x+\hat{b}}^B)^3 \rangle_{\Delta\phi} \right]. \quad (\text{B38})$$

$$j_{AB}^*(\Delta\phi) = \frac{j_{AB}}{r_A(\Delta\phi)} \langle \cos(\delta\theta_{x+\hat{b}}^B - \delta\theta_x^A) \cos \delta\theta_x^A \rangle_{\Delta\phi}, \quad (\text{B34})$$

$$j_{BA}^*(\Delta\phi) = \frac{j_{BA}}{r_B(\Delta\phi)} \langle \cos(\delta\theta_{x+\hat{b}}^A - \delta\theta_x^B) \cos \delta\theta_x^B \rangle_{\Delta\phi}, \quad (\text{B35})$$

where $\langle \dots \rangle_{\Delta\phi} = \int \prod d\delta\theta_x^A d\delta\theta_x^B \rho(\{\delta\theta_x^A, \delta\theta_x^B\}; \Delta\phi) (\dots)$ is the noise average with the configuration $\Delta\phi$ [where $\rho(\{\delta\theta_x^A, \delta\theta_x^B\}; \Delta\phi)$ is the distribution to realize $(\delta\theta_x^A, \delta\theta_x^B)$ with $\Delta\phi$] and $r_a(\Delta\phi) = \langle \cos \delta\theta_x^a \rangle_{\Delta\phi}$. $\bar{\eta}_a$ is a macroscopic noise that has the noise strength of σ/N . We have assumed that the system self-averages.

We proceed by expanding Eqs. (B34) and (B35) in terms of fluctuations as

$$j_{AB}^*(\Delta\phi) \simeq j_{AB} \left[W(\Delta\phi) + \frac{1}{4} \langle (\delta\theta_x^A)^2 \rangle_{\Delta\phi}^2 \right], \quad (\text{B36})$$

$$j_{BA}^*(\Delta\phi) \simeq j_{BA} \left[W(\Delta\phi) + \frac{1}{4} \langle (\delta\theta_x^B)^2 \rangle_{\Delta\phi}^2 \right], \quad (\text{B37})$$

where

The dynamics of $\Delta\phi$, which characterizes the parity-breaking order of the chiral phase [11], then follows:

$$\begin{aligned}\Delta\dot{\phi} &= -[j_{AB}^*(\Delta\phi) + j_{BA}^*(\Delta\phi)] \sin \Delta\phi \\ &\simeq -\left[2j_+ W(\Delta\phi) + \frac{j_-}{2} [(\delta\theta_x^A)^2]_{\Delta\phi}^2 - (\delta\theta_x^B)^2]_{\Delta\phi}^2\right] \sin \Delta\phi.\end{aligned}\quad (\text{B39})$$

Further assuming $j_+ \ll j_-, j_0$, using Eq. (B31), we find

$$\Delta\dot{\phi} \simeq \left[-2j_+ + \frac{j_-^2 \sigma^2 \cos \Delta\phi}{4j_0} \sum_{\mathbf{k}, \mathbf{k}'} \frac{j_0^2 \mathbf{k}'^2 - 2j_-^2 \cos^2 \Delta\phi}{(4j_-^2 \cos^2 \Delta\phi - j_0^2 \mathbf{k}^2)(4j_-^2 \cos^2 \Delta\phi - j_0^2 \mathbf{k}'^2)}\right] \sin \Delta\phi, \quad (\text{B40})$$

which derives Eq. (32) in the main text.

-
- [1] Mark J. Bowick, Nikta Fakhri, M. Cristina Marchetti, and Sriram Ramaswamy, *Symmetry, thermodynamics, and topology in active matter*, *Phys. Rev. X* **12**, 010501 (2022).
- [2] Suraj Shankar, Anton Souslov, Mark J. Bowick, M. Cristina Marchetti, and Vincenzo Vitelli, *Topological active matter*, *Nat. Rev. Phys.* **4**, 380 (2022).
- [3] A. V. Ivlev, J. Bartnick, M. Heinen, C.-R. Du, V. Nosenko, and H. Löwen, *Statistical mechanics where Newton's third law is broken*, *Phys. Rev. X* **5**, 011035 (2015).
- [4] Nariya Uchida and Ramin Golestanian, *Synchronization and collective dynamics in a carpet of microfluidic rotors*, *Phys. Rev. Lett.* **104**, 178103 (2010).
- [5] Rodrigo Soto and Ramin Golestanian, *Self-assembly of catalytically active colloidal molecules: Tailoring activity through surface chemistry*, *Phys. Rev. Lett.* **112**, 068301 (2014).
- [6] Suropriya Saha, Sriram Ramaswamy, and Ramin Golestanian, *Pairing, waltzing and scattering of chemotactic active colloids*, *New J. Phys.* **21**, 063006 (2019).
- [7] Yuval Yifat, Delphine Coursault, Curtis W. Peterson, John Parker, Ying Bao, Stephen K. Gray, Stuart A. Rice, and Norbert F. Scherer, *Reactive optical matter: Light-induced motility in electrostatically asymmetric nanoscale scatterers*, *Light Sci. Appl.* **7**, 105 (2018).
- [8] John Parker, Curtis W. Peterson, Yuval Yifat, Stuart A. Rice, Zijie Yan, Stephen K. Gray, and Norbert F. Scherer, *Optical matter machines: Angular momentum conversion by collective modes in optically bound nanoparticle arrays*, *Optica* **7**, 1341 (2020).
- [9] Jie Zhang, Ricard Alert, Jing Yan, Ned S Wingreen, and Steve Granick, *Active phase separation by turning towards regions of higher density*, *Nat. Phys.* **17**, 961 (2021).
- [10] Tzer Han Tan, Alexander Mietke, Junang Li, Yuchao Chen, Hugh Higinbotham, Peter J. Foster, Shreyas Gokhale, Jörn Dunkel, and Nikta Fakhri, *Odd dynamics of living chiral crystals*, *Nature (London)* **607**, 287 (2022).
- [11] Michel Fruchart, Ryo Hanai, Peter B. Littlewood, and Vincenzo Vitelli, *Non-reciprocal phase transitions*, *Nature (London)* **592**, 363 (2021).
- [12] Michael G. Neubert and Hal Caswell, *Alternatives to resilience for measuring the responses of ecological systems to perturbations*, *Ecology* **78**, 653 (1997).
- [13] Benjamin Kerr, Margaret A. Riley, Marcus W. Feldman, and Brendan J. M. Bohannan, *Local dispersal promotes biodiversity in a real-life game of rock-paper-scissors*, *Nature (London)* **418**, 171 (2002).
- [14] Tobias Reichenbach, Mauro Mobilia, and Erwin Frey, *Mobility promotes and jeopardizes biodiversity in rock-paper-scissors games*, *Nature (London)* **448**, 1046 (2007).
- [15] H. Rieger, *Solvable model of a complex ecosystem with randomly interacting species*, *J. Phys. A* **22**, 3447 (1989).
- [16] Stefano Allesina and Si Tang, *Stability criteria for complex ecosystems*, *Nature (London)* **483**, 205 (2012).
- [17] Guy Bunin, *Ecological communities with Lotka-Volterra dynamics*, *Phys. Rev. E* **95**, 042414 (2017).
- [18] Hyunsuk Hong and Steven H. Strogatz, *Kuramoto model of coupled oscillators with positive and negative coupling parameters: An example of conformist and contrarian oscillators*, *Phys. Rev. Lett.* **106**, 054102 (2011).
- [19] H. R. Wilson and J. D. Cowan, *Excitatory and inhibitory interactions in localized populations of model neurons*, *Biophys. J.* **12**, 1 (1972).
- [20] H. Sompolinsky and I. Kanter, *Temporal association in asymmetric neural networks*, *Phys. Rev. Lett.* **57**, 2861 (1986).
- [21] H. Rieger, M. Schreckenberg, and J. Zittartz, *Glauber dynamics of neural network models*, *J. Phys. A* **21**, L263 (1988).
- [22] H. Rieger, M. Schreckenberg, and J. Zittartz, *Glauber dynamics of the asymmetric SK-model*, *Z. Phys. B* **74**, 527 (1989).
- [23] Martin Brandenbourger, Xander Locsin, Edan Lerner, and Corentin Coulais, *Non-reciprocal robotic metamaterials*, *Nat. Commun.* **10**, 4608 (2019).
- [24] Yuto Ashida, Zongping Gong, and Masahito Ueda, *Non-Hermitian physics*, *Adv. Phys.* **69**, 249 (2021).
- [25] A. Metelmann and A. A. Clerk, *Nonreciprocal photon transmission and amplification via reservoir engineering*, *Phys. Rev. X* **5**, 021025 (2015).
- [26] Naomichi Hatano and David R. Nelson, *Localization transitions in non-Hermitian quantum mechanics*, *Phys. Rev. Lett.* **77**, 570 (1996).

- [27] Ezequiel I. Rodríguez Chiacchio, Andreas Nunnenkamp, and Matteo Brunelli, *Nonreciprocal Dicke model*, *Phys. Rev. Lett.* **131**, 113602 (2023).
- [28] Zhihong You, Aparna Baskaran, and M. Cristina Marchetti, *Nonreciprocity as a generic route to traveling states*, *Proc. Natl. Acad. Sci. U.S.A.* **117**, 19767 (2020).
- [29] Suropriya Saha, Jaime Agudo-Canalejo, and Ramin Golestanian, *Scalar active mixtures: The nonreciprocal Cahn-Hilliard model*, *Phys. Rev. X* **10**, 041009 (2020).
- [30] Cheyne Weis, Michel Fruchart, Ryo Hanai, Kyle Kawagoe, Peter B. Littlewood, and Vincenzo Vitelli, *Coalescence of attractors: Exceptional points in non-linear dynamical systems*, arXiv:2207.11667.
- [31] Lokrshi Prawar Dadhichi, Jitendra Kethapelli, Rahul Chajwa, Sriram Ramaswamy, and Ananyo Maitra, *Non-mutual torques and the unimportance of motility for long-range order in two-dimensional flocks*, *Phys. Rev. E* **101**, 052601 (2020).
- [32] Sarah A.M. Loos, Sabine H.L. Klapp, and Thomas Martynec, *Long-range order and directional defect propagation in the nonreciprocal XY model with vision cone interactions*, *Phys. Rev. Lett.* **130**, 198301 (2023).
- [33] Colin Scheibner, Anton Souslov, Debarghya Banerjee, Piotr Surówka, William T.M. Irvine, and Vincenzo Vitelli, *Odd elasticity*, *Nat. Phys.* **16**, 475 (2020).
- [34] Cynthia J. O. Reichhardt and Charles Reichhardt, *An exceptional view of phase transitions in non-equilibrium systems*, *Nature (London)* **592**, 355 (2021).
- [35] G. Toulouse, *Theory of the frustration effect in spin glasses. I*, *Commun. Phys.* **2**, 115 (1977).
- [36] Roderich Moessner and Arthur P. Ramirez, *Geometrical frustration*, *Phys. Today* **59**, No. 2, 24 (2006).
- [37] J. Villain, R. Bidaux, J.-P. Carton, and R. Conte, *Order as an effect of disorder*, *J. Phys. (Paris)* **41**, 1263 (1980).
- [38] Christopher L. Henley, *Ordering due to disorder in a frustrated vector antiferromagnet*, *Phys. Rev. Lett.* **62**, 2056 (1989).
- [39] R. Moessner and J. T. Chalker, *Properties of a classical spin liquid: The Heisenberg pyrochlore antiferromagnet*, *Phys. Rev. Lett.* **80**, 2929 (1998).
- [40] R. Moessner and J. T. Chalker, *Low-temperature properties of classical geometrically frustrated antiferromagnets*, *Phys. Rev. B* **58**, 12049 (1998).
- [41] Doron Bergman, Jason Alicea, Emanuel Gull, Simon Trebst, and Leon Balents, *Order-by-disorder and spiral spin-liquid in frustrated diamond-lattice antiferromagnets*, *Nat. Phys.* **3**, 487 (2007).
- [42] S. F. Edwards, *Theory of spin glasses*, *J. Phys. F* **5**, 965 (1975).
- [43] David Sherrington and Scott Kirkpatrick, *Solvable model of a spin-glass*, *Phys. Rev. Lett.* **35**, 1792 (1975).
- [44] G. Parisi, *Infinite number of order parameters for spin-glasses*, *Phys. Rev. Lett.* **43**, 1754 (1979).
- [45] H. Sompolinsky and Annette Zippelius, *Relaxational dynamics of the Edwards-Anderson model and the mean-field theory of spin-glasses*, *Phys. Rev. B* **25**, 6860 (1982).
- [46] L. F. Cugliandolo and J. Kurchan, *Analytical solution of the off-equilibrium dynamics of a long-range spin-glass model*, *Phys. Rev. Lett.* **71**, 173 (1993).
- [47] Daniel S. Fisher and David A. Huse, *Equilibrium behavior of the spin-glass ordered phase*, *Phys. Rev. B* **38**, 386 (1988).
- [48] Daniel S. Fisher and David A. Huse, *Nonequilibrium dynamics of spin glasses*, *Phys. Rev. B* **38**, 373 (1988).
- [49] C. Castelnovo, C. Castelnovo, and S.L. Sondhi, *Spin ice, fractionalization, and topological order*, *Annu. Rev. Condens. Matter Phys.* **3**, 35 (2012).
- [50] Lucile Savary and Leon Balents, *Quantum spin liquids: A review*, *Rep. Prog. Phys.* **80**, 016502 (2016).
- [51] Yuan Wan and Roderich Moessner, *Control of the effective free-energy landscape in a frustrated magnet by a field pulse*, *Phys. Rev. Lett.* **119**, 167203 (2017).
- [52] Nikolaj Bittner, Denis Golež, Martin Eckstein, and Philipp Werner, *Effects of frustration on the nonequilibrium dynamics of photoexcited lattice systems*, *Phys. Rev. B* **102**, 235169 (2020).
- [53] Hui-Ke Jin, Andrea Pizzi, and Johannes Knolle, *Prethermal nematic order and staircase heating in a driven frustrated Ising magnet with dipolar interactions*, *Phys. Rev. B* **106**, 144312 (2022).
- [54] Mingxi Yue and Zi Cai, *Prethermal time-crystalline spin ice and monopole confinement in a driven magnet*, *Phys. Rev. Lett.* **131**, 056502 (2023).
- [55] Camille Jorge, Amélie Chardac, Alexis Poncet, and Denis Bartolo, *Active hydraulics laws from frustration principles*, arXiv:2305.06078.
- [56] Kurt Wiesenfeld, Pere Colet, and Steven H. Strogatz, *Synchronization transitions in a disordered Josephson series array*, *Phys. Rev. Lett.* **76**, 404 (1996).
- [57] Kurt Wiesenfeld, Pere Colet, and Steven H. Strogatz, *Frequency locking in Josephson arrays: Connection with the Kuramoto model*, *Phys. Rev. E* **57**, 1563 (1998).
- [58] H. Sakaguchi and Y. Kuramoto, *A soluble active rotator model showing phase transitions via mutual entertainment*, *Prog. Theor. Phys.* **76**, 576 (1986).
- [59] Josef Hofbauer, *Evolutionary dynamics for bimatrix games: A Hamiltonian system?*, *J. Math. Biol.* **34**, 675 (1996).
- [60] Yongjoo Baek, Alexandre P. Solon, Xinpeng Xu, Nikolai Nikola, and Yarif Kafri, *Generic long-range interactions between passive bodies in an active fluid*, *Phys. Rev. Lett.* **120**, 058002 (2018).
- [61] Krzysztof Sacha and Jakub Zakrzewski, *Time crystals: A review*, *Rep. Prog. Phys.* **81**, 016401 (2018).
- [62] Vedika Khemani, Roderich Moessner, and S.L. Sondhi, *A brief history of time crystals*, arXiv:1910.10745.
- [63] Yoshiki Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
- [64] Tosio Kato, *Perturbation Theory for Linear Operators*, 2nd ed. (Springer, Berlin, Heidelberg, 1984).
- [65] Ryo Hanai and Peter B. Littlewood, *Critical fluctuations at a many-body exceptional point*, *Phys. Rev. Res.* **2**, 033018 (2020).
- [66] Ryo Hanai, Alexander Edelman, Yoji Ohashi, and Peter B. Littlewood, *Non-Hermitian phase transition from a polariton Bose-Einstein condensate to a photon laser*, *Phys. Rev. Lett.* **122**, 185301 (2019).
- [67] Carl Philipp Zelle, Romain Daviet, Achim Rosch, and Sebastian Diehl, *Universal phenomenology at critical exceptional points of nonequilibrium $O(N)$ models*, arXiv:2304.09207.

- [68] Juan A. Acebrón, L. L. Bonilla, Conrad J. Pérez Vicente, Félix Ritort, and Renato Spigler, *The Kuramoto model: A simple paradigm for synchronization phenomena*, *Rev. Mod. Phys.* **77**, 137 (2005).
- [69] Edward Ott and Thomas M. Antonsen, *Low dimensional behavior of large systems of globally coupled oscillators*, *Chaos* **18**, 037113 (2008).
- [70] Edward Ott and Thomas M. Antonsen, *Long time evolution of phase oscillator systems*, *Chaos* **19**, 023117 (2009).
- [71] A. Crisanti and H. Sompolinsky, *Dynamics of spin systems with randomly asymmetric bonds: Ising spins and Glauber dynamics*, *Phys. Rev. A* **37**, 4865 (1988).
- [72] G. Parisi, *Asymmetric neural networks and the process of learning*, *J. Phys. A* **19**, L675 (1986).
- [73] J. A. Hertz, G. Grinstein, and S. A. Solla, *Irreversible Spin Glasses and Neural Networks* (Springer, Berlin, Heidelberg, 1987).
- [74] H. Sompolinsky, A. Crisanti, and H. J. Sommers, *Chaos in random neural networks*, *Phys. Rev. Lett.* **61**, 259 (1988).
- [75] Kunihiko Kaneko, *Pattern dynamics in spatiotemporal chaos*, *Physica (Amsterdam)* **34D**, 1 (1989).
- [76] Kunihiko Kaneko, *Overview of coupled map lattices*, *Chaos* **2**, 279 (1992).
- [77] Kurt Wiesenfeld and Peter Hadley, *Attractor crowding in oscillator arrays*, *Phys. Rev. Lett.* **62**, 1335 (1989).
- [78] Tony E. Lee and Ching-Kit Chan, *Heralded magnetism in non-Hermitian atomic systems*, *Phys. Rev. X* **4**, 041001 (2014).
- [79] Kejie Fang, Jie Luo, Anja Metelmann, Matthew H. Matheny, Florian Marquardt, Aashish A. Clerk, and Oskar Painter, *Generalized non-reciprocity in an optomechanical circuit via synthetic magnetism and reservoir engineering*, *Nat. Phys.* **13**, 465 (2017).
- [80] Amikam Aharoni, *Introduction to the Theory of Ferromagnetism* (Oxford University Press, New York, 1996).
- [81] Ernest Montbrió and Diego Pazó, *Shear diversity prevents collective synchronization*, *Phys. Rev. Lett.* **106**, 254101 (2011).
- [82] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, Amsterdam, 2003).
- [83] Fabrizio Lillo and Rosario N. Mantegna, *Drift-controlled anomalous diffusion: A solvable Gaussian model*, *Phys. Rev. E* **61**, R4675 (2000).
- [84] Kwok Sau Fa, *Fokker–Planck equation with linear and time dependent load forces*, *Eur. J. Phys.* **37**, 065101 (2016).